A Comparison of the Wilkie Model and a "Yield-Macro Model" for UK Data

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Abstract

The aim of this study is to compare two stochastic investment models - Wilkie model (Wilkie, 1995) and a "yield-macro" model (Sahin, 2010) constructed using UK yield curves, realised inflation and output gap. First we compare these two models in a philosophical way. We discuss the structures of the models by considering the economic series they cover, the period examined and the nature of the relation between these economic series. Secondly, we compare the models in a variety of empirical ways. We start with the comparison of the simulated series using the models. For this analysis we only use the common economic series such as inflation, bank base rates, consols yields and nominal spot rates. We compare the simulated zero-coupon bond yields obtained from the two models. Finally, we consider a hypothetical pension scheme and compare the real asset values along with the annuity payoffs for different investment scenarios.

KEYWORDS

Wilkie model, term structure, yield curve, output gap, UK.

1 Introduction

The Wilkie stochastic investment model, developed by A. D. Wilkie, is described fully in two papers: the original version is described in 'A Stochastic Investment Model For Actuarial Use' (Wilkie, 1986) and the model is reviewed, updated and extended in 'More On A Stochastic Asset Model For Actuarial Use' (Wilkie, 1995).

The original Wilkie model (1986) was developed from U.K. data over the period 1919-1982, and was made up of four interconnected models for price inflation, share dividend yields, share dividends and long-term interest rates. Wilkie (1995) updated the original model and extended it to include an alternative autoregressive conditional heteroscedastic (ARCH) model for price inflation, and models for wage inflation, short-term interest rates, property yields and income and index-linked yields. Furthermore, these models were fitted to data from numerous developed countries and an exchange rate model was proposed. Sahin, et al. (2008) and Wilkie et al. (2010) reviewed the Wilkie asset model for a variety of UK economic indices in each case by updating the parameters to June 2009. They discussed how the model has performed from 1994 to 2009 and estimated the values of the parameters and their confidence intervals over various sub-periods to study their stability. Their analysis shows that the residuals of many of the series are much fatter-tailed than in a normal distribution. They observe also that besides the stochastic uncertainty built into the model by the random innovations there is also parameter uncertainty arising from the estimated values of the parameters.

Sahin (2010) constructed a yield-macro model, using quarterly UK nominal government spot rates, real spot rates and implied inflation spot rates published on the Bank of England's web page as well as the macroeconomic variables annual realised inflation obtained from Retail Price Index and output gap provided by the OECD Economic Outlook publications. Due to the revision process, the latest available estimate for output gap was the end of 2007. Therefore the quarterly data for the period 1995-2007 were used for the yield-macro model .

In this paper after introducing the yield-macro model birefly we compare the Wilkie model (Wilkie, 1995; Wilkie et al. 2010) with the yield-macro model (Sahin, 2010) in two ways. First we compare these two models in a philosophical way. We discuss the structures of the models by considering the economic series they cover, the period examined and the nature of the relation between these economic series. Secondly, we compare the models in a variety of empirical ways. We start with the comparison of the simulated series using the models. For this analysis we only use the common economic series such as inflation, bank base rates, consols yields and nominal spot rates. We compare the simulated zero-coupon bond yields obtained from the two models. Finally, we consider a hypothetical pension scheme and compare the real asset values along with the annuity payoffs for different investment scenarios. Section 2 introduces the yield-macro model, Section 3 discusses the structural comparison and Section 4 and Section 5 present the empirical analysis. Finally, Section 6 concludes the paper.

2 Yield-Macro Model

2.1 Data

To construct the yield-macro model, we use quarterly UK nominal government spot rates, real spot rates and implied inflation spot rates published on the Bank of England's web page. As for the macroeconomic variables we use annual realised inflation obtained from Retail Price Index and output gap provided by the OECD Economic Outlook publications. Due to the revision process, the latest available estimate for output gap is the end of 2007. Therefore we use the quarterly data for the period 1995-2007 for the yield-macro model.

2.2 Fitting a VAR Model to the Quarterly PCs and the Macroeconomic Variables

After examining the correlations between the yield curves and macro variables we construct a vector autoregressive model for the series. We start with including the first two lags of each variable and eliminate the insignificant ones to obtain the best model. Furthermore, we avoid including simultaneous explanatory variables in to the models because in forecasting we do not want to deal with additional uncertainty rooted by the simultaneous correlations. See Sahin (2010) for the models for each variable and the coefficients of determination.

To construct the 'yield-macro' model, we use quarterly nominal spot rates, implied inflation spot rates, real spot rates, annual realised inflation and output gap over the period 1995 to 2007 [2].

Let X_Q be the matrix of quarterly yield curve data where
X_{Q_N}: Nominal spot rates (52×50)
X_{Q_I}: Implied inflation spot rates (52×46)
X_{Q_R}: Real spot rates (52×46)

Let Q be the matrix of quarterly PCs and macroeconomic variables where:
Q_{N_L}: level component of the nominal spot rates (52×1)
Q_{N_S}: slope component of the nominal spot rates (52×1)
Q_{N_C}: curvature component of the nominal spot rates (52×1)
Q_{I_L}: level component of the implied inflation spot rates (52×1)
Q_{I_S}: slope component of the implied inflation spot rates (52×1)
Q_{I_C}: curvature component of the implied inflation spot rates (52×1)
Q_{R_L}: level component of the real spot rates (52×1)
Q_{R_S}: slope component of the real spot rates (52×1)
Q_{R_C}: curvature component of the real spot rates (52×1)
Q_{RI}: realised inflation (52×1)

[2] Since the output gap data are subject to continuous revision which may take three years to get the latest estimate, the data period in this modelling work is restricted with 2007.

Q_{OG}: output gap (52×1)

The VAR structure of the quarterly model is:

$$Q[t] - \mu_Q \quad = \quad B_1 \ (Q[t-1] - \mu_Q) + B_2 \ (Q[t-2] - \mu_Q) + \epsilon_Q[t]$$

where:

μ_Q is the vector of long run mean of the variables, B_1 and B_2 are the coefficient matrices for the first and second lags of the explanatory variables respectively and $\epsilon_Q[t] \sim N(0, \Sigma_Q)$, i.e. normally distributed residuals with zero mean and Σ_Q variance-covariance matrix.

$$Q = \begin{bmatrix} Q_{N_L} \\ Q_{N_S} \\ Q_{N_C} \\ Q_{I_L} \\ Q_{I_S} \\ Q_{I_C} \\ Q_{R_L} \\ Q_{R_S} \\ Q_{R_C} \\ Q_{RI} \\ Q_{OG} \end{bmatrix}$$

$$\widehat{\mu}_Q^t = \begin{bmatrix} -6.76 & 0 & 0 & -1.47 & 0 & 0 & -6.99 & 0 & 0 & 2.88 & 0 \end{bmatrix}$$

$$\widehat{B}_1 = \begin{bmatrix} 0.92 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.78 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.96 & 0 & 0 & 0 & 0 & 0 & 0 & -0.15 & 0 \\ 0 & 0 & 0 & 0.89 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.56 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.62 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.95 & 0 & 0 & 0 & 0 \\ 0 & 0.27 & 0 & 0 & 0 & 0 & 0 & 0.49 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.86 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.92 & 0 \\ 0 & -0.04 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.89 \end{bmatrix}$$

$$\widehat{B}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1.21 & 0 & -0.41 \\ 0 & 0 & -0.34 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.32 & 0 & 0 & 0 & 1.38 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.09 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.34 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.20 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\widehat{\Sigma}_Q = \begin{bmatrix} 4.54 & & & & & & & & & & \\ -0.62 & 0.76 & & & & & & & & & \\ -0.50 & 0.07 & 0.14 & & & & & & & & \\ 2.51 & -0.22 & -0.30 & 2.28 & & & & & & & \\ -0.17 & 0.25 & 0.07 & -0.18 & 0.27 & & & & & & \\ 0.09 & -0.11 & 0.01 & 0.07 & -0.05 & 0.08 & & & & & \\ 1.88 & -0.30 & -0.19 & 0.20 & 0.05 & 0.00 & 1.61 & & & & \\ -0.53 & 0.30 & 0.05 & -0.10 & -0.02 & 0.02 & -0.40 & 0.29 & & & \\ -0.20 & 0.06 & 0.03 & -0.02 & -0.02 & -0.02 & -0.17 & 0.05 & 0.05 & & \\ 0.07 & 0.10 & -0.05 & 0.07 & 0.03 & -0.01 & 0.02 & 0.02 & -0.01 & 0.17 & \\ -0.03 & 0.03 & 0.01 & 0.04 & 0.01 & 0.01 & -0.06 & 0.02 & 0.01 & 0.02 & 0.06 \end{bmatrix}$$

The negative long run means for the level factors of the yield curves displayed in μ_Q show that these factors have been decreasing since 1995. It should be emphasized that the series we model are not the levels of the yield curves but the factors which affect the levels of the yield curves. Thus, it is not surprising that we obtain negative values for the long run mean of these factors. On the other hand, the long run mean for the realised inflation is about 3%.

When we look at the matrix $\widehat{B}_1$, although there are some off-diagonal values, the diagonal structure of the matrix shows how strong the AR(1) effect is in the models. Similarly, few number of values in $\widehat{B}_2$ shows that the second lags are mostly insignificant.

We display the estimated correlation matrix, $\widehat{\rho}_Q$ for the residuals below. As stated previously, we assume that the coefficients which are greater or less than three standard errors (0.42) are significant. As in the 'yield-only' model, we see several significant correlations between the residuals in matrix $\widehat{\rho}_Q$. Again one reason is that we exclude the simultaneous explanatory variables in the modelling work. The high correlations between the residuals for the level and slope factor models may be due to these strong simultaneous correlations between the level and slope PCs (see Sahin, 2010). Although the PCs themselves are independent within each yield curve, there is a strong negative correlation between the level and the slope factors residuals of the nominal spot rates. This might be some statistical artifact which does not really indicate a correlation between those two set of

residuals.

$$\widehat{\rho}_Q = \begin{bmatrix} \mathbf{1} & & & & & & & & & & \\ -0.34 & \mathbf{1} & & & & & & & & & \\ \mathbf{-0.63} & 0.21 & \mathbf{1} & & & & & & & & \\ \mathbf{0.80} & -0.17 & \mathbf{-0.54} & \mathbf{1} & & & & & & & \\ -0.15 & \mathbf{0.57} & 0.38 & -0.23 & \mathbf{1} & & & & & & \\ 0.15 & \mathbf{-0.50} & 0.03 & 0.20 & -0.41 & \mathbf{1} & & & & & \\ \mathbf{0.68} & -0.28 & -0.40 & 0.10 & 0.09 & -0.06 & \mathbf{1} & & & & \\ \mathbf{-0.49} & \mathbf{0.64} & 0.24 & -0.13 & -0.07 & 0.07 & \mathbf{-0.61} & \mathbf{1} & & & \\ -0.40 & 0.32 & 0.33 & -0.06 & -0.15 & -0.30 & \mathbf{-0.58} & \mathbf{0.48} & \mathbf{1} & & \\ 0.07 & 0.28 & -0.28 & 0.11 & 0.12 & -0.12 & 0.03 & 0.07 & -0.07 & \mathbf{1} & \\ -0.13 & 0.14 & 0.07 & 0.10 & 0.11 & 0.07 & -0.20 & 0.12 & 0.14 & 0.16 & \mathbf{1} \end{bmatrix}$$

3 Structural Comparison of the Models

The frequency of the data used in the models is an important feature which distinguishes the models. The Wilkie model has been constructed on yearly data while the yield-macro model is based on quarterly data. The reason for using quarterly data for our yield-macro model is that the output gap is available on a quarterly frequency. Although we develop different models based on the monthly and the yearly intervals, we would like to compare the Wilkie model with our quarterly yield-macro model because it includes all the variables we intend to model.

The historical data for the series used in the Wilkie model have been available since the 1900s while the term structures of the interest rates and implied inflation and output gap data are available since the 1980s. Using different periods of data for the two models affects the parameters estimated due to different economic conditions experienced in those periods. This also affects the simulations produced for the future years. In order to make the two models exactly comparable, we will introduce 'neutral' initial conditions and 'neutralised' parameters for the models and we will use the same initial values for our state variables to simulate the future in the next sections.

Another distinguishing feature is the output variables the models produce. Figure 1 and Figure 2 display the structures of the Wilkie model and the yield-macro model respectively. When we look at Figure 1 we see that the Wilkie model has a cascade structure and that price inflation is the driving force. It includes wage inflation, share dividend yields, share dividends, share prices, long term and short term interest rates and index-linked yields. On the other hand, the yield macro model in Figure 2 is composed of the term structures of nominal, implied inflation and real spot rates along with the realised inflation and the output gap as macroeconomic variables. Thus, while we exclude the share dividends, dividend yields and share prices and also wage inflation we incorporate two new variables namely implied inflation and the output gap. Additionally, we model the entire term structures rather than just the two ends of the yield curves.

Incorporating new variables has also changed the structure of the model. The price inflation is not the driving force of the yield-macro model because the output gap and the nominal spot rates

have influences on it. Thus, we can see that the use of different variables not only changes the structure of the models but also changes the nature of the relations between the model variables. One of the main features of the yield curve models proposed in this work is the bi-directional relations between the yield-curve factors and the macroeconomic variables.

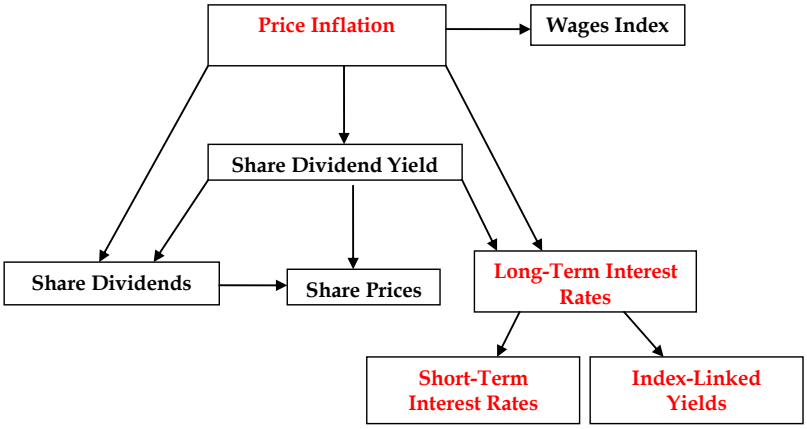

Figure 1: Structure of the Wilkie model

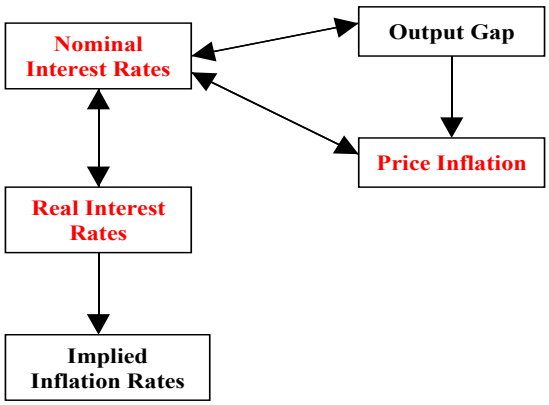

Figure 2: Structure of the Yield-Macro model

When we consider the similarities between these two models, besides indicating some common factors such as price inflation, nominal and index-linked yields we might go further and associate particular variables with the factors used in the yield-macro model. To begin with, both models include the nominal interest rates. The consols yield in the Wilkie model can be considered as

an equivalent of the 'level' and the 'log spread', $BD(t) = \ln C(t) - \ln B(t)$, where $C(t)$ and $B(t)$ long and short term bond yields, as the equivalent of the 'slope' of the nominal yield curve in the yield-macro model. However, we additionally include the 'curvature' factor of the nominal spot rates in our model.

It is possible to discuss the model formulae too. While the nominal slope factor $BD(t)$ has been modelled as an AR(1) process in the Wilkie model, the real curvature factor and output gap have been found significant in the nominal slope model as a part of the yield-macro model. Including two more explanatory variables we see that our model performs significantly better than the AR(1) model of Wilkie.

Wilkie's index-linked yield model might be compared with the 'real level factor' model of the yield-macro model. Wilkie (1995) models the index-linked yields including the residuals obtained from the consols yield model. This is consistent with the significant correlation between the residuals of the level factors of the nominal and real spot rates (see Sahin, 2010)

4 Empirical Comparisons of the Models

4.1 Simulated Economic Series

In this section we compare the Wilkie model and the yield-macro model considering the inflation models, long-term and short-term interest rates and nominal spot rates. To begin with, we simulate the inflation index for 1000 years to study the long run auto-correlation functions of the stationary components of the models. Figure 3 shows the auto- and partial auto-correlation functions of the historical data and the simulated values for the two models over 1000 years in future. The auto-correlation functions decay at different speeds for each model. The auto- and partial auto-correlation functions of the historical data and the simulated values using Wilkie model look similar while the yield-macro model differs showing the first and third lags significant in the partial auto-correlation function. Besides, the auto-correlation function for the yield-macro model decays much slower than the auto-correlation functions of the other two data sets. Since the price inflation model of Wilkie is a strict AR(1) process it has a continuously decreasing auto-correlation function and only the first lag is significant in the partial auto-correlation function. On the other hand, the price inflation part of the yield-macro model incorporates some other factors namely the nominal curvature factor and the output gap as well as depending on its previous value. The nominal curvature factor is an AR(2) process including the price inflation as an explanatory variable as well. Thus relatively complex structure of the yield-macro model produces an auto-correlation function decreasing first, then increasing a little bit and then decreasing again. The significant partial auto-correlation values for the first and the third lags are caused by the structure of the model.

Although we forecast the values in Figure 3 by simulation, it is also possible to calculate them theoretically. The calculations are straight forward for the Wilkie model whereas many matrix multiplications are required for the yield-only model.

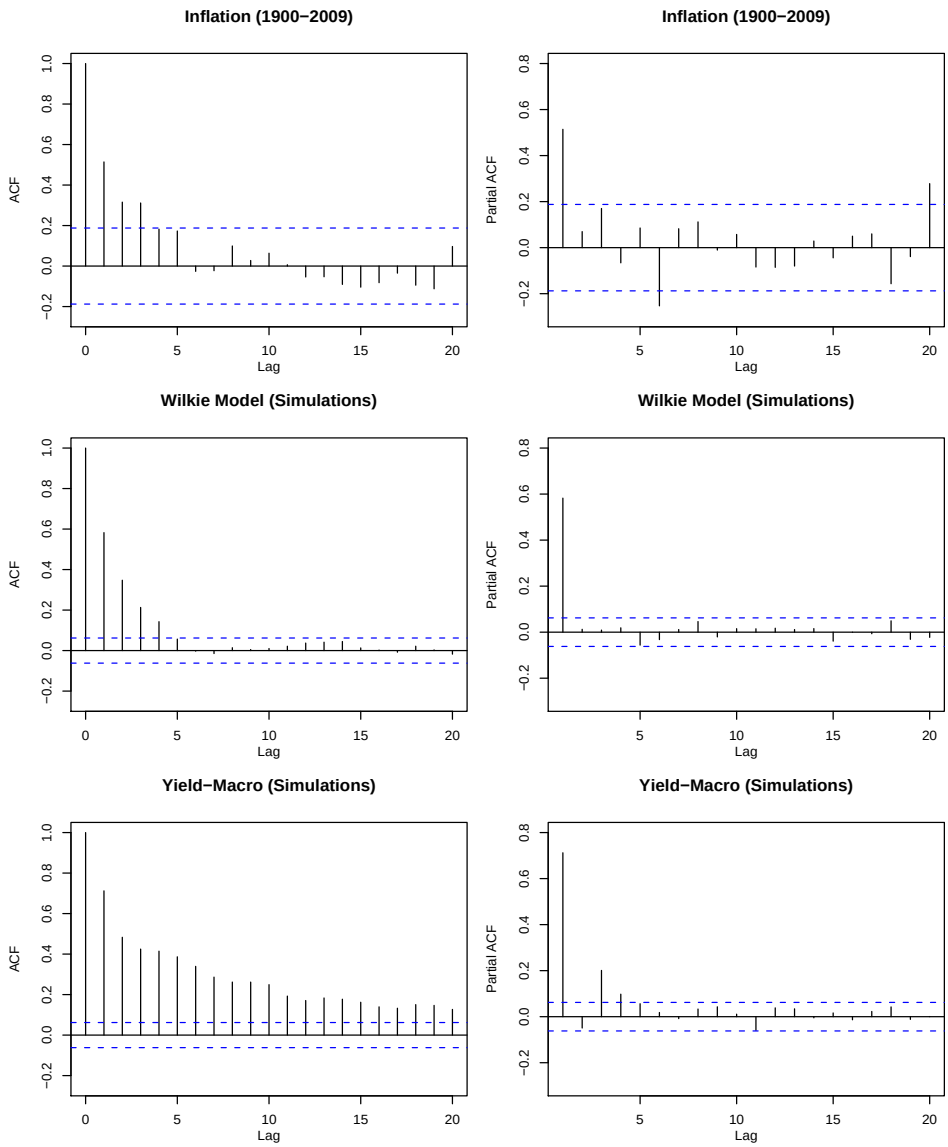

Figure 3: Autocorrelation functions for the historical and simulated price inflation rates

Since the two models are constructed based on different periods the estimated parameters are quite different from each other due to having been affected by the economic conditions of those periods. For example the long-run mean of the Wilkie price inflation model, QMU, is about 4.3% while it is equal to 2.88% for the yield-macro model. All the other means and the standard deviations are different as well. Thus, if we use the model parameters, particularly the means, as they are it is unavoidable that we would find very different economic scnearios for the two models.

All time series models need some initial conditions, that is values of the state space at time $t = 0$. Except in some special cases, the choice of initial conditions affects the short-term properties of the simulations. It is convenient therefore to start with 'unbiased' initial conditions. These unbiased initial conditions are what Wilkie (1995) and Lee and Wilkie (2000) call 'neutral' initial conditions.

For a linear model, these neutral conditions might be the means and for non-linear models these might be long-run expected values, or alternatively, long-run medians. It may also be interesting to see the effect of biased initial conditions, or market condition on a particular date but we do not do this here.

In order to make the two models, the Wilkie model and the yield-macro model, exactly comparable we introduce some 'neutral' initial conditions and 'neutralised' parameters (Lee and Wilkie, 2000). To begin with, we use 'neutral' initial conditions for the yield-macro model by setting the starting values at their long-run means. We obtain these long-run means by setting the standard deviations at zero. By using the neutral initial conditions for the yield-macro model we derive the zero coupon yield curves. Converting the initial zero-coupon yield curve into the par yield curve gives us the initial values for the long-term and short-term bond yields of the Wilkie model. Thus we use the same initial conditions for both models. However, while those initial conditions are neutral starting values for the yield-macro model, they are not neutral for the Wilkie model. Therefore, we adjust (or 'neutralise' (Lee and Wilkie, 2000)) the mean parameters of the Wilkie model according to the initial conditions so that those initial conditions would be neutral for the parameter-adjusted Wilkie model.

For the inflation model the initial value for the yield-macro model is the long-run mean and we use that value as the initial condition for Wilkie's inflation model. When we set the standard deviation of the Wilkie model to zero, we see that the initial condition becomes the long-run mean of the Wilkie inflation model as well. Therefore, for the inflation models, both the initial conditions and the mean parameters are the same and equal to 2.88%. We have done the same for the yield curves too. Note that we start with the same initial conditions for the two models and we adjust only the mean parameters of the Wilkie model based on these initial conditions.

After all these adjustments we can now compare these two models empirically. The economic series have been simulated for the next 35 years in this application.

Once we derive the RPI values after simulating the inflation values for both models we plot the empirical cumulative distribution functions (ECDF) for specific years to compare the distributions of the simulated values in Figure 4. Since the Wilkie inflation model has a higher standard deviation which has been caused by the data period including some extreme values, the distribution of the RPI values are more dispersed than the values obtained from the yield-macro model.

4.2 Simulated Zero-Coupon Yields

We can also compare the zero-coupon bond yields for different maturities and different forecast years obtained from the two models. In order to do such a comparison: First we simulate the short and long-term interest rates of the Wilkie model. Using these simulated values we construct the par yield curve for each year using Equation 1 in Lee and Wilkie (2000) and Wilkie *et al.* (2003).

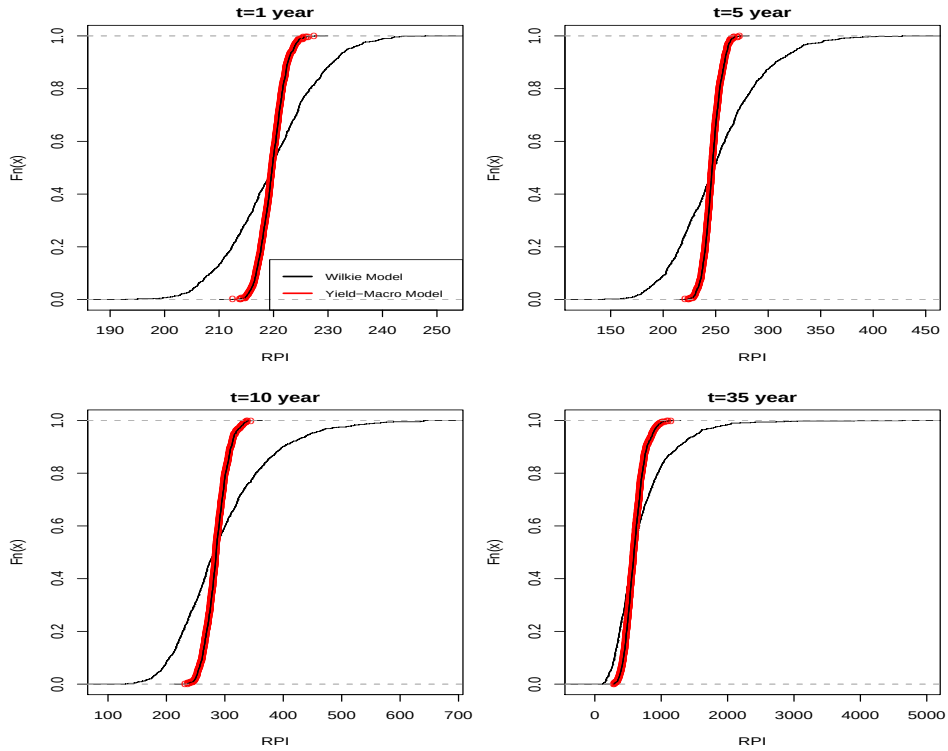

Figure 4: Empirical Cumulative Distribution Functions for the Simulated RPI Values over 35 Years

$$Y(t,n) = C(t) + (B(t) - C(t))\exp(-\beta n) \tag{1}$$

where $Y(t,n)$ is the par yield at time t for term n, $B(t)$ is the base rate, $C(t)$ is the consols yield from the Wilkie model and β is a constant whose value will be given later. We then derive the zero-coupon rates, at annual intervals, recursively, as follows:

Let $v(t,n)$ be the value at time t of a zero-coupon bond of term n.

Then the value of a coupon bond of term n, currently priced at par, with coupon equal to the par yield $Y(t,n)$, and redeemable at par, means that we have, for each n,

$$1 = Y(t,n)\sum_{m=1}^{n} v(t,m) + v(t,n). \tag{2}$$

Given the values of $Y(t,n)$, we can use Equation 2 to derive the $v(t,n)$ recursively.

Starting with $n = 1$, we have

$$1 = Y(t,1)\sum_{m=1}^{1} v(t,m) + v(t,1)$$

whence $v(t,1) = 1/(Y(t,1)+1)$.

We continue year by year:

$$1 = Y(t,n)\sum_{m=1}^{n-1} v(t,m) + (1+Y(t,n))v(t,n)$$

whence $v(t,n) = \left(1 - Y(t,n)\sum_{m=1}^{n-1} v(t,m)\right)/(1+Y(t,n))$.
From the values of $v(t,n)$ we can derive a zero-coupon yield curve:

$$Z(t,n) = \frac{1}{v(t,n)^{1/n}} - 1 \tag{3}$$

Wilkie *et al.* (2003) indicate a problem about this approach which we have encountered in our calculations too. When calculating the zero-coupon discount factor $v(t,n)$, the sum of the values of the coupons from years one to $n-1$, $Y(t,n)\sum_{m=1}^{n-1} v(t,m)$, might exceed unity, so that the calculated value of the zero-coupon discount factor $v(t,n)$ is negative. This unsatisfactory condition happens when, for longer maturities, the par yield is still rising noticeably, and this happens when, with Equation 1, the value of β is too low for the particular values of $B(t)$ and $C(t)$. Therefore we have to choose a value of β that is large enough to prevent this anomaly from happening, at least within the first 35 years (the period for investing in zero-coupon bonds in this application). We find that a value of $\beta = 0.55$ is large enough considering the initial values and the simulations for our calculations. Indeed, Wilkie *et al.* (2003) use $\beta = 0.39$ and Yang (2001) uses a value of β of 0.5. Although we start with the value of 0.1 for β, we have had to increase it up to 0.55 to avoid negative or zero discount factors for the zero-coupon bonds. Using a high value of β produces a very flat yield curve, rather little different from using a constant interest rate of $C(t)$. However, $\beta = 0.55$ is the lowest value that does not give us inconsistencies.

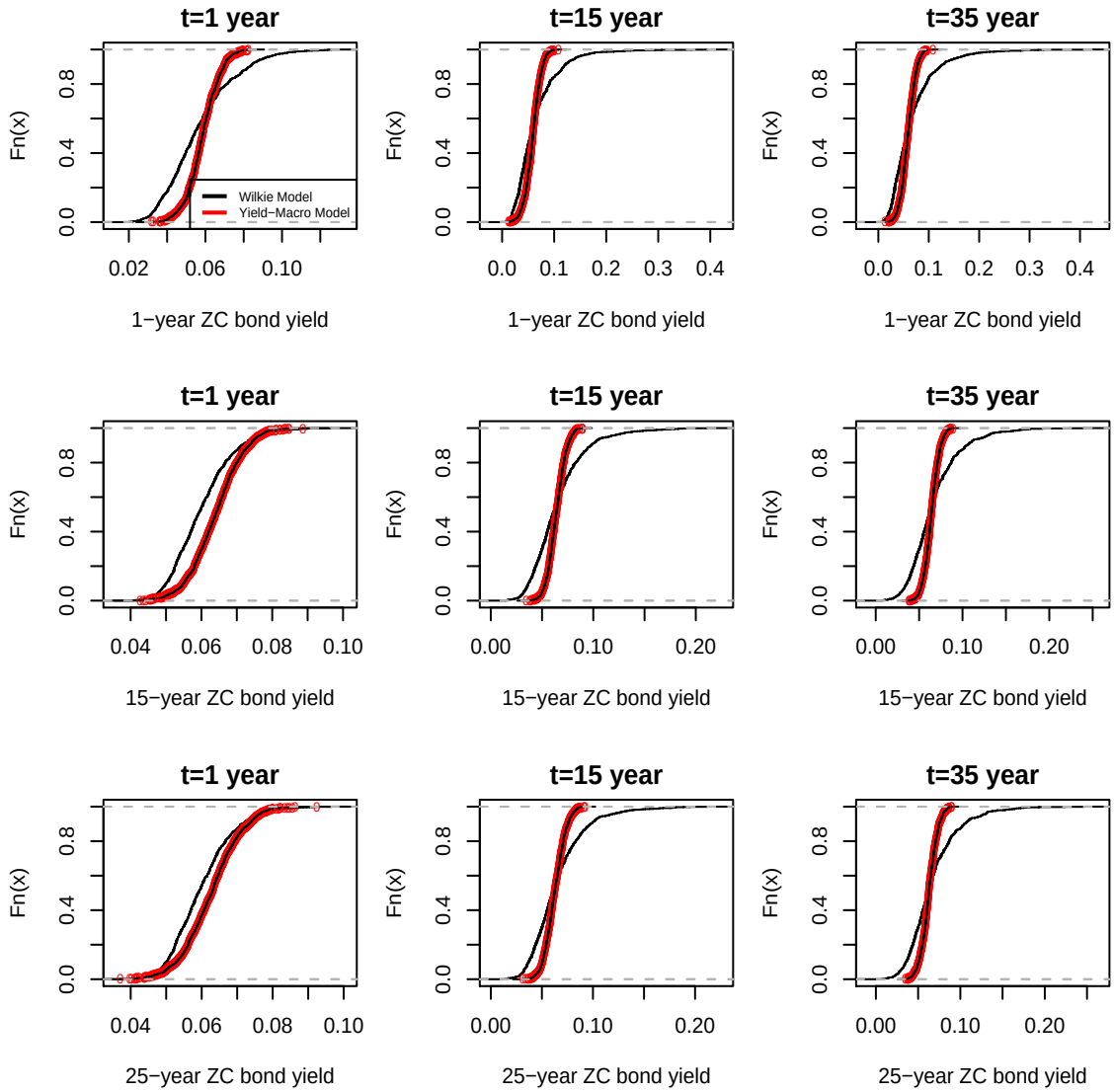

Figure 5: Empirical Cumulative Distribution Functions for the Simulated Zero-Coupon Bond Yields

Figure 5 displays the ECDFs of the zero-coupon yield curves based on 1000 simulations for different maturities and different years from the two models. The ECDFs for the zero-coupon yields for the first forecast year, $t = 1$, seem rather similar for the two models although the simulations obtained from the Wilkie model have a wider spread. At time $t = 1$, as the maturity increases the ECDFs get closer. On the other hand, as we simulate the yield curves for further years the standard deviations decrease for both models while the means remain almost the same. There are some high zero-coupon bond yields for the forecast years $t = 15$ and $t = 35$ in the simulated values using the Wilkie model. Figure 5 indicates that the distributions of the zero-coupon yields obtained from the two models become different as the maturity and the forecasting years increase. The calibration periods and the structures of the models might explain the differences observed in Figure 5. The

parameters of the yield-macro model have been calculated based on a much more stable period. Therefore it is not surprising that the distributions of the ZC bond yields or any other simulated variables are less skewed or humped than the simulated Wilkie model variables. Furthermore, the structural differences between these two models also affect the simulation results. One of the main advantages of the yield-macro model over the Wilkie model is that the yield-macro model forecasts the entire yield curves. When we try to construct the ZC yield curve using the Wilkie model we see that there are some high ZC bond yields for reasons which have been discussed previously.

5 Asset Values and Annuity Payoffs

Another way to compare the Wilkie model and the yield-macro model is to examine the asset values and the annuity payoffs under a hypothetical pension scheme. Although a more realistic application would include mortality, we ignore it during both the investment and the retirement periods for simplicity in this analysis.

We assume an employee at age 30, with an arbitrary initial salary, S. The salary increases according to the simulated RPI index for the next 35 years and the employee retires at age 65. She contributes a constant fraction of her salary, f to a pension fund which is invested into a portfolio of nominal bonds for different maturities. We ignore mortality during both the investment and the retirement period, which is taken as a fixed 25 years, and we analyse the variations in the assets and annuity payoffs.

Let $v(t, n)$ be the price of an n-year zero-coupon bond at time t.

$$v(t, n) = \frac{1}{(1 + Z(t, n))^n} \tag{4}$$

where
$Z(t, n)$ is the n-year spot rate at time t.

Salary rises in line with $RPI(t)$ and contributions are a constant fraction, f, of salary. Thus the yearly contribution C_t is,

$$C_t = S \times f \times \frac{RPI(t)}{RPI(0)}$$

where
$S = 10000$ units
$f = 10\%$
$RPI(t)$ values are simulated using the stochastic models.

Thus, the asset value just before the contribution at time t, A_t, can be calculated as:

$$A_t = (A_{t-1} + C_{t-1}) \underbrace{\frac{v(t, n-1)}{v(t-1, n)}}_{1+R(t)} \tag{5}$$

where $A_0 = 0$ and $R(t)$ is the return at time t. Equation 5 assumes investment in a rolling n-year zero-coupon bond fund.

Once we calculate the asset values over time, we can find the annuity payoffs for the 25 years retirement period using the zero-coupon yield curves at age 65, i.e. the simulated yield curve at year 35. We assume that the annuity is paid yearly in advance.

Let ap be the annuity payoff. Then,

$$A_{35} = ap \times \ddot{a}(35, N) \tag{6}$$

where $\ddot{a}(35, N)$ is the annuity price for 1 unit,

$$\ddot{a}(35, N) = \sum_{m=0}^{N-1} (1 + Z(35, m))^{-m} = \sum_{m=0}^{N-1} v(35, m)$$

$N = 25$ and $Z(35, N)$ is the zero-coupon yield curve at $t = 35$.

We calculate the asset values under different investment strategies for both models. We assume rolling investments in zero-coupon bonds for specific maturities such as 5-year (F1), 10-year (F2), 15-year (F3), 20-year (F4) and 25-year (F5) ZC bonds as described in the previous section. We consider two more scenarios which we invest on decreasing maturity for some years of the investment period. First, we invest in 25-year ZC bonds for the first 10 years, then for the last 25 years instead of a rolling investment we use the zero-coupon yield curve to calculate the returns on decreasing maurities (D1). Second, we again invest in 25-year ZC bonds but for a longer period, 25 years, then for the last 10 years we invest in decreasing maturity bonds (D2). While in D1 the maturity of the assets at time $t = 35$ corresponds to the retirement date, in D2 the maturity of the assets is 15 years at the retirement date. With D2 we try to hedge the risk in the annuity price, $\ddot{a}(35, N)$. On the other hand, a more realistic strategy might be to assume deterministic mortality and an investment policy which aims to match the expected annuity payoffs more exactly by buying small fraction of bonds of different maturities.

Table 1 shows some descriptive statistics for the real asset values calculated using the first 'decreasing maturity' investment strategy (D1) for both models over the next 35 years. Although the mean of the real asset values obtained from the Wilkie model grows faster than the values of the yield macro model, the medians for different years are quite close to each other. The higher standard deviations, skewness and excess kurtosis coefficients indicate that Wilkie model tends to produce some extreme values relative to the yield-macro model. The minimum and maximum values displayed over the years also support this conclusion.

Figure 6 shows the real asset values for different investment strategies over the years. The yield-macro model produces lower mean values than the Wilkie model after the first year but while

Table 1: Real Asset Values, A_t, on a Decreasing Maturity (D1) Investment

Wilkie Model Real Asset Values							
Year	Mean	Sd	Median	Min	Max	Skewness	Kurtosis
1	219.69	8.56	219.63	192.83	247.82	0.10	-0.08
5	251.82	42.45	248.45	142.52	427.77	0.59	0.64
10	295.29	81.53	281.32	130.01	642.74	1.08	1.77
20	413.52	182.87	375.90	103.06	1556.13	1.59	4.05
30	586.74	335.57	510.11	121.58	3157.36	1.95	6.78
35	699.74	444.15	593.25	120.31	4632.24	2.42	11.22
Yield-Macro Model Real Asset Values							
Year	Mean	Sd	Median	Min	Max	Skewness	Kurtosis
1	219.78	2.28	219.75	212.49	227.54	0.07	-0.18
5	247.06	8.25	246.70	220.79	273.45	0.09	-0.09
10	285.79	19.18	285.23	231.86	345.76	0.16	-0.12
20	383.65	54.43	381.28	230.74	565.17	0.30	0.06
30	518.92	107.46	508.83	269.88	961.85	0.47	0.25
35	603.76	142.94	591.03	285.74	1154.34	0.56	0.31

the difference is negligible for 5-year (which has not been displayed in the figure) and 10-year ZC bond investments, the difference increases as the maturity of the invested bond increases. For the investment on the 25-year ZC bond the Wilkie model produces very high values. After 15 years investment the Wilkie model asset values increase sharply which might be related with very low zero-coupon discount factors. As we have discussed in the previous section, choosing $\beta = 0.55$ prevents negative discount factors but some of them are still very close to zero. These low values mean that the ZC bond prices are very low for some specific maturities and years and this causes extreme values in returns considering the rolling investment strategies. The last two plots in Figure 6 show the asset values for the decreasing maturity investments. Since we invest in 25-year ZC bonds only for 10 years, the real annuity payoffs of the models are relatively close in D1 while they are quite different in D2 as a result of much longer investment period on the 25-year ZC bonds.

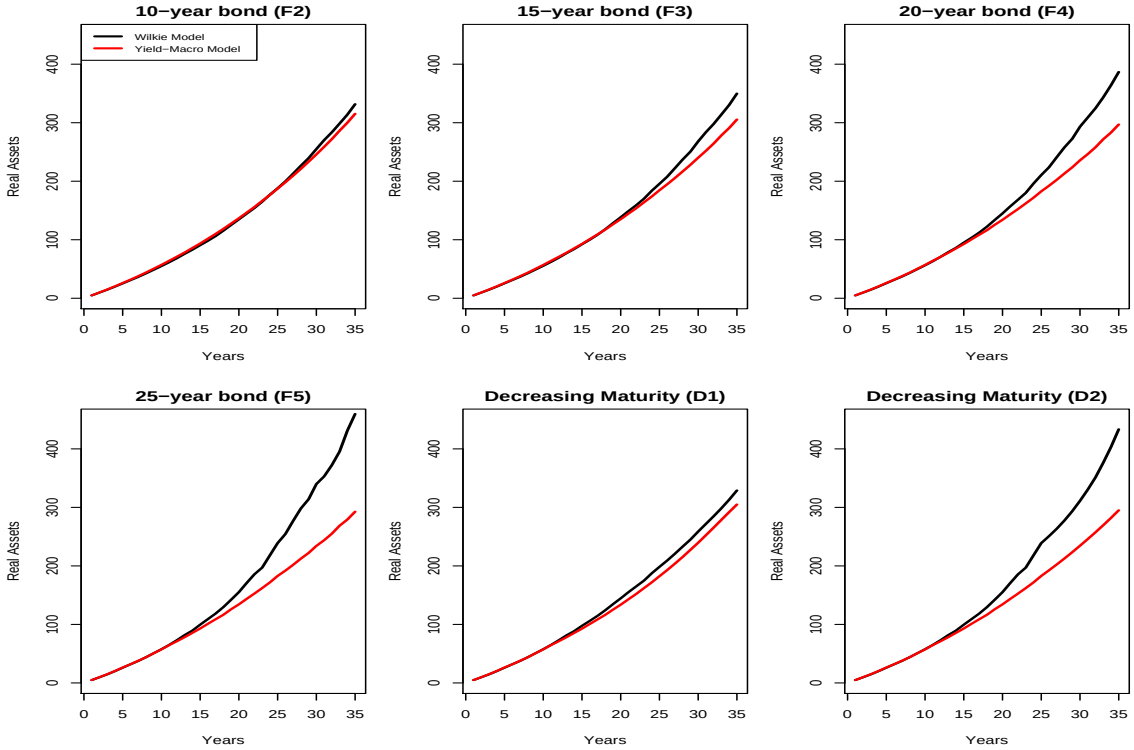

Figure 6: The Mean Amount of Real Assets for Different Investment Strategies

Table 2: Annuity Payoffs as a % of Final Salary

Wilkie Model							
	F1	F2	F3	F4	F5	D1	D2
Mean	63.17%	63.03%	66.18%	73.72%	90.02%	64.92%	88.75%
SD	51%	53%	74%	132%	300%	57%	269%
Median	50.18%	49.63%	49.91%	50.95%	52.34%	53.00	52.69%
Minimum	20.63%	17.51%	15.36%	10.80%	6.03%	20.08%	9.77%
Maximum	637.12%	886.90%	1449.79%	2971.28%	7536.14%	1005.49%	6515.18%
Skewness	5.98	7.49	10.74	15.16	19.20	9.42	17.35
Kurtosis	51.73	87.15	165.34	289.00	426.69	127.27	362.61
Yield-Macro Model							
	F1	F2	F3	F4	F5	D1	D2
Mean	55.11%	55.46%	53.71%	52.11%	51.23%	53.39%	50.61%
SD	13%	15%	16%	16%	17%	13%	12%
Median	53.43%	52.97%	51.55%	49.75%	48.88%	51.77%	48.80%
Minimum	24.94%	23.37%	23.89%	20.73%	17.90%	23.28%	25.89%
Maximum	117.54%	147.68%	155.64%	159.66%	164.92%	106.19%	100.27%
Skewness	0.87	1.14	1.21	1.24	1.29	0.87	0.80
Kurtosis	1.21	2.49	3.02	3.24	3.38	1.06	0.79

Table 2 presents some descriptive statistics for the nominal annuity payoffs as a percentage of final salary for both models. As for the Wilkie model, the mean and the standard deviation of the ratio have been increasing as we use a longer term bond for investment. The significant differences between the means and the medians indicate that there are some extreme values which affect the ratios. The ratios are positively skewed and the excess kurtosis coefficients are exceptionally high. On the other hand, the means and the medians for the yield-macro model are not very different from each other. The standard deviations seem stable and the ratios are slightly positively skewed. Although the excess kurtosis coefficients are much lower than the ones in the Wilkie model, they are significantly high for the ratios obtained from some of the investment strategies.

We might also compare the distributions of these ratios graphically. Figure 7 displays the ECDFs of the annuity payoffs as a percentage of final salary for different investment strategies for the models. Since we know that the annuity payoffs obtained from the Wilkie model have some extreme values we exclude the ratios lower than 5% and higher than 200% to draw the ECDFs. Regardless of the portfolio chosen, the payoff ratios calculated using the Wilkie model are more dispersed than the ratios obtained from the yield-macro model due to more volatile calibration period and the structure of the model.

Figure 8 and Figure 9 show the scatter plots for the asset/salary ratios and annuity prices ($\ddot{a}(35, N)$) on a horizontal log scale for the Wilkie model and the yield-macro model respectively.

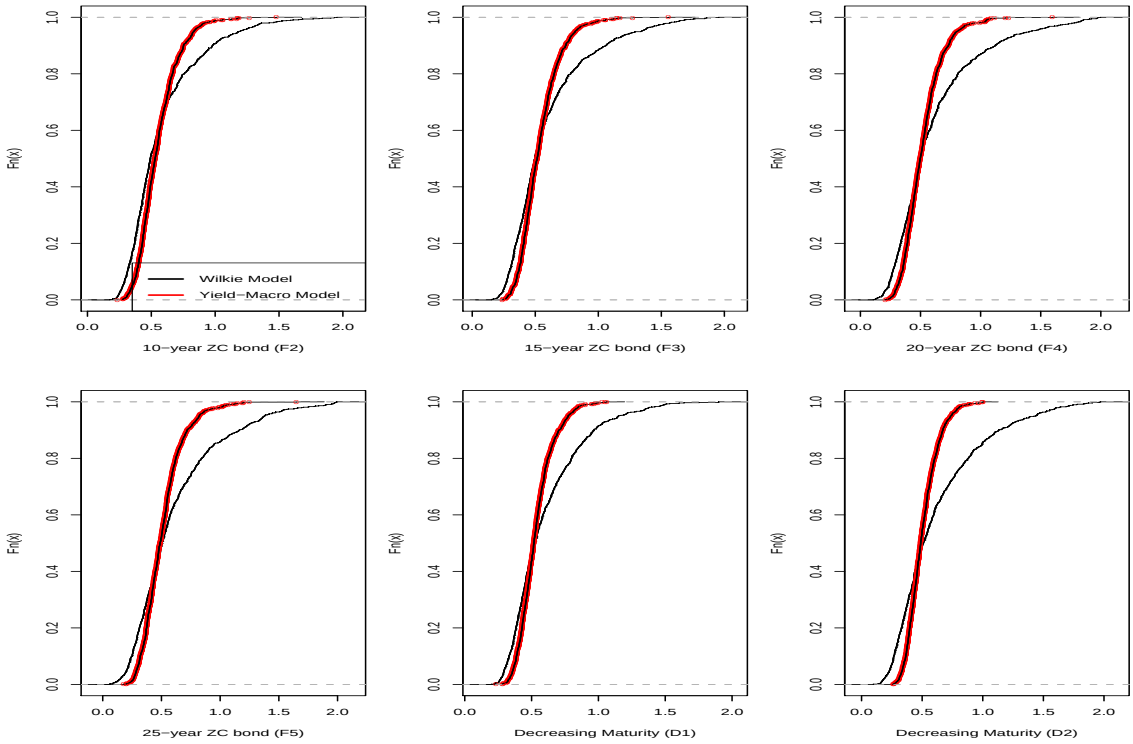

Figure 7: The Empirical Cumulative Distribution Functions of the Annuity Payoffs as a % of Final Salary

We have omitted extremely high values for the Wilkie model in Figure 8 but there are still very high and very low values which increase the spread of the plots. As the maturity of the invested ZC bond extends the correlation between the ratios and the annuity price increases in both figures. As for the decreasing maturity investment strategies, D1 and D2, the correlations seem stronger for D2 at least for the Wilkie model. The reason is that having 15-year ZC bonds as assets at retirement hedges the risk in the annuity price, $\ddot{a}(35, N)$ better. However, the correlations are relatively weak for both D1 and D2 suggesting that this type of strategy does not work all that well, at least looking ahead from time $t = 0$.

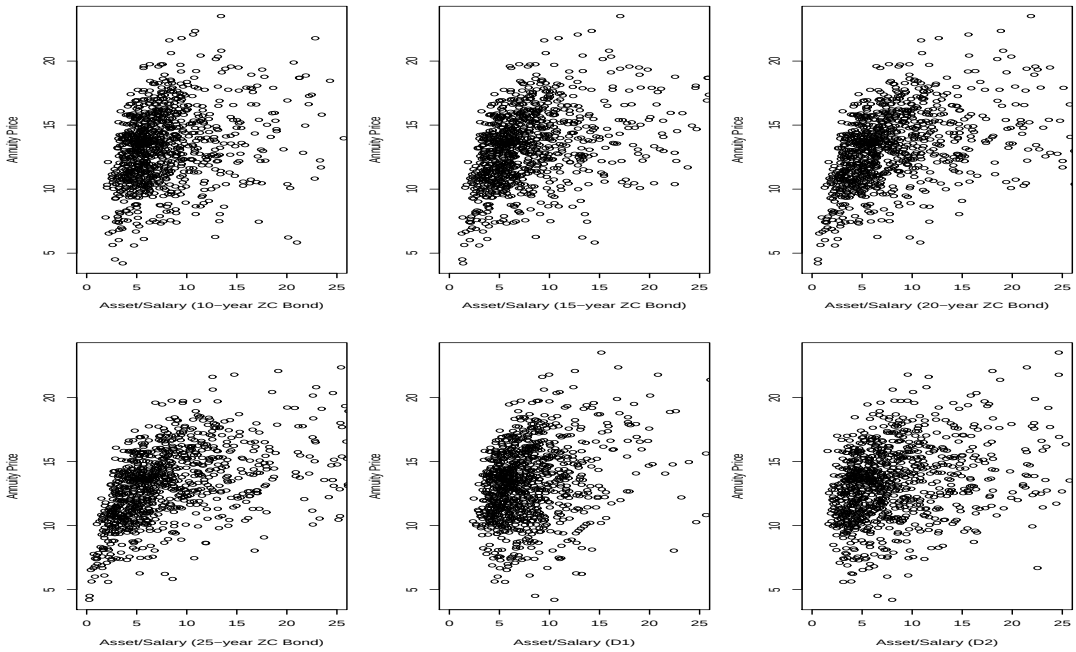

Figure 8: Asset/Salary vs Price (25-Year ZC Bond), Wilkie Model

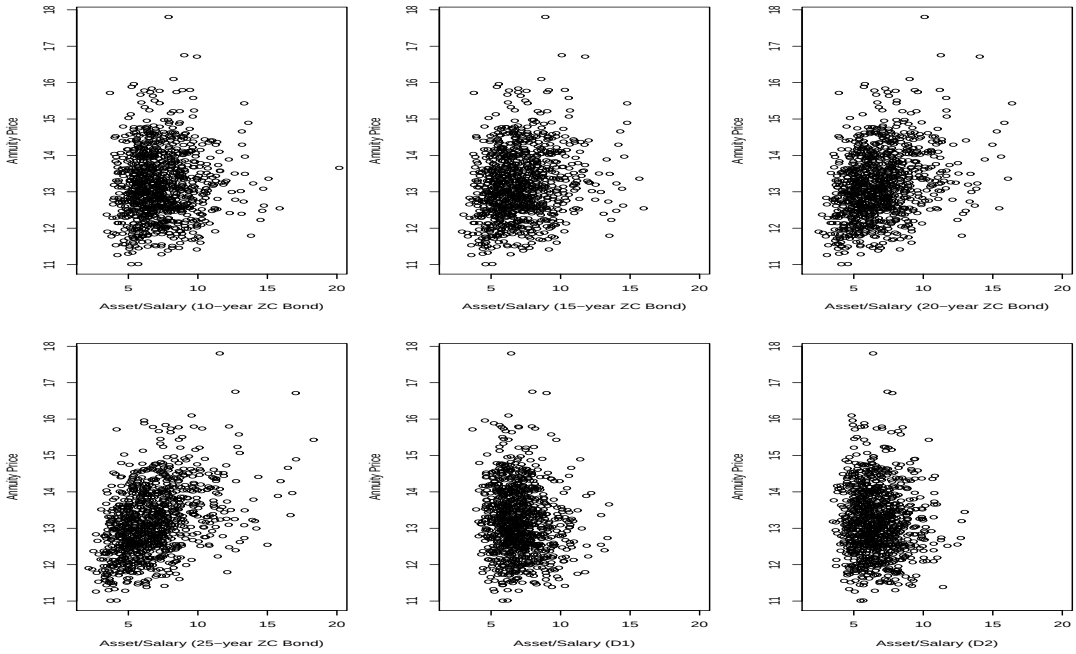

Figure 9: Asset/Salary vs Price (25-Year ZC Bond), Yield-Macro Model

6 Conclusions

In this paper we have compared the Wilkie model and the yield-macro model in both structural and empirical ways. Due to incorporating different input variables, the models have different structures and the nature of the relations between these variables is also different. Since the two models were developed based on different periods of data we use the neutral initial conditions of the yield-macro model for the Wilkie model and we adjust the mean parameters of the inflation and interest rates models of Wilkie according to these initial conditions. Therefore, we have made the two models exactly comparable. Afterwards, we compared the simulated zero-coupon bond yields obtained from the two models.

We have also calculated the asset values and annuity payoffs for the two models under a hypothetical pension scheme. The results show that the Wilkie model produces higher asset values (including some extreme values) for different portfolios and the volatilities have been much higher than the ones obtained from the yield-macro model. This is due to small values of the zero-coupon discount factors which have caused extremely high returns for the chosen investment strategy. The distribution of the ratios are positively skewed with very high kurtosis coefficients while the yield-macro model produce much more stable ratios. Finally, we have compared the annuity payoffs as a percentage of final salary for each model and for each portfolio. When we omit the extreme values for the Wilkie model, the distribution of the ratios seem similar in terms of means but the standard deviations of the ratios from the Wilkie model are still higher. Furthermore the correlation between the asset/salary ratios at retirement and the annuity price increases as the maturity of the bond invested increases.

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