

# VALUATION OF LIFE INSURANCE LIABILITIES UNDER CHANGES OF REGIMES

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## **Abstract**

This paper studies the values of the equity and liabilities of life insurance companies in the presence of regime switching in the economy. Following the contingent claim work of Grosen and Jorgensen [Grosen, A., Jorgensen, P.L., 2002. Life insurance liabilities at market value: an analysis of insolvency risk, bonus policy, and regulatory intervention rules in a barrier option framework, *Journal of Risk and Insurance*, 69 (1), 63-91], where the equity and liability of a life insurance company are evaluated as a barrier option framework, this paper proposes a model where the dynamic evolution of the assets follows a geometric Brownian motion with volatility parameters switching according to a continuous-time Markov chain process with discrete state values. After deriving valuation formulas, numerical implementation is illustrated using US life insurance data, providing strong evidence of switching behavior on the market affecting the contingent claim valuation.

**Keywords:** life insurance contracts; contingent claims; regime switching; barrier options.

**JEL Classification:** G13, G22

## 1. Introduction

Modern life insurance contracts offer to the policy holder several benefits as, an interest rate guaranteed at the end of the period over the initial amount, or a bonus to be paid in case the insurance company shows high returns, for instance. Other contracts are linked to equity or have a participating policy, some of them offer a surrender option. These contracts have a common risk: if the insurance company defaults and is liquidated, policy holders may not be able to collect the promised benefits. Therefore, it is crucial to consider the risk of default of the insurance company when evaluating the market value of policy contracts. The pioneer work of Briys and de Varenne (1997) provided an option pricing framework to study the fair value of life insurance contracts. Grosen and Jorgensen (2002) introduced an exponential barrier in order to monitor the risk of default during the period of the contract. In their work, the value of a life insurance contract with a guarantee payment is decomposed in four terms: the final guarantee payment; a European put option related to the risk of default; a bonus European call option if the insurance assets' value is in surplus at maturity; and a rebate to be paid early in the case the firm defaults before the maturity date. On this direction, several improvements have been done on the literature: Bernard et al. (2005) propose a valuation of life insurance contracts that takes into account the risk of default in the presence of a stochastic term structure of interest rates. Chen and Suchaneki (2007) extended the work of Grosen and Jorgensen (2002) by distinguishing default from liquidation as in the bankruptcy framework of Franois and Morellec (2004). However, these models assume that the life insurance assets' price follows a geometric Brownian motion with constant volatility.

Since life insurance contracts are mostly written on a long term basis (between 20 to 60 years), it is important to capture on the model the structural changes that the firm's asset prices may suffer during long periods. Note for instance that during economic recession periods, financial markets are characterized by low returns while during expansion periods they exhibit high returns. During stable financial periods, the financial market shows low volatility while in booms or crash periods the volatility increases excessively. This paper proposes a regime switching model in order to capture such structural changes. Regime switching models were introduced by Hamilton (1989) to financial econometrics and several applications have been developed. For example, Garcia and Perron (1996) applied it to model real interest rates; Hardy (2001) considers a regime switching process to model the returns of stocks in the long term; Bansal and Zhou (2002) use them to model the term structure of interest rates; Buffington and Elliott (2002) price American options with regime switching; Siu (2005) considers participating policies with surrender options evaluated as American options and more recently Boyle and Draviam (2007) price exotic options.

The goal of this paper is to study the valuation of life insurance contracts that takes into account in a simple way both: the risk of default of the insurance company, and the macroeconomic conditions that may affect the firm's assets' value on a long term. More precisely, this paper follows the framework of Grosen and Jorgensen (2002) where the fair value of a life insurance contract is decomposed into European and knockout barrier options and model the underlying process as a regime switching process. After deriving valuation formulas, an empirical study of US life insurance companies is performed to illustrate the application of the model. The paper is structured as follows: Section 2 provides the life insurance contract specifications as in Grosen and Jorgensen (2002). Section 3 describes the model to price the liabilities of a life insurance company in the presence of a regime switching process. Section 4 describes the econometric procedure to esti-

mate the model parameters. Then, an empirical evaluation to seven US life insurance companies is performed. Finally Section 5 concludes.

## 2. Contract specifications

Inspired by Briys and de Varenne (1997), Grosen and Jorgensen (2002) proposed a framework where the policy holder and the equity holder agree on establish a life insurance company with an initial assets' value  $S(0)$ . The representative policy holder (also named the liability holder) participates with a premium payment  $L(0) = \alpha S(0)$ ,  $\alpha \in (0, 1)$ , which constitutes the liability of the insurance company at time zero. On the other hand, the representative equity holder participates with  $E(0) = (1 - \alpha) S(0)$ , which is the value of equity at time zero. The capital structure of the insurance company at time zero is presented on table 1

Assets	Liabilities
$S(0)$	$E(0) = (1 - \alpha) S(0)$ $L(0) = \alpha S(0)$
$S(0)$	$S(0)$

Table 1: Balance sheet of the insurance company at time 0.

### 2.1. The payoff at maturity time

Most life insurance policies promise the policy holder a continuously compounded return on its initial contribution to be paid at the end of the contract. This promised payment may be written as  $L(T) = L(0) e^{r_g T}$ , where  $r_g$  is the minimum guaranteed interest rate, fixed at the beginning of the contract. Such promise can be honored only if, at maturity date, the insurance company assets' value is enough to cover the promised amount. Otherwise, since policy holders have priority over equity holders, the policy holder will receive the firm's assets' value. In addition to the guaranteed payment at maturity time  $T$ , the policy holder is entitled to receive a bonus in case the insurance company has a surplus at maturity. Denote by  $\delta \in [0, 1]$  the participation rate for the policy holder. Thus, the claim of the policy holder at maturity date may be written as

$$\Upsilon_L(S(T)) = L(T) - [L(T) - S(T)]^+ + \delta [\alpha S(T) - L(T)]^+ \quad (1)$$

This payoff consists on three parts:

1. A deterministic guaranteed payment due at maturity  $L(T)$ ,
2. A European put option payoff  $[L(T) - S(T)]^+$  resulting from the fact that the equity holder has limited liability and
3. A European call option payoff  $\delta [\alpha S(T) - L(T)]^+$  corresponding to the bonus in case the insurance company has a surplus.

The underlying process of the option is the firm's assets' value  $\{S(t)\}; t \in [0, T]$ , while the strike price at maturity is the guaranteed payment  $L(T)$ .

As residual claimants, the value of equity at maturity date consists on the firm's assets' value minus the payment provided to policy holders, that is:

$$\Upsilon_E(S(T)) = [S(T) - L(T)]^+ - \delta [\alpha S(T) - L(T)]^+ \quad (2)$$

which is the difference of two European call options on the firm's assets' value  $S(T)$  and strike price the guaranteed payment  $L(T)$ . Table 2 summarizes the payoff at maturity time.

Assets' value	$S(T)$
Liability holder	$\Upsilon_L(S(T)) = L(T) - [L(T) - S(T)]^+ + \delta [\alpha S(T) - L(T)]^+$
Equity holder	$\Upsilon_E(S(T)) = [S(T) - L(T)]^+ - \delta [\alpha S(T) - L(T)]^+$

Table 2: Payoff at maturity date.

## 2.2. The payoff at liquidation

Insurance companies may be in distress before maturity and consequently not be able to meet its obligations when due. Grosen and Jorgensen (2002) introduce a barrier in order to monitor the risk of default during the life of the contract. Hence, liquidation is declared as soon as the insurance company assets' value is below a given barrier. Liquidation time is defined as the first time the process falls below a barrier provided that it was above it at the beginning. That is,

$$\tau = \inf \{t \in [0, T] \mid S(t) \leq B(t); S(0) > B(0)\} \quad (3)$$

where the barrier is defined as

$$B(t) = \lambda L(0) e^{r_g t}; t \in [0, T] \quad (4)$$

for some specified positive constant  $\lambda$ .

If  $\lambda \geq 1$  and the barrier has been reached before maturity date, the insurance company is in default but still it is able to repay the policy holder his initial deposit compounded at the guaranteed interest rate up to liquidation time. In this case, equity holders have a surplus  $S(\tau) - L(0) e^{r_g \tau} = (\lambda - 1) L(0) e^{r_g \tau}$ , which may be used to pay legal or administrative fees, so this term is not priced on the model.

If  $\lambda < 1$  and the barrier has been reached before maturity date, the insurance company is not able to meet its obligations. Therefore, due to the priority rule, the liability holder will receive the entire value of the assets at liquidation time while equity holders receive nothing.

The payoff at liquidation is summarized on Table 3.

Assets' value	$S(\tau)$
Liability holder	$\Upsilon_L(S(\tau)) = \min\{\lambda, 1\} L(0) e^{rg\tau}$
Equity holder	$\Upsilon_E(S(\tau)) = 0$

Table 3: Payoff at liquidation time  $\tau$ .

### 3. Valuation

In order to price the payoffs of the policy and equity holder, the dynamics of the assets' value must be specified. Briys and de Varenne (1997), Grosen and Jorgensen (2002), Bernard et al. (2005) and Chen and Suchaneki (2007) assume the assets' value follows a geometric Brownian motion process where the parameters are constant<sup>1</sup>. In this paper, instead of that, we propose a model where the parameters of the geometric Brownian motion process switch according to a homogeneous Markov chain.

There are many reasons why we consider a regime switching model instead of a geometric Brownian motion with constant parameters:

- It is well known that the drift and volatility of the assets' value are not constant over time.
- When evaluating investments for long term periods, it is crucial to take into account the main macroeconomic structural changes that may affect the firm's assets' value.
- Empirical evidence shows that financial markets exhibit volatility clustering, meaning by that, periods with low volatility are followed by periods with low volatility until a sudden change arrives, switching to a high volatility period.
- On practice, regime switching models are easier to understand and interpret than stochastic volatility models.

The following section will describe the model.

#### 3.1. Regime switching framework

Let  $\{C(t), t \geq 0\}$  be a homogeneous continuous-time Markov chain defined on a probability space  $(\Omega, \mathcal{F}_t^C, P)$ , with a discrete state space of  $N$  elements, represented by  $E = \{e_1, \dots, e_N\}$ . Each  $e_i$  is an indicator vector having one at the  $i$ -th element and zeros elsewhere, that is,  $e_i = [0, \dots, 0, 1, 0, \dots, 0]'$ , for  $i = 1, \dots, N \in \mathbb{N}$  where the symbol  $'$  denotes transpose. Let  $\mathcal{F}_t^C$  denote the complete filtration generated by the process  $\{C(t), t \geq 0\}$  and let  $\mathbf{A} = [a_{ij}]$  be the  $N \times N$  transition probability matrix or generator of the process. The homogeneous continuous-time Markov chain admits the following semi-martingale representation (see Elliott et al. (1995))

$$C(t) = C(0) + \int_0^t \mathbf{A}C(s) ds + M(t)$$

<sup>1</sup>Note that Bernard et al. (2005) propose a model where the interest rate is stochastic. However, the firm's assets' value follows a geometric Brownian motion with constant volatility.

where  $\{M(t), t \geq 0\}$  is a martingale process with respect to the filtration  $\{\mathcal{F}_t^C\}$ .

A convenient way to describe a Markov chain is to consider the total amount of time the Markov chain stays in state  $i$  over a time interval  $[0, t]$ . Let  $T_i(t)$  be this random variable. It may be expressed as an indicator function or inner product of vectors by

$$T_i(t) = \int_0^t \mathbf{1}_{[C(s)=e_i]} ds \quad \text{or} \quad T_i(t) = \int_0^t \langle e_i, C(s) \rangle ds \quad (5)$$

where  $\mathbf{1}_{[C(t)=e_i]}$  denotes the indicator function having the value of one if the Markov chain is in state  $i$  at time  $t$ , and zero otherwise.

Define the process  $\{\bar{T}_d(t), t \geq 0\}$  as the weighted sum of the time spent on the states  $i = 1, \dots, N$  during the time interval  $[0, t]$ , where  $d_i$  are some given positive weights. Equivalently,

$$\bar{T}_d(t) = \sum_{i=1}^N d_i T_i(t) \quad (6)$$

Suppose the market contains a riskless asset  $B^0(t)$  whose risk-free interest rate evolves according to the continuous-time Markov chain. Let  $r = (r_1, \dots, r_N)'$  be the vector of instantaneous risk-free rates,  $r_i > 0$ , associated to each possible state  $i = 1, \dots, N$ . At time  $t$ , the risk free rate is given by  $r(C(t)) \equiv \langle r, C(t) \rangle$ , where  $\langle, \rangle$  denotes the inner product of vectors. Assume that the dynamics of the risk-free asset is described by

$$dB^0(t) = r(C(t)) B^0(t) dt$$

with solution

$$B^0(t) = B^0(0) e^{\int_0^t r(C(s)) ds}$$

or equivalently

$$B^0(t) = B^0(0) e^{\bar{T}_r(t)} \quad (7)$$

where  $\bar{T}_r(t)$  is the weighted sum of the times spent on the states  $i = 1, \dots, N$  over the interval  $[0, t]$  as defined on equation (6) with weights equal to the instantaneous risk-free rate  $r_i$  associated to each state.

Let  $(\Omega, \mathcal{F}_t, P)$ , be a complete probability space, where  $P$  is the real world or physical probability measure. Let  $W^P(t)$  be a standard Brownian motion on  $(\Omega, \mathcal{F}_t, P)$  and assume that it is independent of the Markov chain  $\{C(t); t \geq 0\}$  defined on the filtration  $\mathcal{F}_t^C$ . Note that  $\mathcal{F}_t$  denotes the enlarged filtration generated by  $\{C(t), t \geq 0\}$  and  $\{W^P(t), t \geq 0\}$ . Thus, assume the dynamics of the firm's assets' value  $\{S(t), t \geq 0\}$  is described by

$$dS(t) = \mu(C(t)) S(t) dt + \sigma(C(t)) S(t) dW^P(t) \quad (8)$$

where the drift parameter  $\mu(C(t))$  and the volatility parameter  $\sigma(C(t))$  of the process switch according to the discrete states of the Markov chain.

More precisely, denote by  $\mu = (\mu_1, \dots, \mu_N)'$  the vector of drift parameters and by  $\sigma = (\sigma_1, \dots, \sigma_N)'$  the vector of volatility parameters  $\sigma_i \geq 0$ , associated to each possible state  $i = 1, \dots, N$ . Then, at time  $t$ , the drift and volatility are, respectively,  $\mu(C(t)) = \langle \mu, C(t) \rangle$  and  $\sigma(C(t)) = \langle \sigma, C(t) \rangle$ .

Denote by  $Z(t)$  the logarithmic return on the firm's assets over the time period  $[0, t]$ ,  $Z(t) = \ln\left(\frac{S(t)}{S(0)}\right)$ . Applying Itô's formula and in terms of occupation times, the process  $Z(t)$  may be written as

$$Z(t) = \bar{T}_\mu(t) - \frac{1}{2}\bar{T}_{\sigma^2}(t) + \varphi(t) \quad (9)$$

where  $\varphi(t)$  follows a Gaussian distribution with mean zero and variance

$$\text{Var}\left(\int_0^t \sigma(C(s)) dW(s)\right) = \int_0^t \sigma^2(C(s)) ds = \sum_{i=1}^N \sigma_i^2 T_i(t) = \bar{T}_{\sigma^2}(t)$$

Thus, conditioning on the sigma algebra generated by  $\{C(t), t \geq 0\}$ , the process  $Z(t)$  is Gaussian with mean  $M = \bar{T}_\mu(t) - \frac{1}{2}\bar{T}_{\sigma^2}(t)$  and variance  $V = \frac{1}{2}\bar{T}_{\sigma^2}(t)$ .

The dynamics described on equation (8) leads to an incomplete market because there is an extra risk introduced by the switching behavior that can not be hedged only with the two financial instruments. Assuming there are no arbitrage opportunities, it exists a risk-neutral probability measure (Harrison and Kreps (1979)). However, since the market is incomplete, the risk-neutral probability measure is not unique (Harrison and Pliska (1983)). By imposing a martingale condition, a risk-neutral probability measure  $Q$  may be derived among many other possibilities. Di Masi et al. (1994) implemented a mean-variance criteria to hedge options when the volatility are functions of a Markov jump process while Siu (2005) applied a Esscher transform measure to evaluate participating life insurance contracts. Conditioning on  $\{C(t), t \geq 0\}$ , the probability measure  $Q$  is unique and must satisfy the martingale condition

$$\mathbb{E}^Q \left[ e^{-[\bar{T}_r(t) - \bar{T}_r(u)]} S(t) \mid S(u); u < t; \mathcal{F}_t^C \right] = S(u) \quad (10)$$

Note that conditioning on the filtration  $\mathcal{F}_t^C$  means that the information related to the hidden Markov chain  $\{C(t), t \geq 0\}$  is observed by all market participants up to time  $t$ . Therefore, conditional on  $\{C(t), t \geq 0\}$  the market is complete and then the *conditional* risk-neutral probability measure is unique. Given the dynamics of the stochastic process  $S(t)$ , equation (10) becomes

$$\begin{aligned} & \mathbb{E}^Q \left[ e^{-[\bar{T}_r(t) - \bar{T}_r(u)]} S(t) \mid S(u); u < t; \mathcal{F}_t^C \right] \\ &= \mathbb{E}^Q \left[ e^{-[\bar{T}_r(t) - \bar{T}_r(u)]} S(u) e^{\int_u^t [\mu(C(s)) - \frac{1}{2}\sigma^2(C(s))] ds + \int_u^t \sigma(C(s)) dW^P(s)} \right. \\ & \quad \left. \mid S(u); u < t; \mathcal{F}_t^C \right] \\ &= e^{-[\bar{T}_r(t) - \bar{T}_r(u)]} S(u) \mathbb{E}^Q \left[ e^{\int_u^t [\mu(C(s)) - \frac{1}{2}\sigma^2(C(s))] ds + \int_u^t \sigma(C(s)) dW^P(s)} \right] \\ &= e^{-[\bar{T}_r(t) - \bar{T}_r(u)]} S(u) e^{[\bar{T}_\mu(t) - \bar{T}_\mu(u) - \frac{1}{2}[\bar{T}_{\sigma^2}(t) - \bar{T}_{\sigma^2}(u)]]} e^{\frac{1}{2}[\bar{T}_{\sigma^2}(t) - \bar{T}_{\sigma^2}(u)]} \\ &= S(u) e^{-[\bar{T}_r(t) - \bar{T}_r(u)]} e^{[\bar{T}_\mu(t) - \bar{T}_\mu(u)]} \end{aligned}$$

Imposing  $\bar{T}_\mu(u) = \bar{T}_r(u)$ , for all  $0 < u < t$ , the martingale condition of equation (10) is satisfied and the dynamics of the underlying switching process may be written as

$$dS(t) = r(C(t)) S(t) dt + \sigma(C(t)) S(t) dW(t) \quad (11)$$

where  $W(t)$  is a  $Q$ -standard Brownian motion and  $Q$  is the risk-neutral probability measure *conditional to*  $\{C(t), t \geq 0\}$ . The conditional risk-neutral probability measure  $Q$  is Gaussian with mean  $\bar{T}_r(t) - \frac{1}{2}\bar{T}_{\sigma^2}(t)$  and variance  $\bar{T}_{\sigma^2}(t)$ , hence

$$dQ = \frac{1}{\sqrt{2\pi\bar{T}_{\sigma^2}(t)}} e^{-\frac{1}{2\bar{T}_{\sigma^2}(t)}[Z(t) - (\bar{T}_r(t) - \frac{1}{2}\bar{T}_{\sigma^2}(t))]^2} dZ(t) \quad (12)$$

where  $\bar{T}_r(t) = \sum_{i=1}^N r_i T_i(t)$  is the sum of the risk-free interest rates and  $\bar{T}_{\sigma^2}(t) = \sum_{i=1}^N \sigma_i^2 T_i(t)$  is the sum of the volatilities, both weighted by the time spent on each state up to time  $t$ .

### 3.2. Conditional value of liabilities and equity

The fair value of a contingent claim is given by the expected discounted payoff under a risk-neutral probability measure. Remark that the risk neutral probability measure on equation (12) has been derived by conditioning on the filtration  $\mathcal{F}_t^C$ . Therefore, the market value of liabilities and equities priced under this probability measure are also conditioned on  $\mathcal{F}_t^C$ . In this section, the conditional value of liabilities and equity are developed. They will be decomposed in two parts: one that has a closed-form solution, and another that requires numerical computations, because it will depend on the hitting time  $\tau$  of the regime switching process. Once the conditional values for liabilities and equities are derived as expected discounted payoffs under  $Q$ , the unconditional evaluation of the contingent claims will be discussed.

#### Conditional Market value of liabilities

Conditional to the filtration  $\mathcal{F}_t^C$ , the expected discounted payoff of the liabilities under the risk neutral measure  $Q$  is

$$V_L(0, T, S(0), L, N | \mathcal{F}_t^C) = \mathbb{E}^Q \left[ e^{-\bar{T}_r(T)} \Upsilon_L(S(T)) \mathbf{1}_{\tau \geq T} | \mathcal{F}_t^C \right] + \mathbb{E}^Q \left[ e^{-\bar{T}_r(\tau)} \Upsilon_L(S(\tau)) \mathbf{1}_{\tau < T} | \mathcal{F}_t^C \right] \quad (13)$$

The first expectation of equation (13) is conditioned to the event that the process does not hit the barrier during the life of the option. By equation (1), this expectation becomes

$$\begin{aligned} & \mathbb{E}^Q \left[ e^{-\bar{T}_r(T)} \Upsilon_L(S(T)) \mathbf{1}_{\tau \geq T} | \mathcal{F}_t^C \right] \\ = & \mathbb{E}^Q \left[ e^{-\bar{T}_r(T)} L(T) \mathbf{1}_{\tau \geq T} | \mathcal{F}_t^C \right] - \mathbb{E}^Q \left[ e^{-\bar{T}_r(T)} [L(T) - S(T)]^+ \mathbf{1}_{\tau \geq T} | \mathcal{F}_t^C \right] \\ & + \mathbb{E}^Q \left[ \alpha \delta e^{-\bar{T}_r(T)} \left[ S(T) - \frac{L(T)}{\alpha} \right]^+ \mathbf{1}_{\tau \geq T} | \mathcal{F}_t^C \right] \end{aligned}$$

The second expectation of equation (13) is conditioned to the event that at some point, prior to maturity, the process hits the barrier. Thus,

$$\mathbb{E}^Q \left[ e^{-\bar{T}_r(\tau)} \Upsilon_L(S(\tau)) \mathbf{1}_{\tau < T} | \mathcal{F}_t^C \right] = \min\{\lambda, 1\} L(0) \mathbb{E}^Q \left[ e^{-\bar{T}_r(\tau) + r_g \tau} \mathbf{1}_{\tau < T} | \mathcal{F}_t^C \right]$$

Summarizing, the conditional value of liabilities given on equation (13) is decomposed as

$$V_L(0, T, S(0), L, N | \mathcal{F}_t^C) = G - PO + BO + RB \quad (14)$$

where

$G \equiv \mathbb{E}^Q [e^{-\bar{T}_r(T)} L(T) \mathbf{1}_{\tau \geq T} | \mathcal{F}_t^C]$  is the guaranteed payment due at maturity,

$PO \equiv \mathbb{E}^Q [e^{-\bar{T}_r(T)} [L(T) - S(T)]^+ \mathbf{1}_{\tau \geq T} | \mathcal{F}_t^C]$  is a put option payoff in case of no default,

$BO \equiv \mathbb{E}^Q \left[ \alpha \delta e^{-\bar{T}_r(T)} \left[ S(T) - \frac{L(T)}{\alpha} \right]^+ \mathbf{1}_{\tau \geq T} | \mathcal{F}_t^C \right]$  is the bonus call option if there is a surplus on the firm's assets, and

$RB \equiv \mathbb{E}^Q [e^{-\bar{T}_r(\tau)} \min\{\lambda, 1\} L(0) e^{r g \tau} \mathbf{1}_{\tau < T} | \mathcal{F}_t^C]$  is the rebate payment in case of early liquidation.

The underlying process of the put and call option is the firm's assets' value  $S(t)$ . The strike price is the promised payment  $L(T)$  for the put option and  $\frac{1}{\alpha} L(T)$  for the bonus option.

### Conditional value of equity

Conditional to the filtration  $\mathcal{F}_t^C$ , the expected discounted payoff of the equity under the risk neutral measure  $Q$  is

$$\begin{aligned} V_E(0, T, S(0), L, N | \mathcal{F}_t^C) &= \mathbb{E}^Q \left[ e^{-\bar{T}_r(T)} \Upsilon_E(S(T)) \mathbf{1}_{\tau \geq T} | \mathcal{F}_t^C \right] \\ &= \mathbb{E}^Q \left[ e^{-\bar{T}_r(T)} [S(T) - L(T)]^+ \mathbf{1}_{\tau \geq T} | \mathcal{F}_t^C \right] \\ &\quad - \mathbb{E}^Q \left[ \alpha \delta e^{-\bar{T}_r(T)} \left[ S(T) - \frac{L(T)}{\alpha} \right]^+ \mathbf{1}_{\tau \geq T} | \mathcal{F}_t^C \right] \end{aligned} \quad (15)$$

The first expectation, denoted by  $CO \equiv \mathbb{E}^Q [e^{-\bar{T}_r(T)} [S(T) - L(T)]^+ \mathbf{1}_{\tau \geq T} | \mathcal{F}_t^C]$  is a call option contract given that there is no default prior to maturity. The second expectation is a short position on the bonus call option  $BO$ .

Summarizing, the conditional value of equity, given by equation (15), is decomposed as

$$V_E(0, T, S(0), L, N | \mathcal{F}_t^C) = CO - BO \quad (16)$$

### Fair valuation

Grosen and Jorgensen (2002) derived closed form solutions for the value of liabilities when the risk-free interest rate and the parameters of the underlying process  $S(t)$  are constant. However, under the regime switching process described by equation (11) and conditional to  $\mathcal{F}_t^C$ , pseudo closed-form solutions are derived.

### Conditional value

**Proposition 3.1** *Under the risk-neutral probability measure  $Q$  and conditional to  $\mathcal{F}_t^C$  the conditional value of liabilities and equities (equation (14) and equation (16) respectively) become*

$$\begin{aligned} V_L(0, T, S(0), L, N | \mathcal{F}_t^C) &= e^{-\bar{T}_r(T)} L(T) [1 - \mathbb{Q}_1 + \mathbb{Q}_2 + \delta \mathbb{Q}_5] \\ &\quad - \mathbb{E}_3^Q - \alpha \delta \mathbb{E}_4^Q + \min\{\lambda, 1\} L(0) \mathbb{E}_6^Q \\ &\quad + \alpha \delta \text{SWCO} \left[ \frac{L(T)}{\alpha} | \mathcal{F}_t^C \right] - \text{SWPO} [L(T) | \mathcal{F}_t^C] \end{aligned} \quad (17)$$

$$\begin{aligned}
V_E(0, T, S(0), L, N | \mathcal{F}_t^C) &= e^{-\bar{T}_r(T)} L(T) [Q_8 - \delta Q_5] \\
&\quad + \alpha \delta \mathbb{E}_4^Q - \mathbb{E}_7^Q \\
&\quad + \text{SWCO} [L(T) | \mathcal{F}_t^C] - \alpha \delta \text{SWCO} \left[ \frac{L(T)}{\alpha} | \mathcal{F}_t^C \right]
\end{aligned} \tag{18}$$

where

$$\begin{aligned}
Q_1 &\equiv \Pr^Q(\tau < T) \\
Q_2 &\equiv \Pr^Q[S(T) < L(T); \tau < T] \\
\mathbb{E}_3^Q &\equiv \mathbb{E}^Q \left[ e^{-\bar{T}_r(T)} S(T) \mathbf{1}_{S(T) < L(T)} \mathbf{1}_{\tau < T} \right] \\
\mathbb{E}_4^Q &\equiv \mathbb{E}^Q \left[ e^{-\bar{T}_r(T)} S(T) \mathbf{1}_{S(T) > \frac{L(T)}{\alpha}} \mathbf{1}_{\tau < T} \right] \\
Q_5 &\equiv \Pr^Q \left[ S(T) > \frac{L(T)}{\alpha}; \tau < T \right] \\
\mathbb{E}_6^Q &\equiv \mathbb{E}^Q \left[ e^{-\bar{T}_r(\tau) + r_g \tau} \mathbf{1}_{\tau < T} \right] \\
\mathbb{E}_7^Q &\equiv \mathbb{E}^Q \left[ e^{-\bar{T}_r(T)} S(T) \mathbf{1}_{S(T) > L(T)} \mathbf{1}_{\tau < T} \right] \\
Q_8 &\equiv \Pr^Q[S(T) > L(T); \tau < T]
\end{aligned}$$

$\text{SWCO} [K | \mathcal{F}_t^C]$  and  $\text{SWPO} [K | \mathcal{F}_t^C]$  denote the price of a conditional switching European call and put option respectively, with strike price  $K$ .

**Proof.** See Appendix A. ■

The value of liabilities and equity on equations (17) and (18) relies on: 1) closed form solutions when the options are evaluated at maturity, and 2) the law of the hitting time  $\tau$  under a regime switching process. The closed-form solutions for the switching options  $\text{SWCO}$  and  $\text{SWPO}$  are provided on Appendix B, while equations  $Q_1$  to  $Q_8$  are evaluated numerically as described on Appendix C.

### 3.3. Unconditional value of liabilities and equity

By taking the mathematical expectation of the conditional value of the liabilities and equity, the following proposition is derived.

**Proposition 3.2** *Under the risk-neutral probability measure  $Q$ , the unconditional value of the liabilities and equity become*

$$\begin{aligned}
&V_L(0, T, S(0), L, N) \\
&= \mathbb{E} [V_L(0, T, S(0), L, N | \mathcal{F}_t^C)]
\end{aligned} \tag{19}$$

$$= \int_0^T \dots \int_0^T V_L(0, T, S(0), L, N | \mathcal{F}_t^C) \psi(t_1(t), \dots, t_N(t)) d_{t_1(t)}, \dots, d_{t_N(t)} \tag{20}$$

$$\begin{aligned}
& V_E(0, T, S(0), L, N) \\
&= \mathbb{E} [V_E(0, T, S(0), L, N | \mathcal{F}_t^C)] \tag{21}
\end{aligned}$$

$$= \int_0^T \dots \int_0^T V_E(0, T, S(0), L, N | \mathcal{F}_t^C) \psi(t_1(t), \dots, t_N(t)) d_{t_1(t)}, \dots, d_{t_N(t)} \tag{22}$$

where  $V_L(0, T, S(0), L, N | \mathcal{F}_t^C)$  is the conditional value of liabilities given in equation (17),  $V_E(0, T, S(0), L, N | \mathcal{F}_t^C)$  is the conditional value of equity given in equation (18) and  $\psi(t_1(t), \dots, t_N(t))$  denotes the joint density function of the occupation times  $T_1, \dots, T_N$ .

#### 4. Empirical study of US life insurance companies

One of the most prolific research fields on empirical finance is the study of assets' returns volatility<sup>2</sup>. Since the seminal ARCH model of Engle (1982) and the GARCH model of Bollerslev (1986), several models have been proposed to study market volatility. However, ARCH models specify volatility as a deterministic function of past observations (squares of lagged residuals), not being able to capture volatility clustering. (For a survey on the literature see Gouriroux (1997)). On the other hand, Stochastic Volatility models (SV) (see Ghysels et al. (1996)) assume that volatility follows some latent stochastic process. Although these models are more sophisticated than ARCH and GARCH models, they are, in general, more difficult to estimate.

It is well known that market returns exhibit abrupt changes with somewhat persistence over some time periods. Those stylized facts are better captured by regime switching models. Since the seminal econometric work of Hamilton (1989)<sup>3</sup>, regime switching models have been applied extensively on modelling business cycles (Hamilton (1989)), exchange rates (Engel and Hamilton (1990)), real interest rates (Garcia and Perron (1996)), term structure of interest rates (Bansal and Zhou (2002)) and stock returns in the long term (Hardy (2001)), among other applications.

This section reviews the methodology to estimate the parameters of the regime switching model via maximum likelihood. Then, the parameters of the US risk-free interest rates and US life insurance stock prices are estimated.

##### 4.1. Econometric specification

The model described in Section 3 results a mixture of Gaussian distributions. We concentrate on estimate only the volatility parameter that switches according to a Markov chain with two possible states. Although more than two states are possible to consider, adding a third state to the Markov chain does not improve the model. (See for example Bergman and Hansson (2005)). When estimating the parameters of the model one must take into account the frequency of the

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<sup>2</sup>For an introduction to financial econometric studies, see Campbell et al. (1997) and Gouriroux and Jasiak (2001).

<sup>3</sup>Statistical work was initiated by Quandt (1958).

data available. It is a common problem and a difficult task to infer the parameters of a continuous time Markov chain from monthly observations. For this reason, the econometric specification considered in this study is based on a discrete time Markov chain.

For simplicity of notation, let  $C(t)$  takes two values: 0 or 1, representing each one the state at time  $t$ . Following Hamilton (1994), the model may be written as

$$R(t) = \mu + [\sigma_0(1 - C(t)) + \sigma_1 C(t)] \varepsilon(t) \quad (23)$$

where  $R(t) = 100 \ln \left( \frac{S(t)}{S(t-1)} \right)$  is the logarithm return between two consecutive observations<sup>4</sup> and  $\varepsilon(t)$  are independent random variables that are normal distributed.

Let  $\{C(t); t \geq 0\}$  be a two-state homogeneous Markov chain, independent of  $\varepsilon(t)$ , with transition probability matrix

$$P = \begin{bmatrix} p_{00} & 1 - p_{00} \\ 1 - p_{11} & p_{11} \end{bmatrix}$$

where  $p_{00} \equiv \Pr[C(t) = 0 | C(t-1) = 0]$  and  $p_{11} \equiv \Pr[C(t) = 1 | C(t-1) = 1]$  are the persistence probabilities. Assuming stationarity, the unconditional probabilities are defined as  $\pi_0 \equiv \Pr[C(t) = 0]$  and  $\pi_1 \equiv \Pr[C(t) = 1]$ , then

$$\begin{aligned} \pi_0 &= p_{00}\pi_0 + (1 - p_{11})\pi_1 \\ \pi_1 &= p_{11}\pi_1 + (1 - p_{00})\pi_0 \end{aligned}$$

therefore,

$$\pi_0 = \frac{1 - p_{11}}{2 - p_{00} - p_{11}}; \quad \pi_1 = \frac{1 - p_{00}}{2 - p_{00} - p_{11}} \quad (24)$$

Denote the vector of the observed past returns by  $\underline{R}(t) = [R(t), R(t-1), \dots, R(1)]'$  and let  $\theta = [\mu, \sigma_0, \sigma_1, \pi_0, \pi_1]'$  be the vector of parameters to be estimated. The density function of the log returns, conditional to  $\{C(t); t \geq 0\}$  (see equation (12)), is

$$f(R(t) | C(t); \theta) = \frac{1}{\sqrt{2\pi} [\sigma_0(1 - C(t)) + \sigma_1 C(t)]} e^{-\frac{1}{2} \frac{[R(t) - \mu]^2}{[\sigma_0(1 - C(t)) + \sigma_1 C(t)]}} \quad (25)$$

therefore, the likelihood function is

$$\begin{aligned} L &= f(\underline{R}(T); \theta) \\ &= f(R(T) | \underline{R}(T-1); \theta) f(R(T-1) | \underline{R}(T-2); \theta) \dots f(R(2) | \underline{R}(1); \theta) \end{aligned}$$

The maximum likelihood estimate is the parameter vector  $\theta$  that maximizes

$$\theta \in \max_{\theta} \ln f(R(T); \theta)$$

then, the likelihood for a given  $\theta$  is computed in a recursive fashion applying Bayes' rule.

### Estimation procedure<sup>5</sup>

<sup>4</sup>Logarithm returns help the series to be stationary, which is a useful property when estimating the model.

<sup>5</sup>See Hamilton (1996), Billio and Sartore (2003) or Roche and Rockinger (2003) for a review on how to estimate regime switching models.

Applying Bayes' rule

$$f(R(t) | \underline{R}(t-1); \theta) \quad (26)$$

$$= \frac{f(\underline{R}(t); \theta)}{f(\underline{R}(t-1); \theta)} \quad (27)$$

$$= \frac{\sum_{C(t)=0}^1 f(\underline{R}(t), C(t); \theta)}{f(\underline{R}(t-1); \theta)}$$

$$= \frac{\sum_{C(t)=0}^1 f(R(t) | \underline{R}(t-1), C(t); \theta) f(\underline{R}(t-1), C(t); \theta)}{f(\underline{R}(t-1); \theta)} \quad (28)$$

but

$$\frac{f(R(t) | \underline{R}(t-1), C(t); \theta)}{f(\underline{R}(t-1); \theta)} = f(R(t) | C(t); \theta)$$

$$\frac{f(\underline{R}(t-1), C(t); \theta)}{f(\underline{R}(t-1); \theta)} = \Pr[C(t) | \underline{R}(t-1); \theta]$$

then equation (28) becomes

$$f(R(t) | \underline{R}(t-1); \theta) = \sum_{C(t)=0}^1 f(R(t) | C(t); \theta) \Pr[C(t) | \underline{R}(t-1); \theta] \quad (29)$$

The first term,  $f(R(t) | C(t); \theta)$  is given by equation (25), while the second term is computed as

$$= \frac{\Pr[C(t) | \underline{R}(t-1); \theta]}{\Pr[C(t), \underline{R}(t-1); \theta]}$$

$$= \sum_{C(t-1)=0}^1 \frac{\Pr[C(t), C(t-1), \underline{R}(t-1); \theta]}{f(\underline{R}(t-1); \theta)}$$

$$= \sum_{C(t-1)=0}^1 \frac{\Pr[C(t) | C(t-1), \underline{R}(t-1); \theta] \Pr[C(t-1), \underline{R}(t-1); \theta]}{f(\underline{R}(t-1); \theta)}$$

$$= \sum_{C(t-1)=0}^1 \Pr[C(t) | C(t-1); \theta] \Pr[C(t-1) | \underline{R}(t-1); \theta] \quad (30)$$

Then,

$$\begin{aligned}
& \Pr [C(t-1) | \underline{R}(t-1); \theta] \\
= & \frac{\Pr [C(t-1), \underline{R}(t-1); \theta]}{f(\underline{R}(t-1); \theta)} \\
= & \frac{f(R(t-1), \underline{R}(t-2), C(t-1); \theta)}{\sum_{C(t-1)=0}^1 f(\underline{R}(t-1), C(t-1); \theta)} \\
= & \frac{f(R(t-1), \underline{R}(t-2), C(t-1); \theta)}{\sum_{C(t-1)=0}^1 f(\underline{R}(t-1), C(t-1); \theta)} \\
= & \frac{f(R(t-1) | C(t-1); \theta) \Pr [C(t-1) | \underline{R}(t-2); \theta]}{\sum_{C(t-1)=0}^1 f(R(t-1) | C(t-1); \theta) \Pr [C(t-1) | \underline{R}(t-2); \theta]} \tag{31}
\end{aligned}$$

Again, the first term,  $f(R(t-1) | C(t-1); \theta)$  is given by equation (25), while  $\Pr [C(t-1) | \underline{R}(t-2); \theta]$  must be calculated recursively.

The maximum likelihood for a given  $\theta$  is obtained by iterating equation (29), which requires the computation of equation (30) that involves equation (31) and so on. The initial probability  $\Pr [C(1) | \underline{R}(0); \theta]$  can be approximate by the unconditional probabilities which satisfy equation (24).

## 4.2. Data description

The parameters of the regime switching model will be estimated for the log returns of different life insurance companies traded on the US market and the risk-free interest rates of the US (using as proxy the US, T-bills, monthly rates). The observation period runs from January 1973 to April 2008; consisting on more than 400 monthly observations.<sup>6</sup>

Figure 1 illustrates the evolution of the risk free rates on US, using the T-Bills as a proxy. Figure 2 illustrates the stock prices of each firm and Figure 3 provide the logarithm returns. The graphs suggests volatility clustering. In other words, it seems that observations with low volatility are followed by observations with low volatility (persistence) until the regime switch to a high volatility state for a short period. Figure 4 summarizes the descriptive statistics of the monthly return series. Some of them show right skewness while others left. All distributions are leptokurtic (peaked relative to the normal distribution). The Jarque-Bera test for testing normality is rejected for all series. Note that the companies Citizens and Pres exhibit extreme values for short periods, and a regime switching model alone is not convenient. Therefore, we concentrate on the remaining seven insurance firms.

[Figure 1, 2, 3 and 4 about here]

<sup>6</sup>Data source: Datastream, Thompson Financial. Kindly provided by the CEDIF at University of Lausanne. A daily data analysis may be added upon request. Then one should take into account the time aggregation differences on the parameter estimation.

### 4.3. Empirical results

The filtered probabilities, as described by equations (29) to (31) are computed for each life insurance company. Figures 5, 7, 9, 11, 13, 15 and 17, show the logarithm return of the stock prices while Figures 6, 8, 10, 12, 14, 16 and 18 show the estimated volatility of the returns (line) and the regime switching process with two states (dotted line) for each firm.

[from Figure 5 to Figure 18 about here]

Figure 19 summarizes the parameter estimation of the stocks' volatilities and the risk-free rate, assuming the logarithm returns of the assets follow a geometric Brownian motion without a regime switching. Last column shows the estimation of the annualized risk-free rate using as a proxy the US Treasury bills.

Figure 20 summarizes the parameter estimation of the stocks' volatilities and the risk-free rate, assuming the logarithm returns of the assets follow a two state regime switching process. The estimated volatility of the logarithm returns on the US life insurance companies are reported on columns 2 to 8. The results show that state 1 is characterized by relative low volatilities going from 7% to 15% while state 2 is characterized by extremely high volatilities going from 122% to 150%, (Natwestern with a 289%).

The transition probabilities indicate that the process stays on the low volatility state for a long time period before switching to a high volatility state: the probability to remain in the low volatility state,  $p_{11}$ , goes from 0.93 to 0.98; while the probability to remain in the high volatility state  $p_{22}$ , goes from 0.11 to 0.23, which is not negligible (except for Torchmark).

The last column in Figure 20 shows the estimated mean of the risk-free rate for a two state regime switching model. State 1 is characterized by a relative low rate (4%), while state 2 exhibits a high rate (8.8%). The persistence probabilities of both states are very high: the probability that the process remains in state 1 given that it is already in state 1 is 0.97, similar than the probability that the process remains in state 2 given it is already in state 2 which is 0.96. However, the switching probabilities to move from state 1 to state 2 or from state 2 to state 1 are very low (0.02 and 0.03 respectively). Finally, Figure 21 shows the estimation of the risk free interest rates on a regime switching process with two states.

[Figure 19, 20 and 21 about here]

### 5. Conclusions

This paper studies the value of liabilities of an insurance company in the presence of regime switching in the economy. It extends the barrier framework of Grosen and Jorgensen (2002) by allowing the dynamics of the life insurance assets' returns switch from one state to another according to a homogeneous Markov chain. After describing the econometric procedure to estimate the model parameters, it is applied to US data. Transition probabilities, parameter estimators and occupation times are estimated for seven US life insurance companies as well as the risk free interest rate, proxy by the Treasury Bills. Empirical results shows that volatility clustering is well captured by

the model. This is important to consider when the evaluation of contingent claims is performed on the long run, as it is the case for life insurance contracts.

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## Appendix A Valuation

From Section 2, consider the liabilities and equity's value equations

$$\begin{aligned} V_L(0, T, S(0), L, N | \mathcal{F}_t^C) &= G - PO + BO + RB \\ V_E(0, T, S(0), L, N | \mathcal{F}_t^C) &= CO - BO \end{aligned}$$

Conditional to  $\mathcal{F}_t^C$ , each term becomes:

$$\begin{aligned} G &= \mathbb{E}^Q [e^{-\bar{T}_r(T)} L(T) \mathbf{1}_{\tau \geq T}] = e^{-\bar{T}_r(T)} L(T) [1 - \Pr^Q(\tau < T)] \\ &= e^{-\bar{T}_r(T)} L(T) [1 - \mathbb{Q}_1] \end{aligned}$$

$$\begin{aligned} PO &= \mathbb{E}^Q [e^{-\bar{T}_r(T)} [L(T) - S(T)]^+ \mathbf{1}_{\tau \geq T}] \\ &= \mathbb{E}^Q [e^{-\bar{T}_r(T)} [L(T) - S(T)]^+] - \mathbb{E}^Q [e^{-\bar{T}_r(T)} [L(T) - S(T)]^+ \mathbf{1}_{\tau < T}] \\ &= \text{SWPO}[L(T)] - e^{-\bar{T}_r(T)} L(T) \Pr^Q[S(T) < L(T); \tau < T] \\ &\quad + \mathbb{E}^Q [e^{-\bar{T}_r(T)} S(T) \mathbf{1}_{S(T) < L(T)} \mathbf{1}_{\tau < T}] \\ &= \text{SWPO}[L(T)] - e^{-\bar{T}_r(T)} L(T) \mathbb{Q}_2 + \mathbb{E}_3^Q \end{aligned}$$

$$\begin{aligned} BO &= \mathbb{E}^Q \left[ \alpha \delta e^{-\bar{T}_r(T)} \left[ S(T) - \frac{L(T)}{\alpha} \right]^+ \mathbf{1}_{\tau \geq T} \right] \\ &= \mathbb{E}^Q \left[ \alpha \delta e^{-\bar{T}_r(T)} \left[ S(T) - \frac{L(T)}{\alpha} \right]^+ \right] - \mathbb{E}^Q \left[ \alpha \delta e^{-\bar{T}_r(T)} \left[ S(T) - \frac{L(T)}{\alpha} \right]^+ \mathbf{1}_{\tau < T} \right] \\ &= \alpha \delta \text{SWCO} \left[ \frac{L(T)}{\alpha} \right] - \alpha \delta \mathbb{E}^Q \left[ e^{-\bar{T}_r(T)} S(T) \mathbf{1}_{S(T) > \frac{L(T)}{\alpha}} \mathbf{1}_{\tau < T} \right] \\ &\quad + \delta e^{-\bar{T}_r(T)} L(T) \Pr^Q \left[ S(T) > \frac{L(T)}{\alpha}; \tau < T \right] \\ &= \alpha \delta \text{SWCO} \left[ \frac{L(T)}{\alpha} \right] - \alpha \delta \mathbb{E}_4^Q + \delta e^{-\bar{T}_r(T)} L(T) \mathbb{Q}_5 \end{aligned}$$

$$RB = \min\{\lambda, 1\} L(0) \mathbb{E}^Q [e^{-\bar{T}_r(\tau) + r_g \tau} \mathbf{1}_{\tau < T}] = \min\{\lambda, 1\} L(0) \mathbb{E}_6^Q$$

$$\begin{aligned} CO &= \mathbb{E}^Q [e^{-\bar{T}_r(T)} [S(T) - L(T)]^+ \mathbf{1}_{\tau \geq T}] \\ &= \mathbb{E}^Q [e^{-\bar{T}_r(T)} [S(T) - L(T)]^+] - \mathbb{E}^Q [e^{-\bar{T}_r(T)} [S(T) - L(T)]^+ \mathbf{1}_{\tau < T}] \\ &= \text{SWCO}[L(T)] - \mathbb{E}^Q [e^{-\bar{T}_r(T)} S(T) \mathbf{1}_{S(T) > L(T)} \mathbf{1}_{\tau < T}] \\ &\quad + e^{-\bar{T}_r(T)} L(T) \Pr^Q [S(T) > L(T); \tau < T] \\ &= \text{SWCO}[L(T)] - \mathbb{E}_7^Q + e^{-\bar{T}_r(T)} L(T) \mathbb{Q}_8 \end{aligned}$$

where

$$\begin{aligned}
\mathbb{Q}_1 &\equiv \Pr^Q (\tau < T) \\
\mathbb{Q}_2 &\equiv \Pr^Q [S(T) < L(T); \tau < T] \\
\mathbb{E}_3^Q &\equiv \mathbb{E}^Q \left[ e^{-\bar{T}_r(T)} S(T) \mathbf{1}_{S(T) < L(T)} \mathbf{1}_{\tau < T} \right] \\
\mathbb{E}_4^Q &\equiv \mathbb{E}^Q \left[ e^{-\bar{T}_r(T)} S(T) \mathbf{1}_{S(T) > \frac{L(T)}{\alpha}} \mathbf{1}_{\tau < T} \right] \\
\mathbb{Q}_5 &\equiv \Pr^Q \left[ S(T) > \frac{L(T)}{\alpha}; \tau < T \right] \\
\mathbb{E}_6^Q &\equiv \mathbb{E}^Q \left[ e^{-\bar{T}_r(\tau) + r_g \tau} \mathbf{1}_{\tau < T} \right] \\
\mathbb{E}_7^Q &\equiv \mathbb{E}^Q \left[ e^{-\bar{T}_r(T)} S(T) \mathbf{1}_{S(T) > L(T)} \mathbf{1}_{\tau < T} \right] \\
\mathbb{Q}_8 &\equiv \Pr^Q [S(T) > L(T); \tau < T] \\
\text{SWCO} [K | \mathcal{F}_t^C] &\equiv \text{switching call option at } T; \text{ strike } K \\
\text{SWPO} [K | \mathcal{F}_t^C] &\equiv \text{switching put option at } T; \text{ strike } K \\
e^{-\bar{T}_r(T)} L(T) &= L(0) e^{-\bar{T}_{\bar{r}_g}(T)} \text{ where} \\
\bar{r}_g &\equiv [r_1 - r_g, \dots, r_N - r_g]'
\end{aligned}$$

It follows that

$$\begin{aligned}
V_L(0, T, S(0), L, N | \mathcal{F}_t^C) &= e^{-\bar{T}_r(T)} L(T) [1 - \mathbb{Q}_1 + \mathbb{Q}_2 + \delta \mathbb{Q}_5] \\
&\quad - \mathbb{E}_3^Q - \alpha \delta \mathbb{E}_4^Q + \min\{\lambda, 1\} L(0) \mathbb{E}_6^Q \\
&\quad + \alpha \delta \text{SWCO} \left[ \frac{L(T)}{\alpha} | \mathcal{F}_t^C \right] - \text{SWPO} [L(T) | \mathcal{F}_t^C] \\
V_E(0, T, S(0), L, N | \mathcal{F}_t^C) &= e^{-\bar{T}_r(T)} L(T) [\mathbb{Q}_8 - \delta \mathbb{Q}_5] \\
&\quad + \alpha \delta \mathbb{E}_4^Q - \mathbb{E}_7^Q \\
&\quad + \text{SWCO} [L(T) | \mathcal{F}_t^C] - \alpha \delta \text{SWCO} \left[ \frac{L(T)}{\alpha} | \mathcal{F}_t^C \right]
\end{aligned}$$

## Appendix B Closed-form formula for regime switching European options

The formulas for the call and put options with underlying switching process, strike price  $K$  and maturity date  $T$  are respectively

$$\begin{aligned}\text{SWCO} [K | \mathcal{F}_t^C] &= S(0) \mathcal{N}(d_1) - K e^{-\bar{T}_r(T)} \mathcal{N}(d_2) \\ \text{SWPO} [K | \mathcal{F}_t^C] &= K e^{-\bar{T}_r(T)} \mathcal{N}(-d_2) - S(0) \mathcal{N}(-d_1)\end{aligned}$$

where

$$\begin{aligned}d_1 &= \frac{\log(S(0)/K) + \bar{T}_r(T) + \frac{1}{2}\bar{T}_{\sigma^2}(T)}{\sqrt{\bar{T}_{\sigma^2}(T)}} \\ d_2 &= \frac{\log(S(0)/K) + \bar{T}_r(T) - \frac{1}{2}\bar{T}_{\sigma^2}(T)}{\sqrt{\bar{T}_{\sigma^2}(T)}} = d_1 - \sqrt{\bar{T}_{\sigma^2}(T)} \\ \bar{T}_r(T) &= \sum_{i=1}^N r_i T_i(T); \quad \bar{T}_{\sigma^2}(T) = \sum_{i=1}^N \sigma_i^2 T_i(T); \end{aligned}$$

$\mathcal{N}$  is the Gaussian cumulative distribution function and  $T_i(T)$  denotes the amount of time the continuous time Markov chain spends on state  $i$  over the time period interval  $[0, T]$ ,  $i = 1, \dots, N$ . See Monter, R. (2008).

## Appendix C Numerical Procedure

The numerical scheme proposed is as follows:

1. Discretize the time interval  $[0, T]$  on  $m$  intervals of size  $\Delta = \frac{T}{m}$ . Note that  $m$  must be large enough such that each sub-interval of time  $\Delta_j$  for  $j = 1, \dots, m$  allows at most one switching of state. Define  $t_j = j\Delta$ , so  $t_1 = \Delta, t_2 = 2\Delta, \dots, t_m = m\Delta = T$ .
2. For each  $t_j$  compute the barrier level by  $B(t_j) \equiv \lambda L(0) e^{r_g t_j}; t_j \in [0, T]$ .
3. Set the number of simulations be equal to  $l$ .
4. Given the probability matrix  $\mathbf{A}(t)$ , simulate a single path of the  $\{C_j(t)\} 0 \leq t \leq T; j = 1, \dots, m$  process taking values on the  $i = 1, \dots, N$  possible states.
5. Compute the occupation time  $T_i(T)$ , for each state  $i = 1, \dots, N - 1$ . Compute  $\bar{T}_r(T) = \sum_{i=1}^N r_i T_i(T)$  and  $\bar{T}_{\sigma^2}(T) = \sum_{i=1}^N \sigma_i^2 T_i(T)$ .
6. For  $k = t_1, \dots, t_k, \dots, t_m = T$ 
  - (a) Compute  $\bar{T}_r(t_k) = \sum_{i=1}^N r_i T_i(t_k)$  and  $\bar{T}_{\sigma^2}(t_k) = \sum_{i=1}^N \sigma_i^2 T_i(t_k)$ .
  - (b) Simulate the process  $Z(t_k) = \bar{T}_r(t_k) - \frac{1}{2} \bar{T}_{\sigma^2}(t_k) + \varphi(t_k)$  where  $\varphi(t_k)$  is normal distributed with mean zero and variance  $\bar{T}_{\sigma^2}(t_k)$ .
  - (c) If  $Z(t_k) \leq B(t_k)$ , set  $\tau = t_k$  which is the liquidation time.
  - (d) Evaluate the expected discounted payoff of the options given  $\tau$ .
7. Compute the conditional switching options  $\text{SWCO}[K | \mathcal{F}_t^C]$  and  $\text{SWPO}[K | \mathcal{F}_t^C]$ .
8. Repeat steps 4 to 7 for  $l$  times and get the average of the expected discounted payoff of the options given  $\tau$ ,  $\text{SWCO}[K | \mathcal{F}_t^C]$  and  $\text{SWPO}[K | \mathcal{F}_t^C]$ .

Finally, the *unconditional value* is computed by simulating the process  $l$  times and getting the average.

## Appendix D Geometric Brownian motion parameter estimation

The firm's assets' value is specified by the equation

$$dS(t) = \mu S(t) dt + \sigma S(t) dW(t) \quad (32)$$

where  $W(t)$  is a standard Brownian motion. Itô's formula implies

$$\log\left(\frac{S(t)}{S(t-1)}\right) = \mu - \frac{\sigma^2}{2} + \sigma(W(t) - W(t-1))$$

Let  $R(t) = \log\left(\frac{S(t)}{S(t-1)}\right)$  denotes the log return over one period of time, then

$$R(t) = \mu - \frac{\sigma^2}{2} + \sigma(\varepsilon(t)) \quad (33)$$

where  $\varepsilon(t)$  follows a standard Gaussian distribution. The mean and variance maximum likelihood estimators of the log increments are, respectively

$$\hat{m}(T) = \frac{1}{T} \sum_{t=1}^T R(t) \quad \hat{s}^2(T) = \frac{1}{T} \sum_{t=1}^T [R(t) - \hat{m}(T)]^2$$

therefore, the maximum likelihood estimators of the volatility and drift parameters are given by

$$\hat{\sigma}^2(T) = \hat{s}^2(T), \quad \hat{\mu}(T) = \hat{m}(T) + \frac{\hat{s}^2(T)}{2}$$

When data is sampled at a time interval  $h$ , we get

$$\hat{\sigma}^2(T) = \frac{1}{h} \hat{s}^2(h, T), \quad \hat{\mu}(h, T) = \frac{\hat{m}(h, T)}{h} + \frac{\hat{s}^2(h, T)}{2h} \quad (34)$$

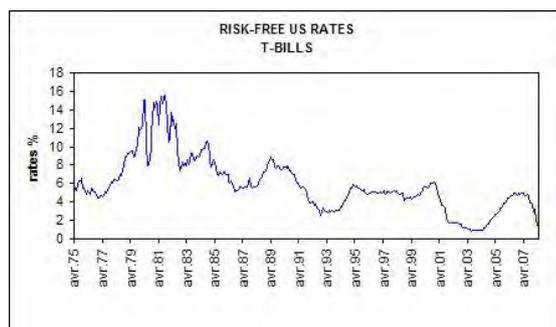


Figure 1: Annual rates of US Treasury Bills. Monthly data. April 1975 April 2008.

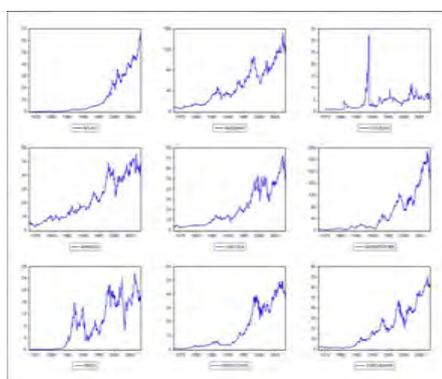


Figure 2: Stock prices. Monthly data. April 1975 to April 2008.

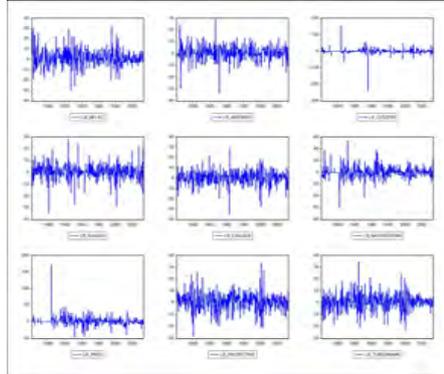


Figure 3: Logarithm returns of stock prices. Monthly data. April 1975 to April 2008.

Log Ret	AFLAC	AMERNAT	CITIZENS	KANSAS	LINCOLN	NATWESTERN	PRES	PROTECTIVE	TORCHMARK
Mean	1.66	0.68	0.57	0.59	0.66	1.04	1.65	1.06	0.82
Median	1.46	0.37	0.00	0.45	0.86	0.00	0.00	1.69	1.10
Maximum	32.21	29.45	154.04	28.22	29.57	54.94	172.86	33.77	34.87
Minimum	-30.15	-33.78	-241.59	-34.69	-35.35	-60.61	-47.85	-28.77	-22.98
Std. Dev.	8.34	6.55	18.82	6.79	7.13	10.37	14.41	7.77	7.36
Skewness	0.22	-0.31	-3.92	-0.43	-0.43	0.13	4.17	-0.09	0.09
Kurtosis	4.67	6.58	82.11	7.11	5.58	9.02	53.01	4.74	4.55
Jarque-Bera	49.14	218.09	104553.42	291.27	122.35	600.90	42527	50.90	40.22
Probability	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
Sum	659	272	227	234	262	414	657	422	327
Sum Sq. Dev.	27519	17012	140291	18259	20121	42597	82192	23882	21426
Observations	397	397	397	397	397	397	397	397	397

Figure 4: Descriptive statistics of monthly returns. Sample: April 1975 to April 2008. 397 observations.

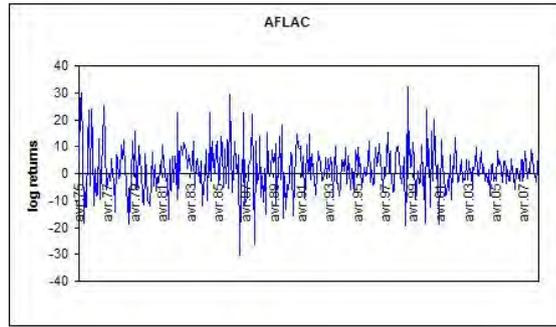


Figure 5: AFLAC log return of stock prices.

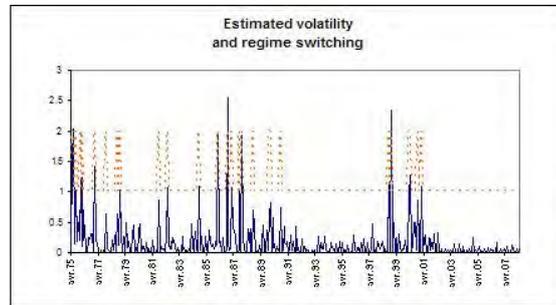


Figure 6: AFLAC estimated volatility and two states regime switching.

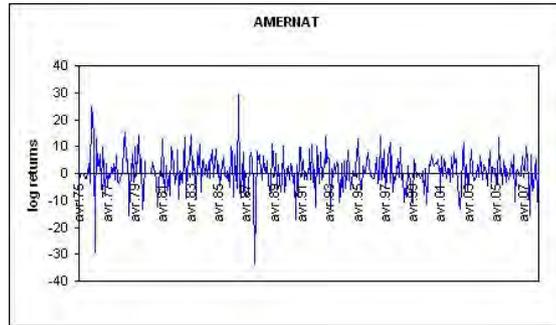


Figure 7: AMERNAT log return of stock prices.

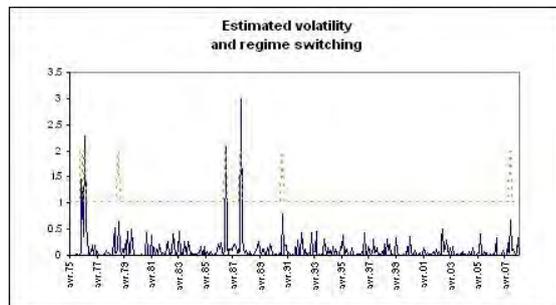


Figure 8: AMERNAT estimated volatility and two states regime switching.

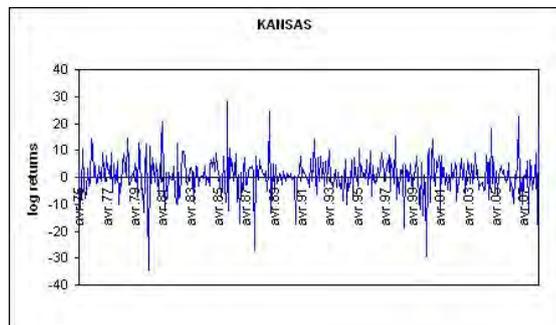


Figure 9: KANSAS log return of stock prices.

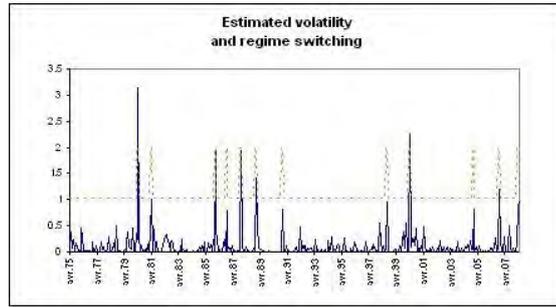


Figure 10: KANSAS estimated volatility and two states regime switching.

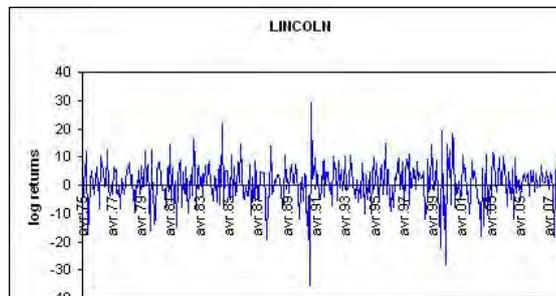


Figure 11: LINCOLN log return of stock prices.

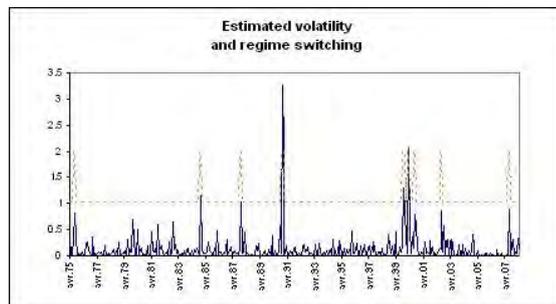


Figure 12: LINCOLN estimated volatility and two states regime switching.

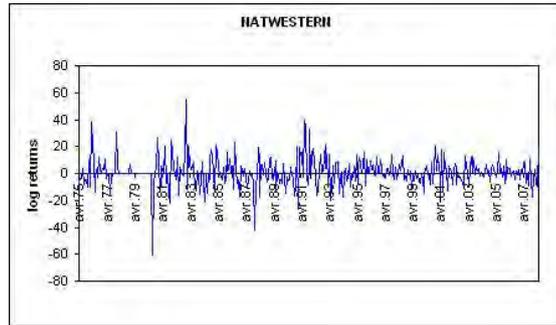


Figure 13: NATWESTERN log return of stock prices.

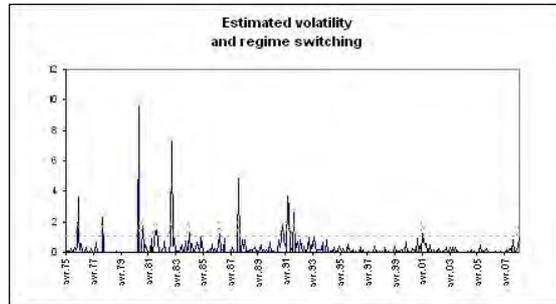


Figure 14: NATWESTERN estimated volatility and two states regime switching.

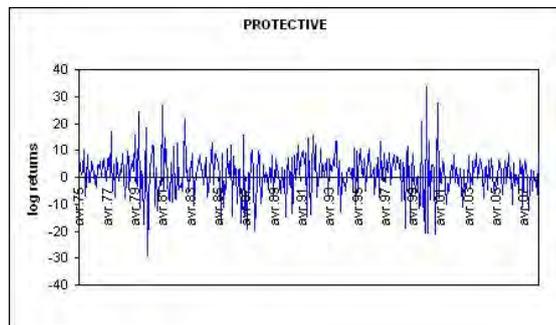


Figure 15: PROTECTIVE log return of stock prices.

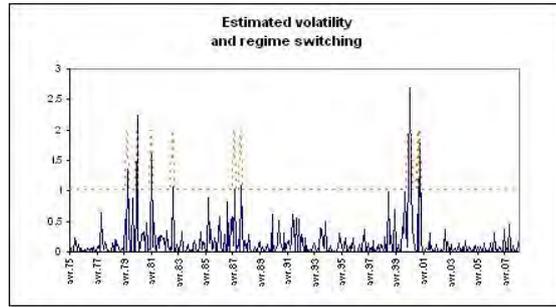


Figure 16: PROTECTIVE estimated volatility and two states regime switching.

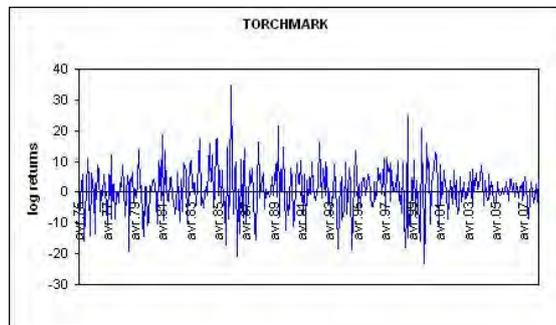


Figure 17: TORCHMARK log return of stock prices.

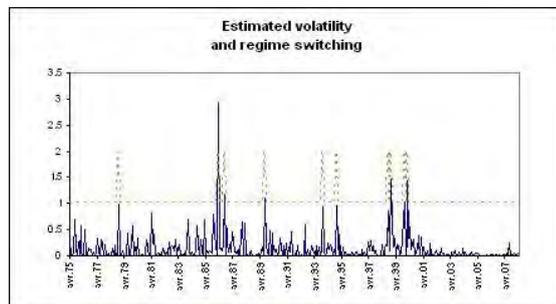


Figure 18: TORCHMARK estimated volatility and two states regime switching.

Volatility	Constant							RISK-FREE
	AFLAC	AMERNAT	KANSAS	LINCOLN	NATWESTERN	PROTECTIVE	TORCHMARK	rate
mean	0.17460	0.10794	0.11585	0.12766	0.27027	0.15153	0.13595	5.89924
var	0.11209	0.06513	0.08218	0.07479	0.58741	0.08619	0.06574	9.12276
std dev	0.33480	0.25520	0.28668	0.27347	0.76643	0.29358	0.25639	3.02039
skew	3.85805	7.10878	6.08085	6.45173	7.77742	4.40329	4.98419	0.92629
exc kurt	17.95265	64.43203	47.32260	57.31936	76.91540	25.69640	39.43100	1.16626
min	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.87000
max	2.54862	2.99202	3.13491	3.26563	9.57566	2.69452	2.91908	15.59000

Figure 19: Estimation results of a non regime switching model for the volatility of the logarithm assets' returns and the mean of the risk-free rate.

Volatility	Regime switching							RISK-FREE
	AFLAC	AMERNAT	KANSAS	LINCOLN	NATWESTERN	PROTECTIVE	TORCHMARK	rate
P11	0.93496	0.98196	0.96867	0.97662	0.96306	0.97662	0.97409	0.97521
P12	0.06504	0.01804	0.03133	0.02338	0.03694	0.02338	0.02591	0.02479
P21	0.88889	0.87500	0.84615	0.81818	0.76471	0.81818	1.00000	0.03896
P22	0.11111	0.12500	0.15385	0.18182	0.23529	0.18182	0.00000	0.96104
T1	0.93182	0.97980	0.96717	0.97222	0.95707	0.97222	0.97475	0.61111
T2	0.06818	0.02020	0.03283	0.02778	0.04293	0.02778	0.02525	0.38889
<b>Volatility state 1</b>								
mean	0.09834	0.07949	0.07287	0.09144	0.15290	0.11297	0.10570	4.05388
var	0.01493	0.01229	0.01188	0.01506	0.05271	0.02750	0.02367	2.15366
std dev	0.12218	0.11085	0.10897	0.12271	0.22959	0.16583	0.15386	1.46753
skew	1.62708	2.00494	2.29629	2.19970	1.97589	2.50716	2.23704	-0.85208
exc kurt	2.05764	3.78973	5.27679	5.27115	3.30337	7.32946	5.12672	-0.55173
min	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.87000
max	0.56770	0.55115	0.54635	0.69280	1.09958	0.97672	0.80835	5.77000
<b>Volatility state 2</b>								
mean	1.22267	1.49590	1.38833	1.37481	2.89220	1.50201	1.28502	8.80078
var	0.27182	0.75786	0.52035	0.61665	5.66210	0.29369	0.36967	6.35888
std dev	0.52136	0.87055	0.72135	0.78527	2.37952	0.54193	0.60800	2.52168
skew	1.19622	0.70337	1.39336	1.68181	1.88854	1.37006	2.57583	1.18252
exc kurt	0.71904	-0.85906	1.44584	2.44292	3.29396	1.08318	7.16256	0.60795
min	0.62847	0.65287	0.79588	0.78786	1.10533	1.02457	0.87098	5.90000
max	2.54862	2.99202	3.13491	3.26563	9.57566	2.69452	2.91908	15.59000

Figure 20: Estimation results of a two state regime switching model for the volatility of the logarithm assets' returns and the mean of the risk-free rate.

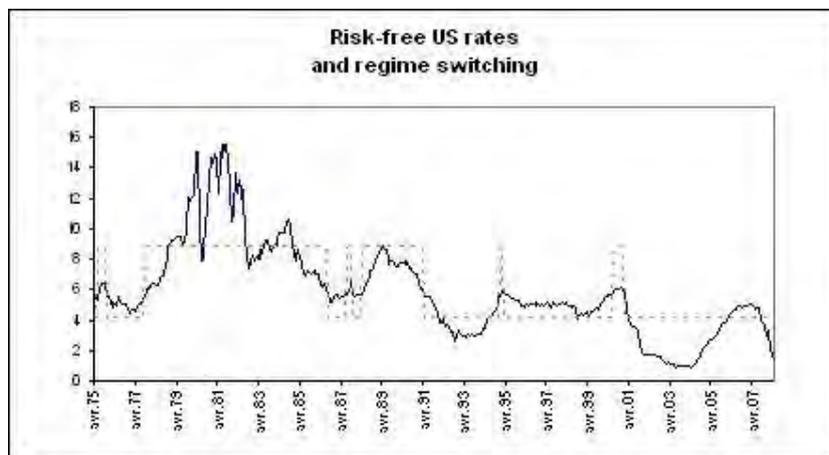


Figure 21: US risk-free rates and the mean estimation of a two states regime switching process.