Modelling Longevity Risk: Generalizations of the Olivier-Smith Model

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Abstract

Stochastic mortality models have been developed for a range of applications from demographic projections to financial management. Financial risk based models build on methods used for interest rates and apply these to mortality rates. They have the advantage of being applied to financial pricing and the management of longevity risk. Olivier and Jeffery (2004) and Smith (2005) proposed a model based on a forward-rate mortality framework with stochastic factors driven by univariate gamma random variables irrespective of age or duration. We assess and further develop this model. We generalized random shocks from a univariate gamma to a univariate Tweedie distribution, thus allowing a broader class of distributions and increasing potential for a better fit to mortality data. Furthermore, since dependence between ages is an observed characteristic of mortality rate improvements, we consider a multivariate Tweedie framework that incorporates age. Such a model provides a more realistic basis for capturing the risk of mortality improvements and serves to enhance longevity risk management for pension and insurance funds.

Keywords: longevity risk, Olivier-Smith model, forward-rate mortality framework, multivariate Tweedie distribution

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1 Introduction

A variety of empirical studies across many developed nations show that mortality trends have been improving stochastically; see, for example, CMI (2005), Blake *et al.* (2006), Blackburn and Sherris (2012), Luciano and Vigna (2005), and Liu (2008). A variety of stochastic mortality models proposed in the literature apply extensions of interest rate term structure modelling, known as *short-rate models*. In particular, the Cox-Ingersoll-Ross (CIR) model, Cox *et al.* (1985), has been adapted in Luciano and Vigna (2005), who use a time inhomogeneous version of the process to model the mortality dynamics. Furthermore, Biffis (2005), Russo *et al.* (2010), and Blackburn and Sherris (2012) develop a variety of affine frameworks extended from interest rate term structure modelling. Blake *et al.* (2006), Cairns *et al.* (2006), Bauer (2006) and Bauer and Ruß (2006) demonstrate that if mortality risk can be traded through securities such as longevity bonds and swaps, then the techniques developed in financial markets for pricing bonds and swaps can be adapted for mortality risk. Finally, Qiao and Sherris (2012) introduce a multifactor stochastic mortality model for group self-annuitization schemes designed to share uncertain future mortality experience including systematic improvements. Such models have the advantage of being able to incorporate a price of longevity risk and have applications to the valuation of various types of mortality linked contracts, which can be used to mitigate longevity risk on life insurance products, including annuities, longevity bonds, and longevity swaps. A typical underlying assumption in these models is that mortality rates are Gaussian.

An alternative to short-rate models is provided by forward-rate approaches that model forward forces of mortality and forward survival probabilities. Olivier and Jeffery (2004) and Smith (2005) apply a forward-rate mortality framework with stochastic factors driven by univariate gamma random variables irrespective of age or duration. Although the model is conceptually interesting, not much has been done to assess the validity of the model assumptions. In addition to restricting the stochastic factors to a gamma distribution, another critical assumption is that of independence amongst these factors across age. We examine these assumptions using England and Wales female mortality data for 1960 to 2009 based on the assumption that population mortality rates are risk neutral prices, which may be confirmed from the data since the martingale property can be assessed by examining the trends in the mortality rates.

Based on the empirical analysis, we generalize the Olivier-Smith model in order to provide a more realistic basis for capturing the risk of mortality improvements. A more realistic risk factor structure that captures features of mortality data has the potential to provide enhanced risk management techniques for mitigating the longevity risk faced by pension funds and annuity providers. We assume random shocks are driven by univariate Tweedie random variables, thus allowing a broader class of distributions. Since dependence between ages is an observed characteristic of mortality rate improvements, we further generalize the model by considering a multivariate Tweedie distribution that incorporates age dependence.

Organization of the paper: In Section 2 we introduce the necessary notation for the forward-rate mortality framework and provide the Olivier-Smith model. We perform an empirical analysis on mortality data from England and Wales in Section 3. In Section 4 we introduce a univariate Tweedie generalization and in Section 5 commence consideration of a multivariate framework. We summarize the project in Section 6.

2 Notation

We closely follow the notation provided in Cairns *et al.* (2006), Cairns (2007).

Let $C(t)$ be the cash account at time t. Define the survivor index as

$$S(t,x) = \exp \int_{t_0}^{t} -\mu(u, x+u)du,$$

where $\mu(t,x)$ is the force of mortality at time t for an individual aged x. Henceforth, we use the convention $t_0 = 0$. Therefore, the survival index $S(t,x)$ is the probability an individual aged x at time zero survives to time t, in other words, sees age $x+t$.

Let $LB(T,x)$ be a zero-coupon longevity bond that pays $C(t)S(T,x)$ at maturity T. Let $D(t,T,x)$ be the price and $\widetilde{D}(t,T,x)$ be the discounted (to time zero) price of $LB(T,x)$ at time t. That is,

$$\widetilde{D}(t,T,x) = \frac{D(t,T,x)}{C(t)}.$$

We assume the market is arbitrage-free and apply the Fundamental Theorem of Asset Pricing. That is, there exists probability measure Q such that $\widetilde{D}(t,T,x)$ are Q-martingales. Furthermore, we define $\mathcal{M}_t$ and $\mathcal{H}_t$ to be the filtrations representing the evolution of solely the force of mortality, and the evolution of both the force of mortality and interest rates, respectively, up to and including time t. We have that

$$\begin{aligned} D(t,T,x) &= E_Q[C(t)\widetilde{D}(t,T,x)|\mathcal{H}_t] \\ &= C(t)\,E_Q[C(T)S(T,x)/C(T)|\mathcal{H}_t] \\ &= C(t)\,E_Q[S(T,x)|\mathcal{M}_t] \\ &= C(t)\,S(t,x)\,p_Q(t,t,T,x), \end{aligned}$$

where

$$\begin{aligned} p_Q(t,T_1,T_2,x) &= P_Q[\tau_x > T_2|\tau_x > T_1, \mathcal{M}_t] \\ &= \frac{E_Q[S(T_2,x)|\mathcal{M}_t]}{E_Q[S(T_1,x)|\mathcal{M}_t]}, \end{aligned}$$

and where τ_x is the remaining lifetime random variable for an individual aged x at time zero. Therefore, $p_Q(t,T_1,T_2,x)$ is the mortality forward-rate; that is, the probability of an individual age x at time zero surviving to time T_2 given survival to time T_1, based on mortality information $\mathcal{M}_t$. For clarity, we describe the indices of p_Q in greater detail:

- t is the time-period under consideration,
- T_1 is the *forward-time*, that is, the time to which survival is conditioned from time zero,
- T_2 the *maturity-time*, that is, the time to which survival is measured from time zero; a result is that $T_2 - T_1$ is the *time-horizon* of the forward mortality rate,
- and x is the age at time zero, which implies that $x + T_1$ is the *effective* age of the forward mortality rate, we refer to this age as the *forward-age*.

The martingale property implies

$$p_Q(t,t,T,x) = E_Q[p_Q(t+1,t,T,x)|\mathcal{M}_t]. \tag{1}$$

The Olivier-Smith model is given below; see Olivier and Jeffery (2004), Smith (2005).

Model 1 (The Olivier-Smith Model) *For all ages x and forward-times $T = t, t+1, \ldots$*

$$p_Q(t+1,T,T+1,x) = p_Q(t,T,T+1,x)^{b(t+1,T,T+1,x)G(t+1)},$$

where $G(1), G(2), \ldots$ are independent and identically distributed gamma random variables with shape and rate parameter α, that is, such that $E_Q[G(t)] = 1$ and $Var_Q(G(t)) = 1/\alpha$. Furthermore, the $b(t+1, T, T+1, x)$ are $\mathcal{M}_t$-measurable bias correction functions given by

$$b(t+1, T, T+1, x) = -\frac{\alpha p_Q(t, t, T, x)^{-1/\alpha}(p_Q(t, T, T+1, x)^{-1/\alpha} - 1)}{\ln p_Q(t, T, T+1, x)}.$$

For an arbitrage-free market, we require the martingale property, that is, equation (1) to be satisfied. Consequently, we obtain

$$p_Q(t, t, T, x) \quad = \quad \frac{\alpha^\alpha}{(\alpha - \sum_{u=t}^{T-1} b(t+1, u, u+1, x) \ln p_Q(t, u, u+1, x))^\alpha};$$

see, for example, Cairns (2007) for the details of this derivation, which rests on standard properties of the gamma distribution and the use of moment generating functions.

The model restricts the stochastic nature of mortality evolution by imposing gamma, $\Gamma(\alpha, \alpha)$, random variables. Henceforth, we generalize this stochastic component and make use of the notation $Z(t+1, T, T+1, x)$; we have that

$$Z(t+1, T, T+1, x) = \frac{\ln p_Q(t+1, T, T+1, x)}{\ln p_Q(t, T, T+1, x)};$$

note that this represents a deviation from the notation used in Cairns (2007).

3 Empirical Analysis

The Olivier-Smith model makes two assumptions regarding the stochastic mortality component. The first is that the distribution that generates stochastic mortality does not depend on time t, forward-time T, or age at time zero x. The second is that the distribution is gamma. We presently investigate the validity of these assumptions using real data. In this setting, we assume population mortality rates are risk neutral and set the bias correction terms, b, equal to one. This is validated by the data, since resulting observations of the stochastic factors are centered around one.

In a risk-neutral framework with constant annual forces of mortality and one-year forward rates, we have, for all t, all x, and $T = t-1, \ldots, \omega - 1 - x$,

$$p_P(t, T, T+1, x) = \exp\{-\mu(t, x+T)\},$$

where ω is the maximum attainable age. We define for all t, all x, and $T = t, \ldots, \omega - 1 - x$,

$$Z_P(t+1, T, T+1, x) = \frac{\ln p_P(t+1, T, T+1, x)}{\ln p_P(t, T, T+1, x)} = \frac{\mu(t+1, x+T)}{\mu(t, x+T)}. \qquad (2)$$

We use female central mortality rates from the Human Mortality Database (2013) for *England and Wales Total Population* for the period 1960-2009, and ages 0-105 ($\omega = 106$). We extract observations of $Z_P(t+1, T, T+1, x)$ from the data.

See Figure 1 for plots and companion histograms of observations of Z over time t for various fixed combinations of T and x. The histograms also include a gamma density fit using the method of moments. Over time t, the Z resemble a random sample. However, the underlying distribution changes with respect to the *forward-age* of the mortality rates, $x + T$. Notice the high volatility of the first set of plots, this is not surpising given that this set represents a forward-age of 105.

See Figure 2 for plots of observations of Z over age x for various fixed t. The value of T is of trivial consequence. Recall that we have no observations of Z for $T < t$; for $T > t$, we obtain a similar plot as that of $T = t$, where the only difference is a shift in the x-axis, representative of a translation of the forward-age. We present the plots for $T = t$ as they are most informative.

Figure 2 reinforces what is observed in Figure 1. The volatility of the stochastic component clearly varies with age.

In Figure 3 we fix the year-of-birth and plot over time, which is representative of following a cohort. We notice that volatility varies with forward-age, which is visible to a greater or lesser extent depending on which forward-age-range the cohort traverses.

Figure 1: The Stochastic Mortality Component over Time t for Fixed x

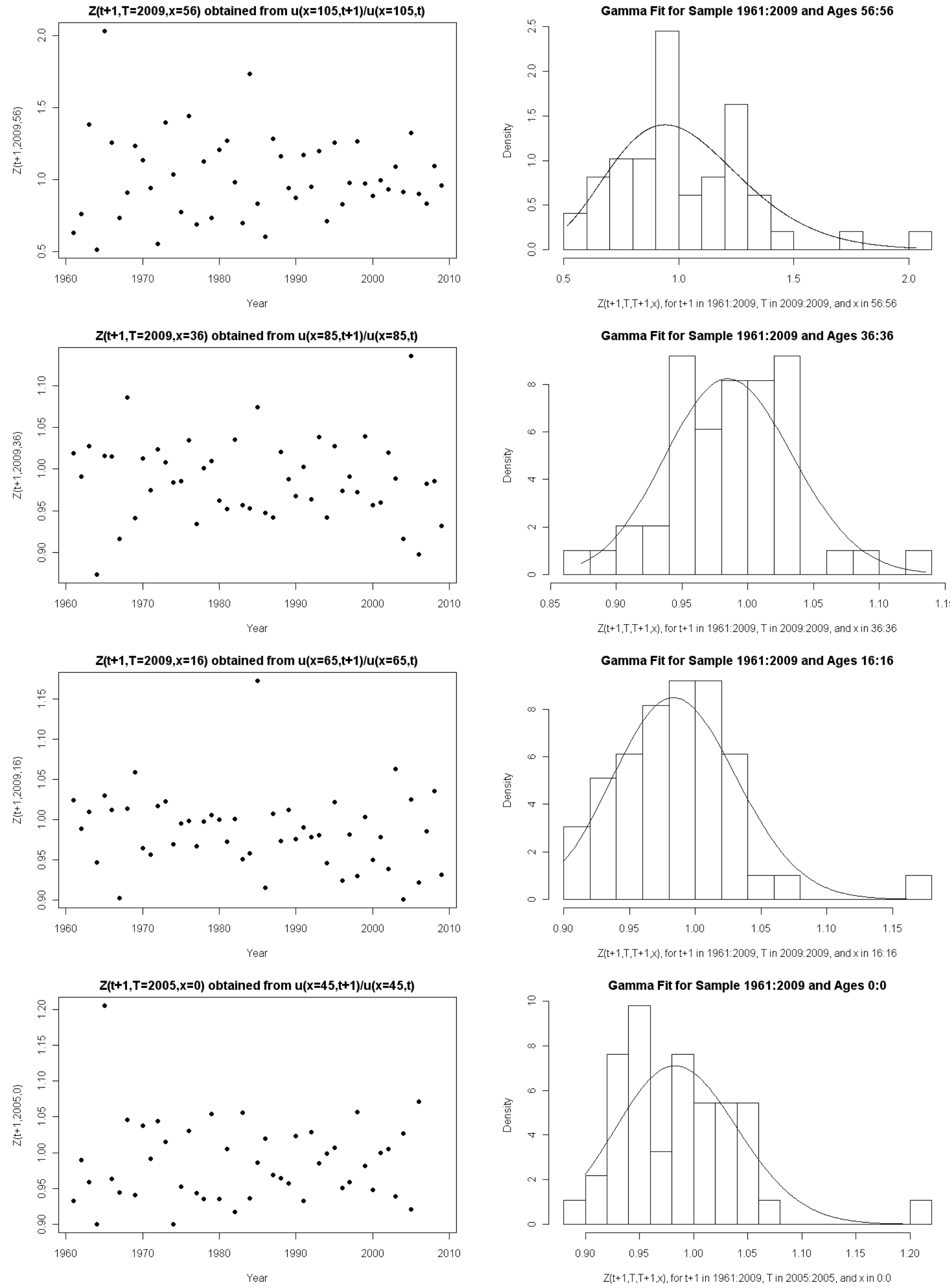


Figure 2: The Stochastic Mortality Component over Age x for Fixed t

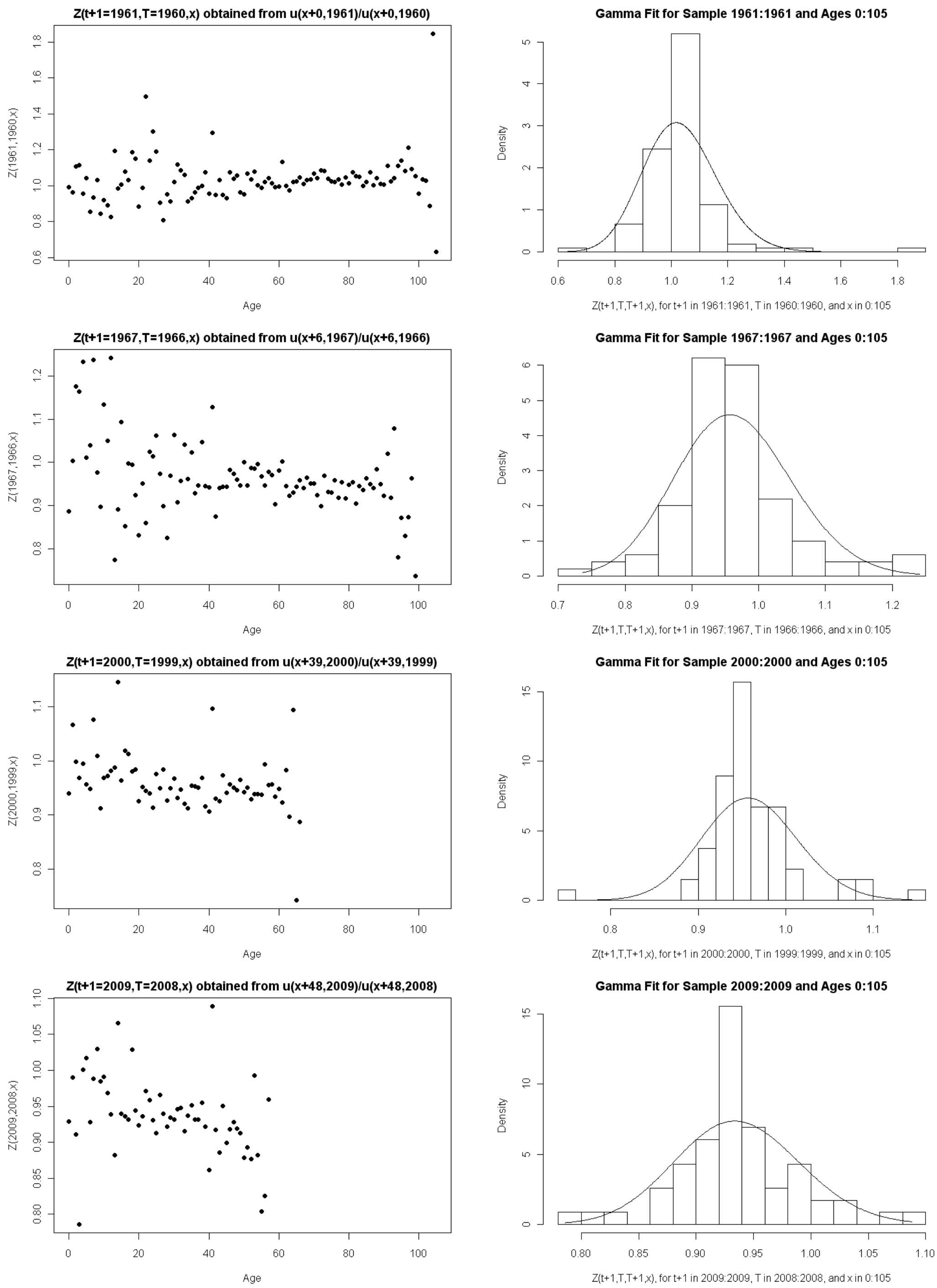
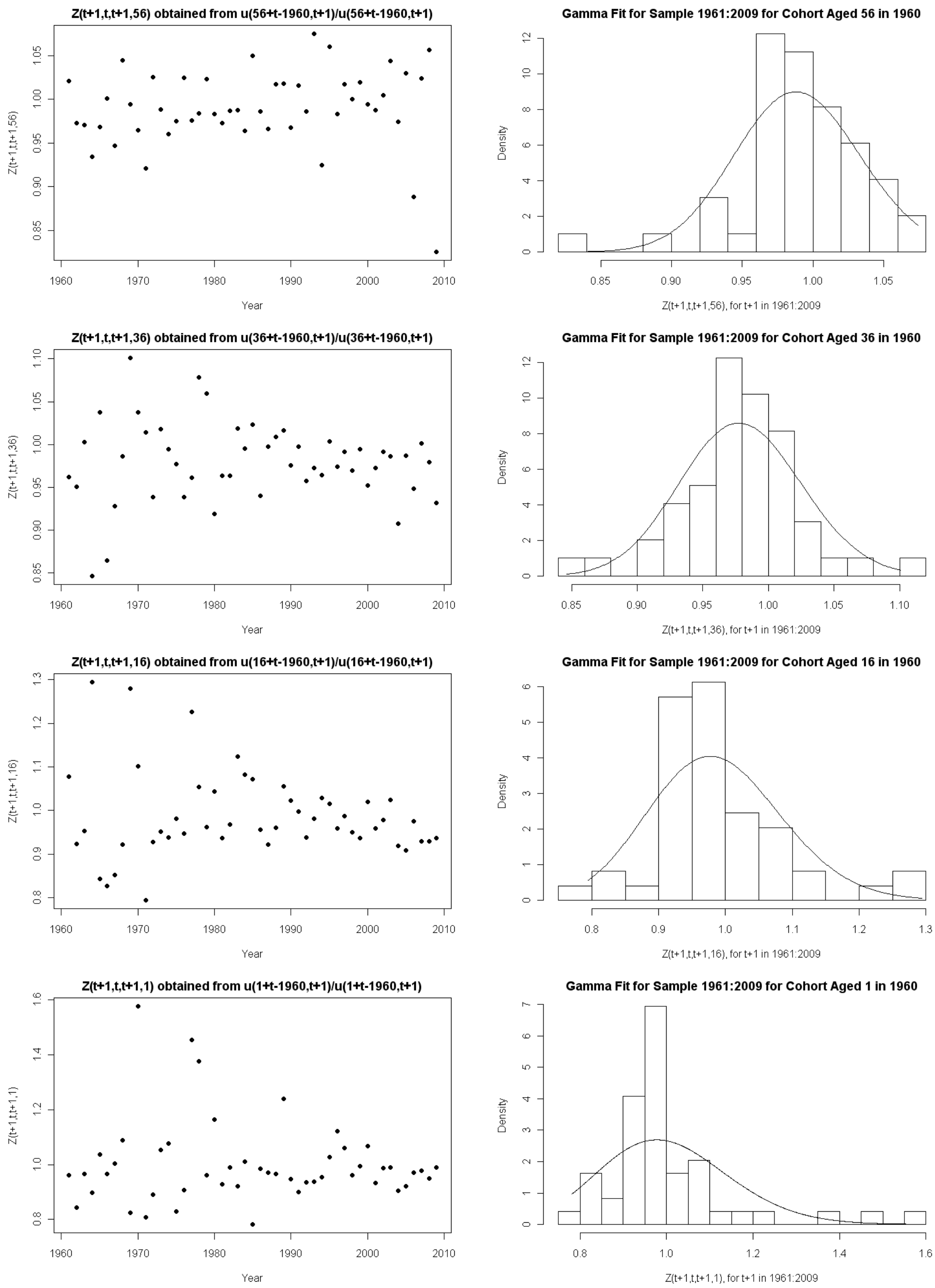


Figure 3: The Stochastic Mortality Component for Fixed Year-of-Birth



3.1 Goodness-of-Fit Tests

From the fitted densities shown in Figures 1, 2, and 3, it is difficult to conclude whether the gamma distribution provides a good fit to the data. Furthermore, fitting the data presumes independent and identically distributed observations, both of which may be violated.

However, we present quantile-to-quantile (QQ) plots and formal distributional (goodness-of-fit) tests to investigate the suitability of the gamma distribution. Figures 4, 5, and 6 show the QQ plots corresponding to Figures 1, 2, and 3, respectively. They contrast the quantiles of the empirical distribution with those from the estimated gamma distribution. A deviation from the 45 degree line indicates a departure from the assumed distribution. The results indicate the gamma distribution provides a reasonable fit for the stochastic mortality component over time t; see Figure 4. Figures 5 and 6 are less favourable.

Figure 4: The Stochastic Mortality Component over Time t for Fixed x; QQ plot corresponding to Figure 1

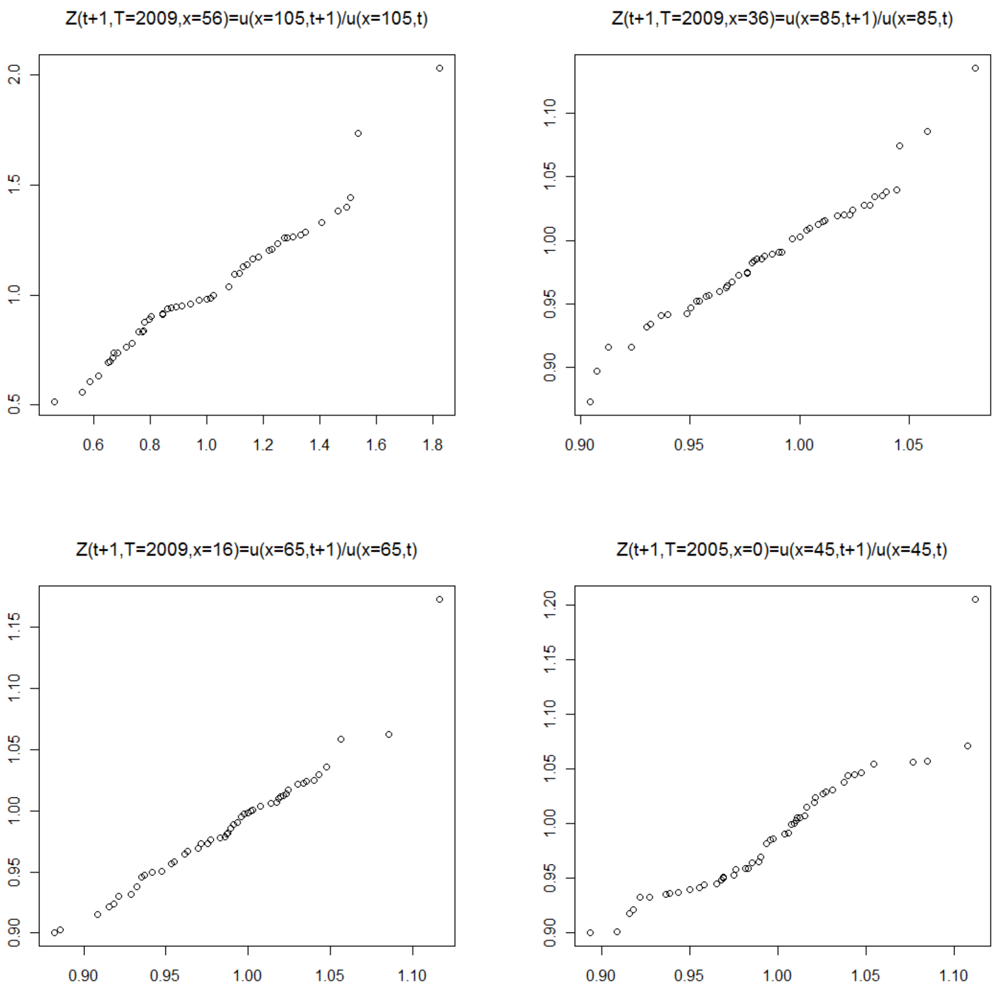


Figure 5: The Stochastic Mortality Component over Age x for Fixed t;
QQ plot corresponding to Figure 2

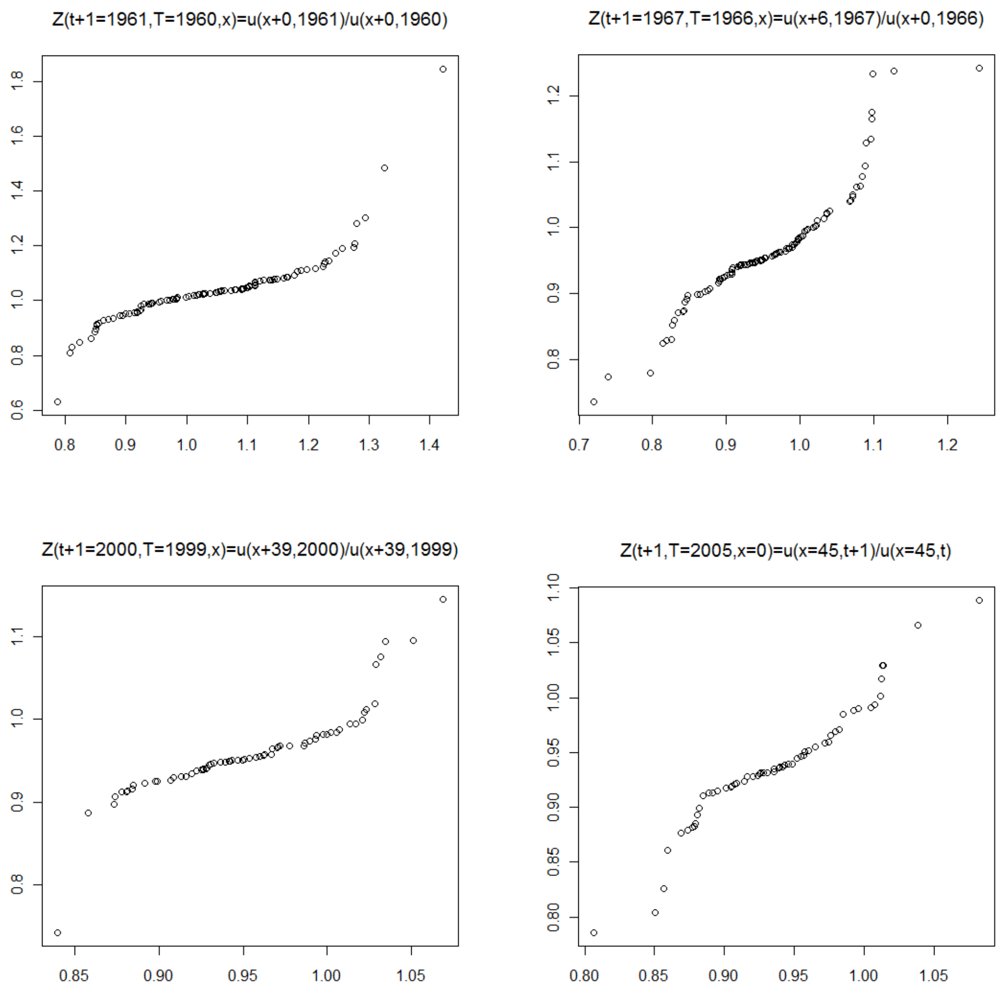
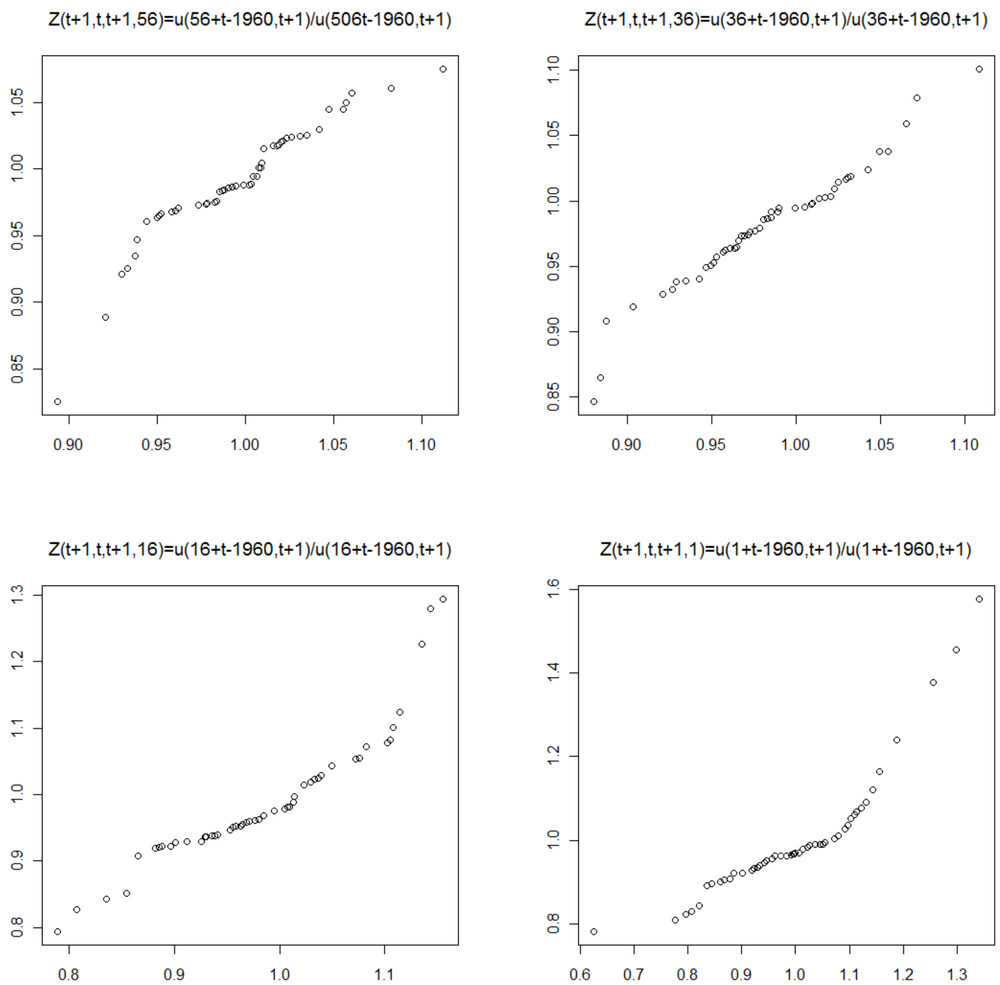


Figure 6: The Stochastic Mortality Component for Fixed Year-of-Birth; QQ plot corresponding to Figure 3



We apply the goodness-of-fit tests suggested in D'Agostino and Stephens (1986) and Stephens (1974). We test the null hypothesis that the data belongs to the theoretical (hypothesized) gamma distribution, that is, $H_0 : F_n(x) = \hat{F}_n(x)$, where $F_n(x)$ represents the empirical cumulative distribution function (cdf) and $\hat{F}_n(x)$, the theoretical cdf with estimated parameters obtained from maximum likelihood. Let the x_i be ordered observations, and $z_i = \hat{F}_n(x_i)$, the resulting estimated quantiles. We consider the following test statistics:

1. The Anderson-Darling statistic A^2, given by

$$A^2 = -n - 1/n \sum_{i=1}^{n} (2i-1)(\ln(z_i) + \ln(1 - z_{n+1-i})).$$

2. The Kolmogorov statistic D, given by $D = \max(D^+, D^-)$, where

$$\begin{aligned} D^+ &= \max_{1 \le i \le n} [(i/n) - z_i], \\ D^- &= \max_{1 \le i \le n} [z_i - (i-1)/n]. \end{aligned}$$

   The modified form statistics which is proposed in Stephens (1974) together with the critical values corresponds to $D(\sqrt{n} + 0.12 + 0.11/\sqrt{n})$.

3. The Carmér-von Mises statistic W^2, given by

$$W^2 = 1/12n + \sum_{i=1}^{n} (z_i - (2i-1)/2n)^2.$$

   The modified form statistic reported in Stephens (1974) is given by $(W^2 - 0.4/n + 0.6/n^2)(1 + 1/n)$.

The critical values for the above tests are specified in Table 1, as provided in D'Agostino and Stephens (1986) and Stephens (1974).

Table 1: Critical Values for the Goodness-of-Fit Tests

Stat.	Crit.5%	Crit.1%
A^2	2.492	3.857
D	1.358	1.628
W^2	0.461	0.743

Table 2: Distributional Test Results

	α	β	A^2	D	W^2
Panel A: The Stochastic Mortality Component over Time t for Fixed x					
Z(t+1,T=2009,x=56)	12.943646	12.624253	0.2181661**	0.5312711**	0.1075524**
Z(t+1,T=2009,x=36)	415.72498	421.08874	0.2475575**	0.5700078**	0.1057159**
Z(t+1,T=2009,x=16)	452.03843	458.56811	0.4509184**	0.6866471**	0.1270422**
Z(t+1,T=2005,x=0)	331.04846	335.43804	0.6111859**	0.7313687**	0.1552846**
Panel B: The Stochastic Mortality Component over Age x for Fixed t					
Z(t+1=1961,T=1960,x)	71.337103	69.005897	3.843054*	1.571849*	0.7661192
Z(t+1=1967,T=1966,x)	127.56511	132.22064	2.624833*	1.256487**	0.5699055**
Z(t+1=2000,T=1999,x)	312.29324	325.31700	2.905791*	1.167076**	0.5625603*
Z(t+1=2009,T=2008,x)	297.92308	317.92973	1.075193**	1.010486**	0.2789330**
Panel C: The Stochastic Mortality Component for Fixed Year-of-Birth					
Z(t+1,t,t+1,56)	497.3803	502.4013	0.8524845**	0.9784892**	0.2081623**
Z(t+1,t,t+1,36)	444.92502	454.13746	0.5299603**	0.5954966**	0.1529977**
Z(t+1,t,t+1,16)	107.80345	109.21021	1.463862**	1.020509**	0.332074**
Z(t+1,t,t+1,1)	53.05649	53.04923	2.440254*	1.390677*	0.520064*

Table 2 summarizes the results for the goodness-of-fit tests; ** and * indicate the gamma assumption is not rejected at a 5% and 1% significance level, respectively, where the gamma density is given by

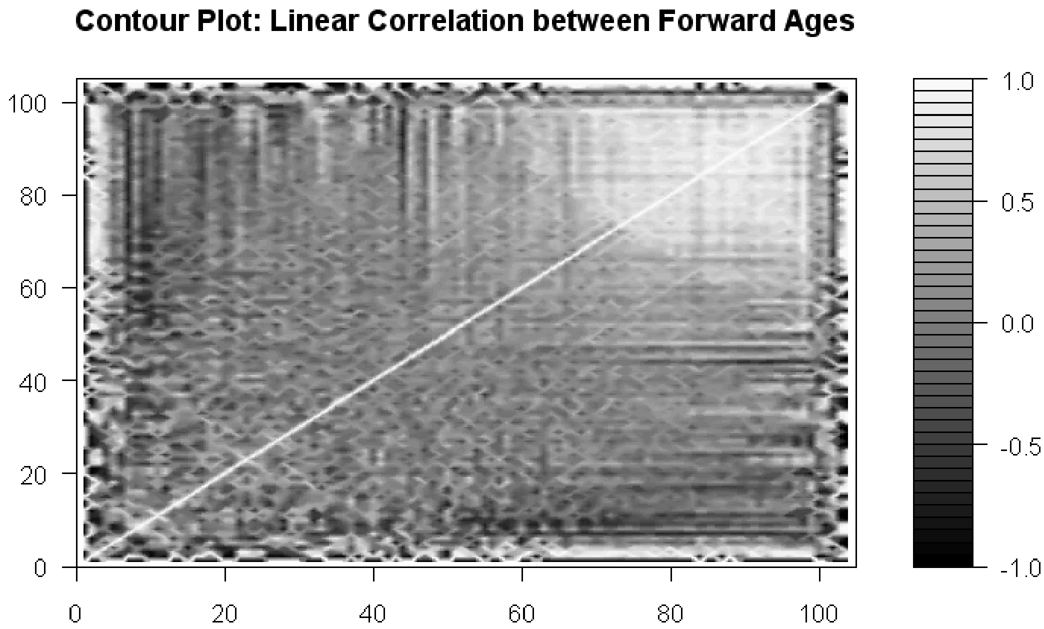
$$f(x) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}.$$

The distributional tests confirm the results observed from the QQ plots. The gamma distribution is not rejected at either significance level for the observations of stochastic mortality component over time (Panel A). The results for stochastic mortality component over age for fixed calendar year (Panel B) and over time for fixed year-of-birth (Panel C) are mixed. Finally, note from Table 2 that the estimates of the shape and rate parameters, α and β, are approximately equal, which confirms our assumption that the expected value of the stochastic component is one.

3.2 Correlation Structure

Figures 1, 2, and 3 confirm that the stochastic component Z depends on the forward-age $x + T$, which can also be seen from Equation (2). Of further interest is whether there is any correlation between the mortality rates at forward-ages. To investigate this, we provide contour plots of linear correlation between the stochastic components by forward-age; see Figure 7. From the plots we do notice varying levels of correlation, especially amongst the older ages.

Figure 7: Correlations between $Z(t+1,t,t+1,x-t)$ and $Z(t+1,t,t+1,y-t)$



4 Univariate Tweedie Generalization

We briefly introduce the Tweedie distribution, first formulated in Tweedie (1984). See, for example, Aalen (1992), Jørgensen and De Souza (1994), Smyth and Jørgensen (2002), Furman and Landsman (2010) for applications of the model to actuarial science.

Recall that the random variable X is said to belong to the Exponential Dispersion Family (EDF) of distributions in the additive form if its probability measure $P_{\theta,\lambda}$ is absolutely continuous with respect to some measure Q_λ and can be represented as follows for some function $\kappa(\theta)$ called the cumulant:

$$dP_{\theta,\lambda}(x) = e^{[\theta x - \lambda\kappa(\theta)]} dQ_\lambda(x);$$

see Jørgensen (1997), Section 3.1. The parameter θ is named the canonical parameter and λ the index or dispersion parameter belonging to the set of positive real numbers $\Lambda = (0,\infty) = \mathbb{R}_+$. We denote by $X \sim ED(\theta,\lambda)$ a random variable belonging to the additive EDF.

To define the Tweedie family, we notice that for regular EDF, cumulant $\kappa(\theta)$ is a twice differentiable function and, for the additive form, the expectation is given by

$$\mu = \lambda\kappa'(\theta).$$

Moreover, function $\kappa'(\theta)$ is one-to-one map and there exists inverse function

$$\theta = \theta(\mu) = (\kappa')^{-1}(\mu).$$

Function $V(\mu) = \kappa''(\theta(\mu))$ is called the unit variance function and provides the classification of members of the EDF. In particular, the Tweedie subclass is the class of EDF with power unit variance function.

$$V(\mu) = \mu^p,$$

where p is called the power parameter. Specific values of p correspond to specific distributions, for example when $p = 0, 1, 2, 3$, we recover the normal, overdispersed Poisson, gamma, and inverse Gaussian distributions, respectively. The cumulant $\kappa_p(\theta) = \kappa(\theta)$ for a Tweedie subclass has the form

$$\kappa(\theta) = \begin{cases} e^\theta, & p = 1, \\ -\ln(-\theta), & p = 2, \\ \frac{\alpha-1}{\alpha}\left(\frac{\theta}{\alpha-1}\right)^\alpha, & p \neq 1, 2, \end{cases}$$

where $\alpha = (p-2)/(p-1)$. Furthermore, the canonical parameter belongs to set Θ_p, given by

$$\Theta_p = \begin{cases} [0,\infty), & \text{for } p < 0, \\ \mathbb{R}, & \text{for } p = 0, 1, \\ (-\infty, 0), & \text{for } 1 < p \leq 2, \\ (-\infty, 0], & \text{for } 2 < p < \infty. \end{cases}$$

We denote by $X \sim Tw_p(\theta, \lambda)$ a random variable belonging to the additive Tweedie family.

Model 2 (The Univariate Tweedie Generalization) *For all ages x and forward-times $T = t, t+1, \dots$*

$$p_Q(t+1, T, T+1, x) = p_Q(t, T, T+1, x)^{b(t+1,T,T+1,x)Z(t+1)},$$

where $Z(1), Z(2), \dots$ are independent and identically distributed Tweedie random variables, $Tw_p(\theta, \lambda)$ with $E_Q[Z] = \lambda\kappa_p'(\theta)$ and $Var_Q(Z) = \lambda\kappa_p''(\theta)$. Furthermore, the $b(t+1, T, T+1, x)$ are $\mathcal{M}_t$-measurable bias correction functions given by

$$\begin{aligned} & b(t+1, T, T+1, x) \\ & \quad = \frac{\kappa_p^{-1}(\ln p_Q(t, t, T+1, x)/\lambda + \kappa_p(\theta)) - \kappa_p^{-1}(\ln p_Q(t, t, T, x)/\lambda + \kappa_p(\theta))}{\ln p_Q(t, T, T+1, x)}. \end{aligned}$$

For an arbitrage-free market, we require the martingale property, that is, equation (1) to be satisfied. Consequently, we obtain

$$\begin{aligned} p_Q(t, t, T, x) &= E_Q[p_Q(t+1, t, T, x)|\mathcal{M}_t] \\ &= E_Q\left[\prod_{u=t}^{T-1} p_Q(t+1, u, u+1, x)\middle|\mathcal{M}_t\right] \\ &= E_Q\left[\prod_{u=t}^{T-1} p_Q(t, u, u+1, x)^{b(t+1,u,u+1,x)Z(t+1)}\middle|\mathcal{M}_t\right] \\ &= E_Q\left[\exp\left\{Z(t+1)\sum_{u=t}^{T-1} b(t+1, u, u+1, x)\ln p_Q(t, u, u+1, x)\right\}\middle|\mathcal{M}_t\right] \\ &= M_{Z|\mathcal{M}_t}\left(\sum_{u=t}^{T-1} b(t+1, u, u+1, x)\ln p_Q(t, u, u+1, x)\right), \end{aligned}$$

where

$$M_{Z|\mathcal{M}_t}(y) = E_Q[e^{Zy}|\mathcal{M}_t] = \exp\{\lambda(\kappa_p(\theta + y) - \kappa_p(\theta))\},$$

is the moment generating function of $Z \sim Tw_p(\theta, \lambda)$. This reduces to the Olivier-Smith model if we select $p = 2$, $\lambda = \alpha$, and $\theta = -\alpha$.

The b functions are derived following a recursive procedure as shown in Cairns (2007). First, we investigate the case with a maturity of $t + 1$. From the above, we have

$$p_Q(t, t, t+1, x) = M_{Z|\mathcal{M}_t}(b(t+1, t, t+1, x) \ln p_Q(t, t, t+1, x)),$$

which yields

$$b(t+1, t, t+1, x) = \frac{\kappa_p^{-1}(\ln p_Q(t, t, t+1, x)/\lambda + \kappa_p(\theta)) - \theta}{\ln p_Q(t, t, t+1, x)}.$$

From this it is clear that,

$$\begin{aligned}
\sum_{u=t}^{T} b(t+1, u, u+1, x) \ln p_Q(t, u, u+1, x) &= \kappa_p^{-1}(\ln p_Q(t, t, T+1, x)/\lambda + \kappa_p(\theta)) - \theta \\
\sum_{u=t}^{T-1} b(t+1, u, u+1, x) \ln p_Q(t, u, u+1, x) &= \kappa_p^{-1}(\ln p_Q(t, t, T, x)/\lambda + \kappa_p(\theta)) - \theta.
\end{aligned}$$

Substracting the two equations from one another leaves

$$\begin{aligned}
&b(t+1, T, T+1, x) \\
&\quad = \frac{\kappa_p^{-1}(\ln p_Q(t, t, T+1, x)/\lambda + \kappa_p(\theta)) - \kappa_p^{-1}(\ln p_Q(t, t, T, x)/\lambda + \kappa_p(\theta))}{\ln p_Q(t, T, T+1, x)}.
\end{aligned}$$

Note that $\ln p_Q(t, t, t, x) = 0$, consequently, the expression above holds for the case with maturity $t + 1$. Therefore, this is the general expression and we no longer require the recursive argument.

The variance of the forward rates is given by

$$\begin{aligned}
&Var_Q(p_Q(t+1, T, T+1, x)|\mathcal{M}_t) \\
&\qquad = M_{Z|\mathcal{M}_t}(2b(t+1, T, T+1, x) \ln p_Q(t, T, T+1, x)) \\
&\qquad\quad - M_{Z|\mathcal{M}_t}(b(t+1, T, T+1, x) \ln p_Q(t, T, T+1, x))^2.
\end{aligned}$$

5 Multivariate Tweedie Generalization

We begin by formulating the most general model.

Model 3 (The General Model) *For all ages x and forward-times $T = t, t+1, \ldots$*

$$p_Q(t+1, T, T+1, x) = p_Q(t, T, T+1, x)^{b(t+1,T,T+1,x)Z(t+1,T,T+1,x)},$$

where the Z follow some multivariate Tweedie distribution. The $b(t+1, T, T+1, x)$ are some $\mathcal{M}_t$-measurable bias correction functions.

To preserve the martingale property, we have

$$
\begin{aligned}
p_Q(t,t,T,x) &= E_Q[p_Q(t+1,t,T,x)|\mathcal{M}_t] \\
&= E_Q\left[\exp\left\{\sum_{u=t}^{T-1} Z(t+1,T,T+1,x)b(t+1,u,u+1,x)\ln p_Q(t,u,u+1,x)\right\}\Big|\mathcal{M}_t\right].
\end{aligned}
$$

To proceed further, a multivariate distribution must be specified for the Z. Assuming independence between the components, whilst allowing for the distributions to depend on forward-age is an intermediate step, resulting in forward probabilities given by

$$
p_Q(t,t,T,x) = \prod_{u=t}^{T-1} M_{Z_{x+T}|\mathcal{M}_t}(b(t+1,T,T+1,x)\ln p_Q(t,T,T+1,x)),
$$

with bias correction functions suitably defined and stochastic components generated by distributions $Z_{x+T} \sim Tw_p(\theta_{x+T},\lambda_{x+T})$.

6 Summary

We investigate the Olivier-Smith model and show that, using population mortality data for England and Wales, the model requires a more general framework, with additional emphasis on forward-ages. The gamma distribution provides a reasonable fit, but is a restrive assumption. We improve the model by specifying a more general distribution, namely the Tweedie class of the exponential dispersion family. A careful investigation of potential multivariate generalizations is forthcoming.

Acknowledgements

The authors would like to acknowledge the financial support of ARC Linkage Grant Project LP0883398 Managing Risk with Insurance and Superannuation as Individuals Age with industry partners PwC, APRA and the World Bank as well as the support of the Australian Research Council Centre of Excellence in Population Ageing Research (project number CE110001029).

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