31 May – 03 June 2016
at ISEG – Lisbon School of Economics
and Management

Survival Probability in the Classical Risk Model with a Franchise or a Liability Limit: Exponentially Distributed Claim Sizes

Olena Ragulina

Taras Shevchenko National University of Kyiv, Ukraine

Overview

1. Classical Risk Model

Premium arrivals:

- $c > 0$ is a constant premium intensity.

Claim arrivals:

- **claim sizes** form a sequence $(Y_i)_{i \geq 1}$ of nonnegative i.i.d. random variables with c.d.f. $F(y) = \mathbb{P}[Y_i \leq y]$ and finite expectation μ;
- the **number of claims** on the time interval $[0, t]$ is a Poisson process $(N_t)_{t \geq 0}$ with constant intensity $\lambda > 0$;
- the random variables Y_i, $i \geq 1$, and the process $(N_t)_{t \geq 0}$ are independent;
- the **total claims** on $[0, t]$ equal $\sum_{i=1}^{N_t} Y_i$; we set $\sum_{i=1}^{0} Y_i = 0$ if $N_t = 0$.

1. Classical Risk Model

Let the **net profit condition** hold, i.e. $c > \lambda\mu$.
The insurance company uses the **expected value principle** for premium calculation: $c = \lambda\mu(1+\theta)$, where $\theta > 0$ is a safety loading.

Surplus:

- an insurance company has a nonnegative initial surplus x;
- $X_t(x)$ is a surplus at the time $t \geq 0$ provided that the initial surplus equals x.

The surplus process $\big(X_t(x)\big)_{t\geq 0}$ follows

$$X_t(x) = x + ct - \sum_{i=1}^{N_t} Y_i, \quad t \geq 0. \tag{1}$$

1. Classical Risk Model

- The **infinite-horizon ruin probability** is given by

$$\psi(x) = \mathbb{P}\big[\inf_{t\geq 0} X_t(x) < 0\big].$$

- The **infinite-horizon survival probability** are given by

$$\varphi(x) = 1 - \psi(x).$$

2.1. Classical Risk Model with a Franchise and a Liability Limit

The function $\varphi(x)$ is a solution to the integro-differential equation

$$c\varphi'_+(x) = \lambda\varphi(x) - \lambda \int_0^x \varphi(x-y)\, dF(y). \qquad (2)$$

If the claim sizes are exponentially distributed, then the closed form solution to (2) is

$$\varphi(x) = 1 - \frac{1}{1+\theta} \exp\left(\frac{-\theta x}{\mu(1+\theta)}\right).$$

2.1. Classical Risk Model with a Franchise and a Liability Limit

A **franchise** is a provision in an insurance policy whereby an insurer does not pay unless damage exceeds the franchise amount. It is applied to deter a large number of trivial claims.

A **liability limit** determines the maximum amount that is paid by an insurer. It is used to restrict the insurer's liability to the insured.

- d and L are franchise and liability limit amounts: $0 \le d < L \le +\infty$ (we choose them at the initial time and do not change them later);
- in particular, if $d = 0$, then a franchise is not used; if $L = +\infty$, then a liability limit is not used;
- $Y_i^{(d,L)}$, $i \ge 1$, is an insurance compensation for the ith claim;
- $F^{(d,L)}(y)$ is the c.d.f. of $Y_i^{(d,L)}$.

2.1. Classical Risk Model with a Franchise and a Liability Limit

Normally, a franchise and a liability limit also imply reduction of insurance premiums.
We suppose that the safety loading $\theta > 0$ is constant.
The premium intensity is given by

$$c^{(d,\,L)} = \lambda(1+\theta)\,\mathbb{E}\big[Y_i^{(d,\,L)}\big].$$

Let $X_t^{(d,\,L)}(x)$ be the surplus of the insurance company at time t provided that its initial surplus is x, and the franchise and liability limit amounts are d and L, respectively, then

$$X_t^{(d,\,L)}(x) = x + c^{(d,\,L)}t - \sum_{i=1}^{N_t} Y_i^{(d,\,L)}, \quad t \geq 0. \tag{3}$$

Let $\varphi^{(d,\,L)}(x)$ be the corresponding infinite-horizon survival probability.

2.2. Survival Probability in the Classical Risk Model with a Franchise

Theorem 1

Let the surplus process $\left(X_t^{(d,+\infty)}(x)\right)_{t\geq 0}$ follow (3) under the above assumptions with $0 < d < +\infty$ and $L = +\infty$, and the claim sizes be exponentially distributed with mean μ. Then

$$\varphi^{(d,+\infty)}(x) = \varphi_{n+1}^{(d,+\infty)}(x) \quad \text{for all} \quad x \in [nd,(n+1)d), \quad n \geq 0,$$

where

2.2. Survival Probability in the Classical Risk Model with a Franchise

Theorem 1

$$\varphi_1^{(d,\,+\infty)}(x) = C_{1,\,1}\, e^{x/\gamma},$$

$$\varphi_2^{(d,\,+\infty)}(x) = \left(C_{2,\,1} + A_{2,\,0}\, x\right) e^{x/\gamma} + C_{2,\,2}\, e^{-x/\mu},$$

$$\varphi_{n+1}^{(d,\,+\infty)}(x) = \left(C_{n+1,\,1} + \sum_{i=0}^{n-1} A_{n+1,\,i}\, x^{i+1}\right) e^{x/\gamma} + \left(C_{n+1,\,2} + \sum_{i=0}^{n-2} B_{n+1,\,i}\, x^{i+1}\right) e^{-x/\mu}, \quad n \geq 2.$$

2.2. Survival Probability in the Classical Risk Model with a Franchise

The constants in Theorem 1 are defined in the following way:

$$\gamma = (1+\theta)(\mu + d),$$

$$C_{1,1} = \frac{\theta}{1+\theta},$$

$$A_{2,0} = -\frac{\theta}{(1+\theta)(\gamma+\mu)}\, e^{-d/\gamma},$$

$$C_{2,1} = \frac{\theta}{1+\theta}\left(1 + \frac{\gamma\mu + d(\gamma+\mu)}{(\gamma+\mu)^2}\, e^{-d/\gamma}\right),$$

$$C_{2,2} = -\frac{\theta\gamma\mu}{(1+\theta)(\gamma+\mu)^2}\, e^{d/\mu};$$

the constants $A_{n+1,\,i}$, $0 \le i \le n-1$, are given in a recurrent way by formulas

$$A_{n+1,\,n-1} = -\frac{A_{n,\,n-2}}{n(\gamma+\mu)}\, e^{-d/\gamma}, \quad n \ge 2,$$

$$A_{n+1,j} = -\frac{1}{\gamma+\mu}\left[(j+2)\gamma\mu A_{n+1,\,j+1} + \frac{1}{j+1} \times \left(\sum_{i=j-1}^{n-2} A_{n,\,i}\,(-d)^{i-j+1}\binom{i+1}{j}\right) e^{-d/\gamma}\right],$$

$$1 \le j \le n-2, \quad n \ge 3,$$

$$A_{n+1,\,0} = -\frac{1}{\gamma+\mu}\left[2\gamma\mu A_{n+1,\,1} + \left(C_{n,\,1} + \sum_{i=0}^{n-2} A_{n,\,i}\,(-d)^{i+1}\right) e^{-d/\gamma}\right], n \ge 2;$$

the constants $B_{n+1,\,i}$, $0 \le i \le n-2$, are given in a recurrent way by formulas

$$B_{3,\,0} = \frac{C_{2,\,2}}{\gamma+\mu}\, e^{d/\mu},$$

$$B_{n+1,\,n-2} = \frac{B_{n,\,n-3}}{(n-1)(\gamma+\mu)}\, e^{d/\mu}, \qquad n \ge 3,$$

$$B_{n+1,j} = \frac{1}{\gamma+\mu}\Bigg[(j+2)\gamma\mu B_{n+1,\,j+1} + \frac{1}{j+1} \times \left(\sum_{i=j-1}^{n-3} A_{n,\,i}\,(-d)^{i-j+1}\binom{i+1}{j}\right) e^{d/\mu}\Bigg],$$

$$1 \le j \le n-3, \qquad n \ge 4,$$

$$B_{n+1,\,0} = \frac{1}{\gamma+\mu}\left[2\gamma\mu B_{n+1,\,1} + \left(C_{n,\,2} + \sum_{i=0}^{n-3} B_{n,\,i}\,(-d)^{i+1}\right) e^{d/\mu}\right], n \ge 3;$$

2.2. Survival Probability in the Classical Risk Model with a Franchise

the constants $C_{n+1,\,1}$ and $C_{n+1,\,2}$ are given by formulas

$$C_{n+1,\,1} = C_{n,\,1} + \frac{\gamma\mu(A_{n,\,0} - A_{n+1,\,0})}{\gamma+\mu}$$
$$+ \sum_{i=0}^{n-3}\left(A_{n,\,i} - A_{n+1,\,i} + \frac{(i+2)\gamma\mu(A_{n,\,i+1} - A_{n+1,\,i+1})}{\gamma+\mu}\right)(nd)^{i+1}$$
$$+ \left(A_{n,\,n-2} - A_{n+1,\,n-2} - \frac{n\gamma\mu A_{n+1,\,n-1}}{\gamma+\mu}\right)(nd)^{n-1}$$
$$- A_{n+1,\,n-1}\,(nd)^{n} + \frac{\gamma\mu}{\gamma+\mu}$$
$$\times \left(\sum_{i=0}^{n-3}(i+1)(B_{n,\,i} - B_{n+1,\,i})(nd)^{i} - (n-1)B_{n+1,\,n-2}\,(nd)^{n-2}\right)$$
$$\times \exp\left(-nd\,\frac{\gamma+\mu}{\gamma\mu}\right), \quad n \geq 2,$$

2.2. Survival Probability in the Classical Risk Model with a Franchise

$$C_{3,2} = C_{2,2} + \frac{\gamma\mu B_{3,0}}{\gamma+\mu} - 2dB_{3,0} + \frac{\gamma\mu(A_{3,0} - A_{2,0} + 4dA_{3,1})}{\gamma+\mu} \exp\left(2d\,\frac{\gamma+\mu}{\gamma\mu}\right),$$

2.2. Survival Probability in the Classical Risk Model with a Franchise

$$C_{n+1,2} = C_{n,2} + \frac{\gamma\mu(B_{n+1,0} - B_{n,0})}{\gamma+\mu}$$
$$+ \sum_{i=0}^{n-4}\left(B_{n,i} - B_{n+1,i} + \frac{(i+2)\gamma\mu(B_{n+1,i+1} - B_{n,i+1})}{\gamma+\mu}\right)(nd)^{i+1}$$
$$+ \left(B_{n,n-3} - B_{n+1,n-3} + \frac{(n-1)\gamma\mu B_{n+1,n-2}}{\gamma+\mu}\right)(nd)^{n-2}$$
$$- B_{n+1,n-2}(nd)^{n-1} + \frac{\gamma\mu}{\gamma+\mu}$$
$$\times\left(\sum_{i=0}^{n-2}(i+1)(A_{n+1,i} - A_{n,i})(nd)^{i} + nA_{n+1,n-1}(nd)^{n-1}\right)$$
$$\times \exp\left(nd\,\frac{\gamma+\mu}{\gamma\mu}\right), \quad n \geq 3.$$

2.2. Survival Probability in the Classical Risk Model with a Franchise

Theorem 2

Let the surplus processes $\left(X_t(x)\right)_{t\geq 0}$ and $\left(X_t^{(d,+\infty)}(x)\right)_{t\geq 0}$ follow (1) and (3), respectively, under the above assumptions with $0 < d < +\infty$ and $L = +\infty$. Moreover, let $\varphi(x)$ and $\varphi^{(d,+\infty)}(x)$ be the corresponding survival probabilities, and the claim sizes be exponentially distributed with mean μ.

2.2. Survival Probability in the Classical Risk Model with a Franchise

Theorem 2

(i) If
$$x \in \left[0,\ \min\left\{\frac{\mu(1+\theta)}{\theta}\ln\left(1+\frac{\theta d}{\mu(1+\theta)}\right),\ d\right\}\right],$$
then $\varphi^{(d,+\infty)}(x) < \varphi(x)$ for any $0 < d < +\infty$.

(ii) For
$$d \in \left(0,\ \frac{\mu(1+\theta)\ \ln(1+\theta)}{\theta}\right)$$
and large enough initial surpluses, we have $\varphi^{(d,+\infty)}(x) > \varphi(x)$.

2.3. Survival Probability in the Classical Risk Model with a Liability Limit

Theorem 3

Let the surplus process $\left(X_t^{(0,\,L)}(x)\right)_{t\geq 0}$ follow (3) under the above assumptions with $d = 0$ and $0 < L < +\infty$, and the claim sizes be exponentially distributed with mean μ. Then

$$\varphi^{(0,\,L)}(x) = \varphi_{n+1}^{(0,\,L)}(x) \quad \text{for all} \quad x \in [nL, (n+1)L), \quad n \geq 0,$$

where

$$\varphi_1^{(0,\,L)}(x) = \bar{C}_{1,\,1} + \bar{C}_{1,\,2}\, e^{\bar{\gamma}_1 x/\bar{\gamma}_2},$$

$$\varphi_2^{(0,\,L)}(x) = \bar{C}_{2,\,1} + \left(\bar{C}_{2,\,2} + \bar{A}_{2,\,0}\, x\right) e^{\bar{\gamma}_1 x/\bar{\gamma}_2},$$

$$\varphi_{n+1}^{(0,\,L)}(x) = \bar{C}_{n+1,\,1} + \left(\bar{C}_{n+1,\,2} + \sum_{i=0}^{n-1} \bar{A}_{n+1,\,i}\, x^{i+1}\right) e^{\bar{\gamma}_1 x/\bar{\gamma}_2}, \quad n \geq 2.$$

2.3. Survival Probability in the Classical Risk Model with a Liability Limit

The constants in Theorem 3 are defined in the following way:

$$\bar{\gamma}_1 = 1 - (1+\theta)\left(1 - e^{-L/\mu}\right),$$

$$\bar{\gamma}_2 = \mu(1+\theta)\left(1 - e^{-L/\mu}\right),$$

$$\bar{C}_{1,1} = -\frac{\theta\left(1 - e^{-L/\mu}\right)}{\bar{\gamma}_1},$$

$$\bar{C}_{1,2} = \frac{\theta}{\bar{\gamma}_1(1+\theta)},$$

$$\bar{A}_{2,0} = -\frac{\theta}{\bar{\gamma}_1\bar{\gamma}_2(1+\theta)} \exp\left(-L\left(\frac{1}{\mu} + \frac{\bar{\gamma}_1}{\bar{\gamma}_2}\right)\right),$$

$$\bar{C}_{2,1} = \frac{\theta}{\bar{\gamma}_1(1+\theta)} \left(\left(1 - \frac{1}{\bar{\gamma}_1} \right) e^{-L/\mu} - (1+\theta)\left(1 - e^{-L/\mu}\right) \right),$$

$$\bar{C}_{2,2} = \frac{\theta}{\bar{\gamma}_1(1+\theta)} \left(1 + \left(\frac{1}{\bar{\gamma}_1} + \frac{L}{\bar{\gamma}_2} - 1 \right) \exp\left(-L \left(\frac{1}{\mu} + \frac{\bar{\gamma}_1}{\bar{\gamma}_2} \right) \right) \right).$$

All other constants are given in a recurrent way.

2.3. Survival Probability in the Classical Risk Model with a Liability Limit

Theorem 4

Let the surplus processes $\left(X_t(x)\right)_{t\geq 0}$ and $\left(X_t^{(0,\,L)}(x)\right)_{t\geq 0}$ follow (1) and (3), respectively, under the above assumptions with $d = 0$ and $0 < L < +\infty$. Moreover, let $\varphi(x)$ and $\varphi^{(0,\,L)}(x)$ be the corresponding survival probabilities, and the claim sizes be exponentially distributed with mean μ. Then $\varphi^{(0,\,L)}(x) > \varphi(x)$ for any $0 < L < +\infty$ and for small enough and large enough initial surpluses.

2.4. Survival Probability in the Classical Risk Model with both a Franchise and a Liability Limit

Theorem 5

Let the surplus process $\left(X_t^{(d,\,L)}(x)\right)_{t\geq 0}$ follow (3) under the above assumptions with $0 < d < +\infty$ and $L = md$, where $m \geq 3$ is integer, and the claim sizes be exponentially distributed with mean μ. Then

$$\varphi^{(d,\,L)}(x) = \varphi_{n+1}^{(d,\,L)}(x) \quad \text{for all} \quad x \in [nd, (n+1)d), \quad n \geq 0,$$

where

2.4. Survival Probability in the Classical Risk Model with both a Franchise and a Liability Limit

Theorem 5

$$\varphi_1^{(d,\,L)}(x) = \tilde{C}_{1,\,1}\, e^{x/\tilde{\gamma}},$$

$$\varphi_2^{(d,\,L)}(x) = \left(\tilde{C}_{2,\,1} + \tilde{A}_{2,\,0}\, x\right) e^{x/\tilde{\gamma}} + \tilde{C}_{2,\,2}\, e^{-x/\mu},$$

$$\varphi_{n+1}^{(d,\,L)}(x) = \left(\tilde{C}_{n+1,\,1} + \sum_{i=0}^{n-1} \tilde{A}_{n+1,\,i}\, x^{i+1}\right) e^{x/\tilde{\gamma}} + \left(\tilde{C}_{n+1,\,2} + \sum_{i=0}^{n-2} \tilde{B}_{n+1,\,i}\, x^{i+1}\right) e^{-x/\mu}, \quad n \geq 2.$$

2.4. Survival Probability in the Classical Risk Model with both a Franchise and a Liability Limit

The constants in Theorem 5 are defined in the following way:

$$\tilde{\gamma} = (1+\theta)\big(\mu + d - \mu e^{(d-L)/\mu}\big),$$
$$\tilde{C}_{1,1} = \theta/(1+\theta),$$
$$\tilde{A}_{2,0} = -\frac{\theta}{(1+\theta)(\tilde{\gamma}+\mu)}\, e^{-d/\tilde{\gamma}},$$
$$C_{2,1} = \frac{\theta}{1+\theta}\left(1 + \frac{\tilde{\gamma}\mu + d(\tilde{\gamma}+\mu)}{(\tilde{\gamma}+\mu)^2}\, e^{-d/\tilde{\gamma}}\right),$$
$$C_{2,2} = -\frac{\theta\tilde{\gamma}\mu}{(1+\theta)(\tilde{\gamma}+\mu)^2}\, e^{d/\mu}.$$

All other constants are given in a recurrent way.

2.4. Survival Probability in the Classical Risk Model with both a Franchise and a Liability Limit

Theorem 6

Let the surplus processes $\left(X_t(x)\right)_{t\geq 0}$ and $\left(X_t^{(d,\,L)}(x)\right)_{t\geq 0}$ follow (1) and (3), respectively, under the above assumptions with $0 < d < +\infty$ and $L = md$, where $m \geq 3$ is integer. Moreover, let $\varphi(x)$ and $\varphi^{(d,\,L)}(x)$ be the corresponding survival probabilities, and the claim sizes be exponentially distributed with mean μ.

(i) If $d - \mu e^{(d-L)/\mu} > 0$, then $\varphi^{(d,\,L)}(x) < \varphi(x)$ for

$$x \in \left[0,\, \min\left\{\frac{\mu(1+\theta)}{\theta}\,\ln\left(1 + \frac{\theta\left(d - \mu e^{(d-L)/\mu}\right)}{\mu(1+\theta)}\right),\, d\right\}\right].$$

(ii) If $d - \mu e^{(d-L)/\mu} < 0$, then $\varphi^{(d,\,L)}(x) > \varphi(x)$ for $x \in [0, d]$.

3. Optimal Control by a Franchise Amount in the Classical Risk Model

Additional assumptions:

- the insurance company adjusts the franchise amount d_t at every time $t \geq 0$ on the basis of the information available up to time t, i.e. every admissible strategy $(d_t)_{t\geq 0}$ ((d_t) for brevity) of the franchise amount choice is a predictable process w.r.t. the natural filtration generated by $(N_t)_{t\geq 0}$ and $(Y_i)_{i\geq 1}$
- $0 \leq d_t \leq d_{\max}$, where $d_{\max}$ is the maximum allowed franchise amount such that $0 < F(d_{\max}) < 1$; in particular, if $d_t = 0$, then the franchise is not used at time t
- the safety loading $\theta > 0$ is constant

The premium intensity at time t depends on the franchise amount at this time and it is given by

$$c(d_t) = \lambda(1+\theta)\int_{d_t}^{+\infty} y\, dF(y).$$

3. Optimal Control by a Franchise Amount in the Classical Risk Model

Let $X_t^{(d_t)}(x)$ be the surplus of the insurance company at time t provided its initial surplus is x and the strategy (d_t) is used. Then

$$X_t^{(d_t)}(x) = x + \int_0^t c(d_s)\, ds - \sum_{i=1}^{N_t} Y_i \, \mathbb{I}_{\{Y_i > d_{\tau_i}\}}, \quad t \geq 0. \qquad (4)$$

The **ruin time** under the admissible strategy (d_t) is defined as

$$\tau^{(d_t)}(x) = \inf\{t \geq 0 \colon X_t^{(d_t)}(x) < 0\}.$$

The corresponding **infinite-horizon survival probability** is given by

$$\varphi^{(d_t)}(x) = \mathbb{P}\left[\tau^{(d_t)}(x) = \infty\right].$$

3. Optimal Control by a Franchise Amount in the Classical Risk Model

Our **aim** is to maximize the survival probability over all admissible strategies (d_t), i.e. to find

$$\varphi^*(x) = \sup_{(d_t)} \varphi^{(d_t)}(x),$$

and show that there is an optimal strategy (d_t^*) such that $\varphi^*(x) = \varphi^{(d_t^*)}(x)$ for all $x \geq 0$.

The optimal strategy will be a function of the initial surplus only.

3. Optimal Control by a Franchise Amount in the Classical Risk Model

Optimal control problems:

- **optimal control by investments:** C. Hipp, M. Plum (2000); C. Hipp, M. Plum (2003); C. S. Liu, H. Yang (2004); P. Azcue, N. Muler (2009)
- **optimal control by reinsurance:** H. Schmidli (2001); C. Hipp, M. Vogt (2003)
- **optimal control by investments and reinsurance:** H. Schmidli (2002); M. I. Taksar, C. Markussen (2003); S. D. Promislow, V. R. Young (2005); C. Hipp, M. Taksar (2010)

3. Optimal Control by a Franchise Amount in the Classical Risk Model

Proposition 1

Let the surplus process $\left(X_t^{(d_t)}(x)\right)_{t\geq 0}$ follow (4) under the above assumptions. If $\varphi^*(x)$ is differentiable on $\mathbb{R}_+$, then it satisfies the Hamilton-Jacobi-Bellman equation

$$\sup_{d\in[0,\, d_{\max}]}\Bigg((1+\theta)\int_d^{+\infty} y\, dF(y)\, \left(\varphi^*(x)\right)' + \left(F(d)-1\right)\varphi^*(x) + \int_d^{d\vee x} \varphi^*(x-y)\, dF(y)\Bigg) = 0, \tag{5}$$

which is equivalent to

$$\left(\varphi^*(x)\right)' = \inf_{d\in[0,\, d_{\max}]}\left(\frac{\left(1-F(d)\right)\varphi^*(x) - \int_d^{d\vee x} \varphi^*(x-y)\, dF(y)}{(1+\theta)\int_d^{+\infty} y\, dF(y)}\right). \tag{6}$$

3. Optimal Control by a Franchise Amount in the Classical Risk Model

Remark 1

Note that if there is one solution to (5) or (6), then there are infinitely many solutions to these equations which differ with a multiplicative constant.

3. Optimal Control by a Franchise Amount in the Classical Risk Model

Theorem 7

If the random variables Y_i, $i \geq 1$, have a p.d.f. $f(y)$, then there is a solution $G(x)$ to (6) with $G(0) = \theta/(1+\theta)$, which is nondecreasing and continuously differentiable on $\mathbb{R}_+$, and $\theta/(1+\theta) \leq \lim_{x \to +\infty} G(x) \leq 1$.

The solution $G(x)$ to (6) that satisfies the conditions of Theorem 1 can be found as the limit of the sequence of functions $\big(G_n(x)\big)_{n \geq 0}$ on $\mathbb{R}_+$, where $G_0(x) = \varphi^{(0)}(x)$ is the survival probability provided that $d_t = 0$ for all $t \geq 0$, and

$$G_n'(x) = \inf_{d \in [0,\, d_{\max}]} \left(\frac{\big(1 - F(d)\big)\, G_{n-1}(x) - \int_d^{d \vee x} G_{n-1}(x-y)\, dF(y)}{(1+\theta) \int_d^{+\infty} y\, dF(y)} \right),$$
$$G_n(0) = \theta/(1+\theta), \quad n \geq 1. \tag{7}$$

3. Optimal Control by a Franchise Amount in the Classical Risk Model

Theorem 8

Let the surplus process $\left(X_t^{(d_t)}(x)\right)_{t\geq 0}$ follow (4) and $G(x)$ be the solution to (6) that satisfies the conditions of Theorem 7. Then for any $x \geq 0$ and arbitrary admissible strategy (d_t), we have

$$\varphi^{(d_t)}(x) \leq \frac{G(x)}{\lim_{x\to+\infty} G(x)}, \qquad (8)$$

and equality in (8) is attained under the strategy $(d_t^*) = \left(d_t^*\left(X_{t_-}^{(d_t^*)}(x)\right)\right)$, where $\left(d_t^*(x)\right)$ minimizes the right-hand side of (6), i.e.

$$\varphi^*(x) = \varphi^{(d_t^*)}(x) = \frac{G(x)}{\lim_{x\to+\infty} G(x)}.$$

3. Optimal Control by a Franchise Amount in the Classical Risk Model

Remark 2

In Theorem 8 we used any solution to (6) that satisfies the conditions of Theorem 7. However, Theorem 8 also implies uniqueness of such a solution. The corresponding strategy (d_t^*) may not be unique in the general case.

Theorem 9

Let the surplus process $\left(X_t^{(d_t)}(x)\right)_{t\geq 0}$ follow (4), the claim sizes be exponentially distributed with mean μ, and $d_{\max} = \mu$. Then the strategy (d_t) with $d_t = 0$ for all $t \geq 0$ is not optimal.

Remark 3

Theorem 18 implies that we can always increase the survival probability adjusting the franchise amount if the claim sizes are exponentially distributed.

3. Optimal Control by a Franchise Amount in the Classical Risk Model

Example 1

If the claim sizes are exponentially distributed with mean $\mu = 10$, $d_{\max} = \mu$, and $\theta = 0.1$, then

$$\varphi^{(0)}(x) \approx 1 - 0.9090909\, e^{-x/110}, \quad x \geq 0,$$

$$\varphi^{*}(x) \approx \begin{cases} 0.111048767\, e^{x/22} & \text{if} \quad x \leq 8.93258, \\ 1 - 0.90382792\, e^{-x/110} & \text{if} \quad x > 8.93258, \end{cases}$$

$$d_t^{*}(x) = \begin{cases} 10 & \text{if} \quad x \leq 8.93258, \\ 0 & \text{if} \quad x > 8.93258. \end{cases}$$

References

C. Hipp, M. Plum (2000)
Optimal investment for insurers
Insurance: Mathematics and Economics 27(2), 215–228.

C. Hipp, M. Plum (2003)
Optimal investment for investors with state dependent income, and for insurers
Finance and Stochastics 7(3), 299–321.

C. S. Liu, H. Yang (2004)
Optimal investment for an insurer to minimize its probability of ruin
North American Actuarial Journal 8(2), 11–31.

P. Azcue, N. Muler (2009)
Optimal investment strategy to minimize the ruin probability of an insurance company under borrowing constrains
Insurance: Mathematics and Economics 44(1), 26–34.

References

H. Schmidli (2001)
Optimal proportional reinsurance policies in a dynamic setting
Scandinavian Actuarial Journal 2001(1), 55–68.

C. Hipp, M. Vogt (2003)
Optimal dynamic XL reinsurance
ASTIN Bulletin 33(2), 193–207.

H. Schmidli (2002)
On minimizing the ruin probability by investment and reinsurance
The Annals of Applied Probability 12(3), 890–907.

M. I. Taksar, C. Markussen (2003)
Optimal dynamic reinsurance policies for large insurance portfolios
Finance and Stochastics 7(1), 97–121.

References

S. D. Promislow, V. R. Young (2005)
Minimizing the probability of ruin when claims follow Brownian motion with drift
North American Actuarial Journal 9(3), 109–128.

C. Hipp, M. Taksar (2010)
Optimal non-proportional reinsurance control
Insurance: Mathematics and Economics 47(2), 246–254.

O. Ragulina (2014)
Maximization of the survival probability by franchise and deductible amounts in the classical risk model
Springer Optimization and Its Applications 90, 287–300.

Thank you very much for your attention!