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The application of copulas to the modelling of the marriage reverse annuity contract[1]

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OUTLINE OF THE TALK

1. PURPOSE OF PRESENTATION

2. DESCRIPTION APPLIED MODEL

- probabilistic structure
- joint distribution of spouses lifetime
- matrix representation of benefits

3. NUMERICAL EXAMPLES AND CONCLUSIONS

- copula
- rate of interest
- results

4. LITERATURE

1. PURPOSE OF PRESENTATION

The aim of this presentation is to model the probabilistic structure and cash flows arising from marriage reverse annuity contracts. We consider two survival statuses of the marriage reverse annuity contract:

- the joint-life status (**JLS**) - the benefit is paid only until the death of one spouses,
- the last surviving status (**LSS**) - the benefit is paid until the death of the other spouse.

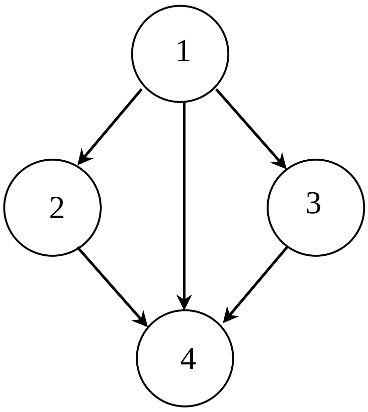
In contrast to the classical approach which assumes that future lifetimes of the wife and the husband are independent, dependence of lifetimes between the spouses is assumed. The structure of the dependence of the length of the spouses' lives is modelled by copulas.

We analyse the impact of degree of the dependence between the future lifetimes of married partners on actuarial values. We calculate the benefit of the marriage reverse annuity contract based on matrix formulas for the first moments of cash flows arising from multistate insurance contracts. We compare the results obtained for dependence and independence of future lifetimes of spouses.

2. DESCRIPTION APPLIED MODEL

probabilistic structure

We investigate the following Markov model based on the stationary Markov chain with $N = 4$ states, which is presented in Figure 1.



State space:
1 - both spouses are alive
2 - husband is dead
3 - wife is dead
4 - both spouses are dead

Figure 1. The space of states of model
Source: Own elaboration

Let us denote:

- $S = \{1, 2, 3, 4\}$ – *a state space,*
- $T = \{(1, 2), (1,3), (1,4), (2,4), (3,4)\}$ – *a set of direct transitions* between states of the state space,
- (S, T) – *a multiple state model* (it describes all possible insured risk events up to the end of contract).

Moreover, $\{X(k), k \geq 0\}$ is a discrete-time process which determines the state of spouses' life (x, y). $X(k) = i$, where $i \in S$, means that spouses are in state *i* at the moment *k*, $k = 0, 1, 2, 3, \ldots$.

2. DESCRIPTION APPLIED MODEL

probabilistic structure

In order to determine the probabilistic structure of model we assume that $\{X(k), k \geq 0\}$ is nonhomogeneous Markov chain (see, e.g. [Wolthuis, Van Hoeck 1986]) and we introduce the following matrix **D** (cf. [Dębicka 2013])

$$\mathbf{D} = \begin{pmatrix} \mathbf{P}(0) \\ \mathbf{P}(1) \\ \vdots \\ \mathbf{P}(n) \end{pmatrix} \in R^{(n+1)\times 4},$$

where

$$\mathbf{P}(0) = (1,0,0,0)^T$$

$$\mathbf{P}(k) = \left(p_1(k), p_2(k), p_3(k), p_4(k)\right)^T, \qquad k = 0,1,2,3\ldots, \quad i \in S = \{1,2,3,4\}$$

$$p_i(k) = \mathrm{P}\left(X(k) = i\right).$$

2. DESCRIPTION APPLIED MODEL

joint distribution of spouses lifetime

T_x^M - future lifetime of x-year-old man, $T_x^M \in [0, \omega_x^M]$,

$$\omega_x^M = \text{limit age of the man - man's age at entry } x,$$

T_y^W - future lifetime of y-year-old woman, $T_y^W \in [0, \omega_y^W]$,

$$\omega_y^W = \text{limit age of the woman - woman's age at entry } y.$$

In order to derive the probability $p_i(k) = \mathrm{P}(X(k) = i)$, for i = 1, 2, 3, 4, we need the joint distribution of lifetimes (T_x^M, T_y^W).

We assume that we know:

1) the cumulative distribution function $F(w, z)$ of pair $(T_{x_0}^M, T_{y_0}^W)$, where x_0 and y_0 be a reference age for men and women, such that $x = x_0 + s$, $y = y_0 + t$ and $s, t \geq 0$.
2) the copula $C(u, v)$, a link between the joint and marginal cumulative distribution function, connected with this joint distribution, i.e.

$$F(w, z) = C\left(F^M(w), F^W(z)\right),$$

where $F^M(w) = P\left(T_{x_0}^M \leq w\right)$ and $F^M(z) = P\left(T_{y_0}^W \leq z\right)$ are the marginal distributions of the lifetimes $T_{x_0}^M$ and $T_{y_0}^W$.

To the actuarial calculation we need the following survival function

$$S(w, z) = P(T_{x_0}^M > w, T_{y_0}^W > z).$$

The joint survival function $S(w, z)$ can be determined by the use of the survival copula $C^*(w, z)$ as follows

$$S(w, z) = C^*\left(S^M(w), S^W(z)\right),$$

where $S^M(w) = P\left(T_{x_0}^M > w\right)$, $S^W(z) = P\left(T_{y_0}^W > z\right)$ and (cf. [Nelson 1999])

$$C^*(w, z) = w + z - 1 + C(1 - w, 1 - z).$$

2. DESCRIPTION APPLIED MODEL

joint distribution of spouses lifetime

Theorem

Assume that the copula C^* is the link between the joint and marginal survival function for *x*-years old men and *y*-years old women. In addition, $\{X(t), t \in \mathrm{T}\}$ is a nonhomogeneous Markov chain described evolution of the insured risk in the multistate model for the marriage reverse annuity contract with $(S,T) = (\{1,2,3,4\}, \{(1,2),(1,3),(1,4),(2,4),(3,4)\})$. Then the elements of the matrices $\mathbf{P}(k) = (p_1(k), p_2(k), p_3(k), p_4(k))^T$ have the following form:

$$p_1(k) = \mathrm{P}(X(k)=1) = \frac{C^*\left(S^M(x+k), S^W(y+k)\right)}{C^*\left(S^M(x), S^W(y)\right)},$$

$$p_2(k) = \mathrm{P}(X(k)=2) = \frac{C^*\left(S^M(x), S^W(y+k)\right) - C^*\left(S^M(x+k), S^W(y+k)\right)}{C^*\left(S^M(x), S^W(y)\right)},$$

$$p_3(k) = \mathrm{P}(X(k)=3) = \frac{C^*\left(S^M(x+k), S^W(y)\right) - C^*\left(S^M(x+k), S^W(y+k)\right)}{C^*\left(S^M(x), S^W(y)\right)}$$

$$p_4(k) = \mathrm{P}(X(k)=4) =$$

$$= \frac{C^*\left(S^M(x), S^W(y)\right) - C^*\left(S^M(x), S^W(y+k)\right) - C^*\left(S^M(x+k), S^W(y)\right) + C^*\left(S^M(x+), S^W(y+k)\right)}{C^*\left(S^M(x), S^W(y)\right)}.$$

2. DESCRIPTION APPLIED MODEL

matrix representation of benefits

The benefit is calculated on the basis of the value of real estate W and the percentage α of value W $(\alpha \in (0\%, 50\%])$.

We have to introduced some matrices.

$$\mathbf{S}^* = (1,1,1,1)^T \in R^4, \qquad \mathbf{I}_{k+1} = \Big(0,0,\ldots,\underbrace{1}_{k+1},\ldots,0\Big)^T \in R^{n+1}, \text{ for each } k = 0,1,\ldots,n,$$

$$\mathbf{J}_1 = (1,0,0,0)^T \in R^4, \qquad \mathbf{J}_4 = (0,0,0,1)^T \in R^4.$$

Moreover the vector of discounting factors is denoted by

$$\mathbf{V} = (1, v, \ldots, v^n)^T \in R^{n+1},$$

where v^k for $k = 1, 2, \ldots, n$ is given by the following formulae

$$v^k = \exp(-k \cdot R_{0,k}),$$

and $R_{0,k}$ is the short term rate (it is the function depending on time k).

2. DESCRIPTION APPLIED MODEL

matrix representation of benefits

The benefit $\ddot{b}$ of whole life marriage reverse annuity contract, which is paid immediately, is calculated by the following formulae (cf. [Dębicka, Marciniuk 2014]):

- **the joint-life status (JLS)**

$$\ddot{b} = \frac{\alpha W}{1 + \mathbf{V}^T \cdot \left(\sum_{i=1}^{m-1} \mathbf{I}_{k+1} \cdot \mathbf{I}_{k+1}^T \right) \cdot \mathbf{D} \cdot \left(\mathbf{S}^* - \mathbf{J}_4 \right)},$$

where $m = \max\left\{ \omega_x^M, \omega_y^W \right\}$.

- **the status of last surviving (LSS)**

$$\ddot{b} = \frac{\alpha W}{\mathbf{V}^T \cdot \left(\mathbf{I} - \mathbf{I}_{n+1} \cdot \mathbf{I}_{n+1}^T \right) \cdot \mathbf{D} \cdot \mathbf{J}_1},$$

where **I** is the identity matrix, which size is $(n+1) \times (n+1)$.

3. NUMERICAL EXAMPLES AND CONLCUSIONS

copula

We have to set of data:

- from Main Statistical Office – for Lower Silesia from year 2011,
- from Wroclaw cemeteries – from year 2011.

We investigate the Markov model based on the stationary Markov chain (see [Wolthuis, Van Hoeck 1986], [Norberg 1989], [Denuit et al. 2001]). This model let us establish the joint distribution of the lifetimes of spouses, i.e. $\left(T_x^M, T_y^W\right)$, for fixed ages *x* and *y*. Heilpern in [Heilpern 15] derived the joint cumulative distribution function $F(w,z) = C\left(P\left(T_{x_0}^M \le w\right), P\left(T_{y_0}^W \le z\right)\right)$ for the reference age $x_0 = y_0 = 60$ and the Kendall's tau coefficient of correlation $\tau = 0.076$ between random variables $T_{x_0}^M$ and $T_{y_0}^W$. Heilpern also designated the following Gumbel copula *C*(*u*, *v*), which described the dependence structure of lifetimes $T_{x_0}^M$ and $T_{y_0}^W$

$$C(u,v) = \exp\left(-\left((-\ln u)^\alpha + (-\ln v)^\alpha\right)^{\frac{1}{\alpha}}\right) \qquad \text{for} \qquad \alpha = 1.0786 .$$

For cemeteries data the Kendall's tau coefficient of correlation $\tau = 0.156$ between random variables $T_{x_0}^M$ and $T_{y_0}^W$ for $x_0 = y_0 = 0$. The best copula (from chosen function) for this data is the following AMH copula

$$C(u,v) = \frac{uv}{1-\alpha(1-u)(1-v)} \qquad \text{for } \alpha = 0.5867 .$$

The marginal survival function $S^M(w)$ and $S^W(z)$ can be calculated by using the values of $l_{x_0}^M$ and $l_{y_0}^W$ from Life Tables as follows

$$S^M(w) = P(T_{x_0}^M > w) = P\left(T_0^M > x_0 + w \mid T_0^M > x_0\right) = \frac{l_{x_0+w}^M}{l_{x_0}^M},$$

$$S^W(z) = P(T_{y_0}^W > z) = P\left(T_0^W > y_0 + z \mid T_0^W > y_0\right) = \frac{l_{y_0+z}^W}{l_{y_0}^W}.$$

The values $l_{x_0+w}^M$, $l_{x_0}^M$, $l_{y_0+z}^W$, $l_{y_0}^W$ are taken from Lower Silesia Life Table for man and women from year 2011.

3. NUMERICAL EXAMPLES AND CONLCUSIONS

interest rate

The parameters of function $R_{0,k}$ are estimated by using the least-squares method on the basis of real Polish market data, related to the yield to maturity on fixed interest bonds from 03.03.2015[2] (cf. [Dębicka, Marciniuk]). The estimation was made by the use of the Solver in Microsoft Excel. We applied three models of short term rate, i.e. Svensson model (sv), Nelson-Siegel model (ns) and Bliss model (bl). In Figure 2 the data and all function of interest rate models are presented.

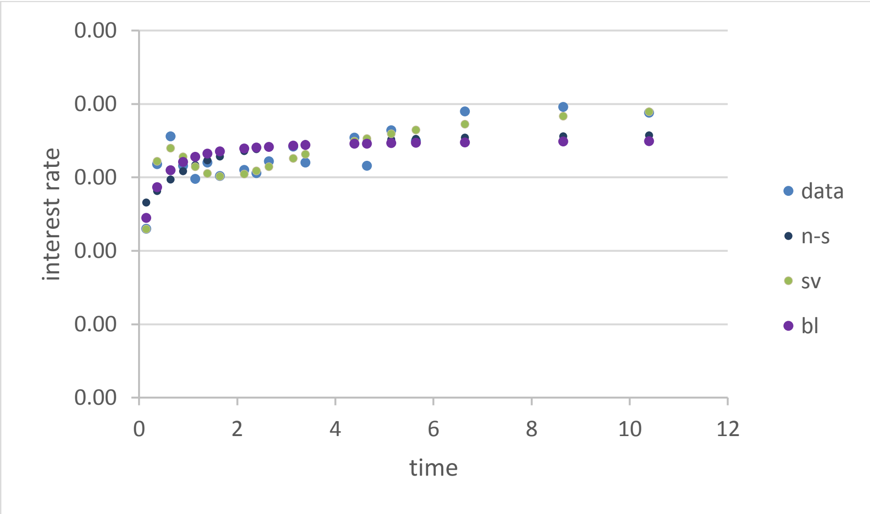


Figure 2. The model of short term rate
Source: Own elaboration

The best fitted model is Svensson model, therefore it is used to the actuarial calculation. The function $R_{0,k}$ has the following form

$$R_{0,k} = \beta_0 + \beta_1 \frac{\tau_1}{k}\left(1 - e^{-\frac{k}{\tau_1}}\right) + \beta_2\left(\frac{\tau_1}{k}\left(1 - e^{-\frac{k}{\tau_1}}\right) - e^{-\frac{k}{\tau_1}}\right) + \beta_3\left(\frac{\tau_2}{k}\left(1 - e^{-\frac{k}{\tau_2}}\right) - e^{-\frac{k}{\tau_2}}\right),$$

where

$$\beta_0 = 0.02096, \quad \beta_1 = -0.01684, \quad \beta_2 = 0.05844,$$
$$\beta_3 = -0.05069, \quad \tau_1 = 0.33388, \quad \tau_2 = 0.57974.$$

[2] Sources: http://bossa.pl/notowania/stopy/rentownosc_obligacji/

3. NUMERICAL EXAMPLES AND CONLCUSIONS

results - LSS

The numerical calculations presented in this section are made by the use of own programs, written in MATLAB. We assume for simplicity that the value of the property W is equally to 100000 PLN and $\alpha = 50\,\%$. If we want to calculate the benefit for another real values W_1 and α_1, the benefits should be multiplied by $\frac{W_1 \alpha_1}{W \alpha}$.

The value of benefit in the case of LSS for woman' age depending on the man's age is presented in Figure 3 for $x \in \{60, 65, 70, 75, 80, 85\}$ and $y \in \{60, 70, 80\}$.

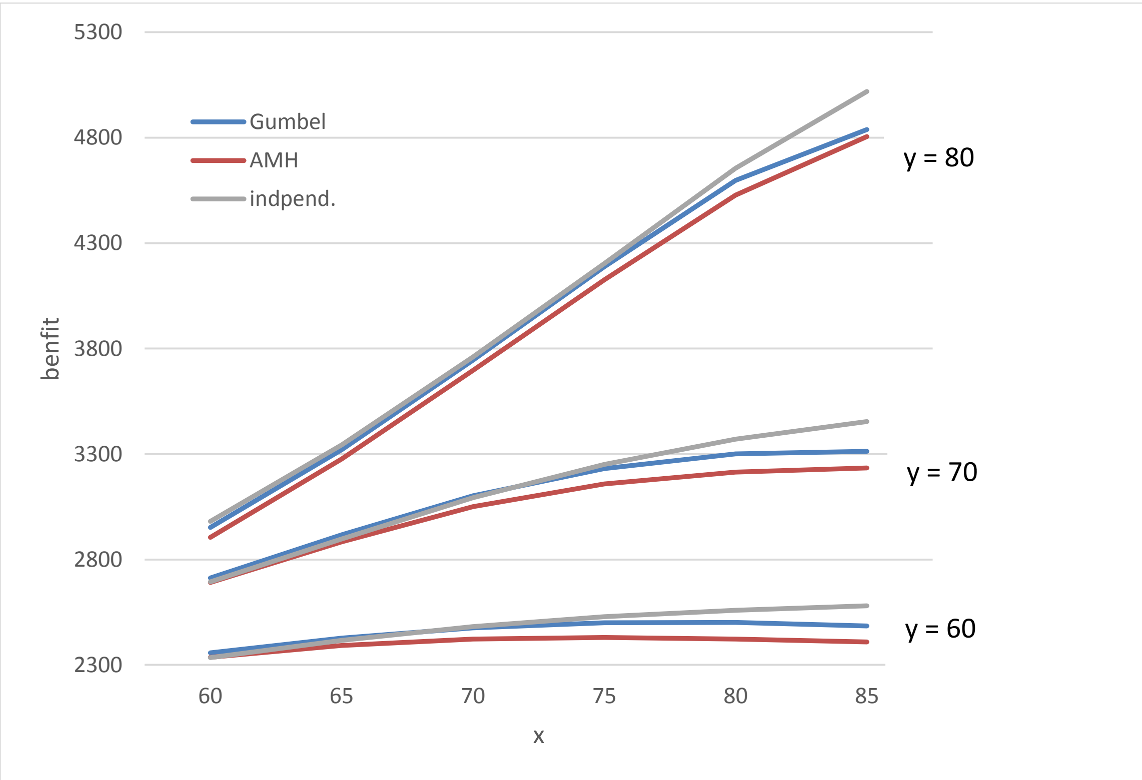


Figure 3: The benefit for woman in the case LSS.
Source: Own elaboration.

3. NUMERICAL EXAMPLES AND CONLCUSIONS

results - LSS

The value of benefit for man' age depending on the woman's age is presented in Figure 4 for $y \in \{60, 65, 70, 75, 80, 85\}$ and $x \in \{60, 70, 80\}$.

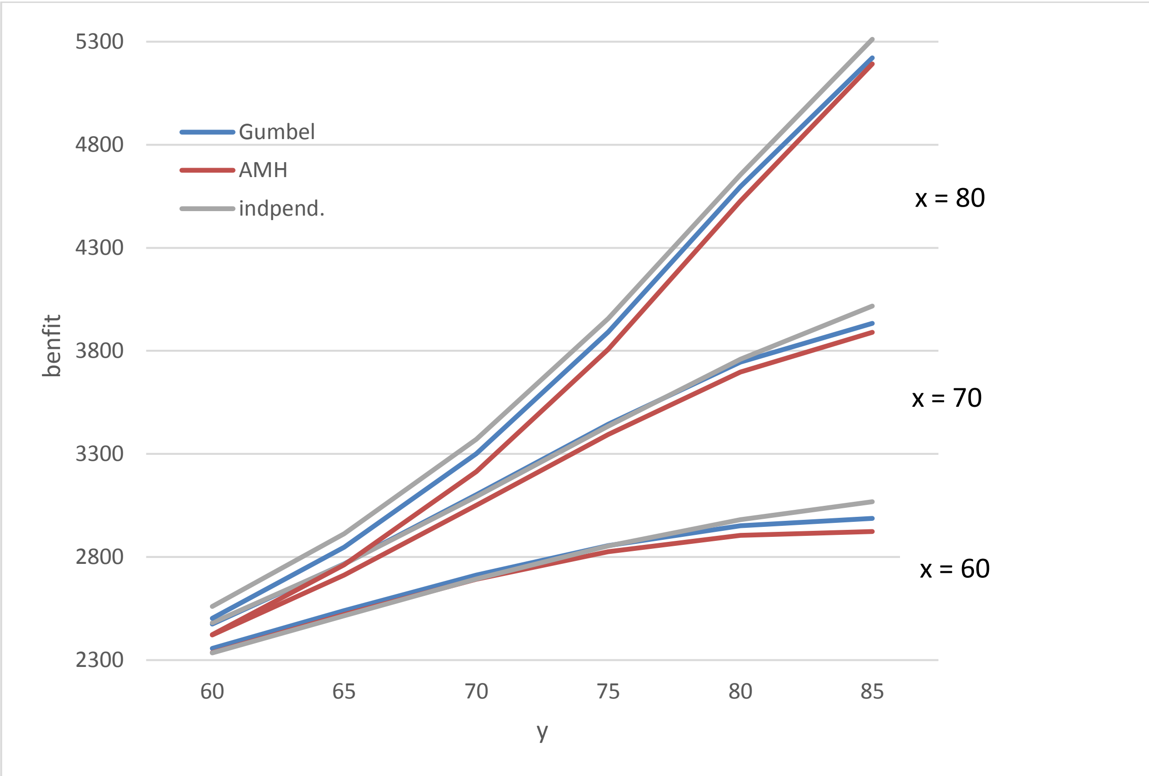


Figure 4: The benefit for man in the case LSS.
Source: Own elaboration.

3. NUMERICAL EXAMPLES AND CONLCUSIONS

results - LSS

- The benefit increases with the age of the spouses.
- Its value grows faster for the elderly people.
- A greater impact on the amount of benefit has the woman's age.
- For independent future lifetimes of spouse, the benefit is almost always higher than in other cases.
- Only for younger people this benefit is some lower than in the case of Gumbel function (about 1% for the 60-year-old men and about 0.5% for 65-year-old men and 60-year-old women).
- The benefit for AMH function is the lowest. Although for spouses at age over 85 years, this benefit is higher than benefit for the Gumbel function, what we can see in Figure 5.

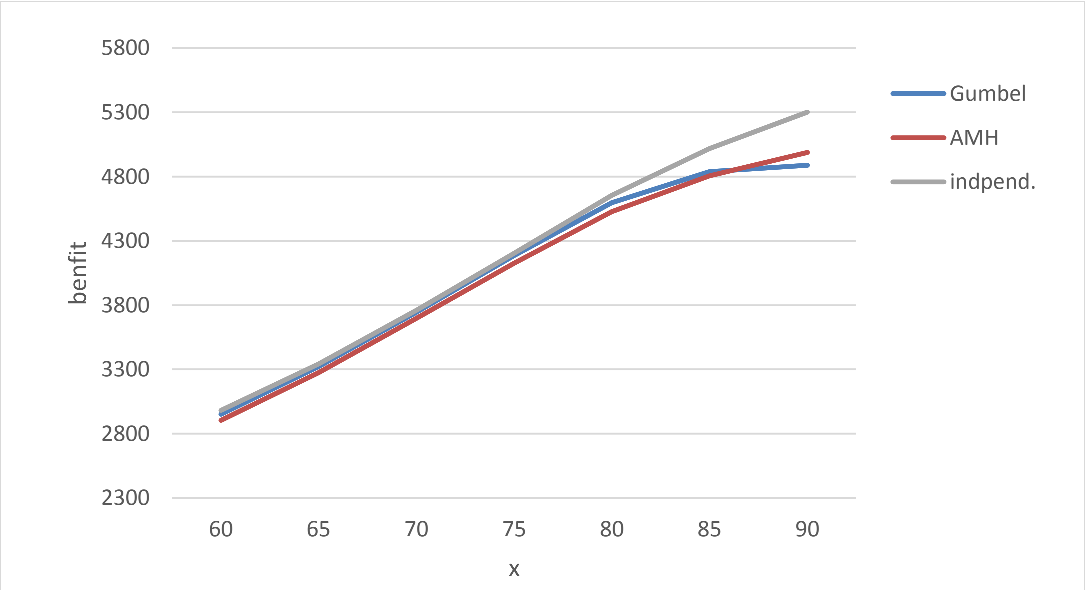


Figure 5: The benefit for 80-years-old woman at age in the case LSS.
Source: Own elaboration.

3. NUMERICAL EXAMPLES AND CONLCUSIONS

results - JLS

Let us consider the **Joint-Life Status**.

- In this case the benefit also increases with the age of the spouses.
- Similarly its value grows faster for the elderly people, however a greater impact on the amount of benefit has the man's age, what we can see in Figure 6 (only for man – for the woman is similar). The benefit is the convex function otherwise than in the case of LSS.

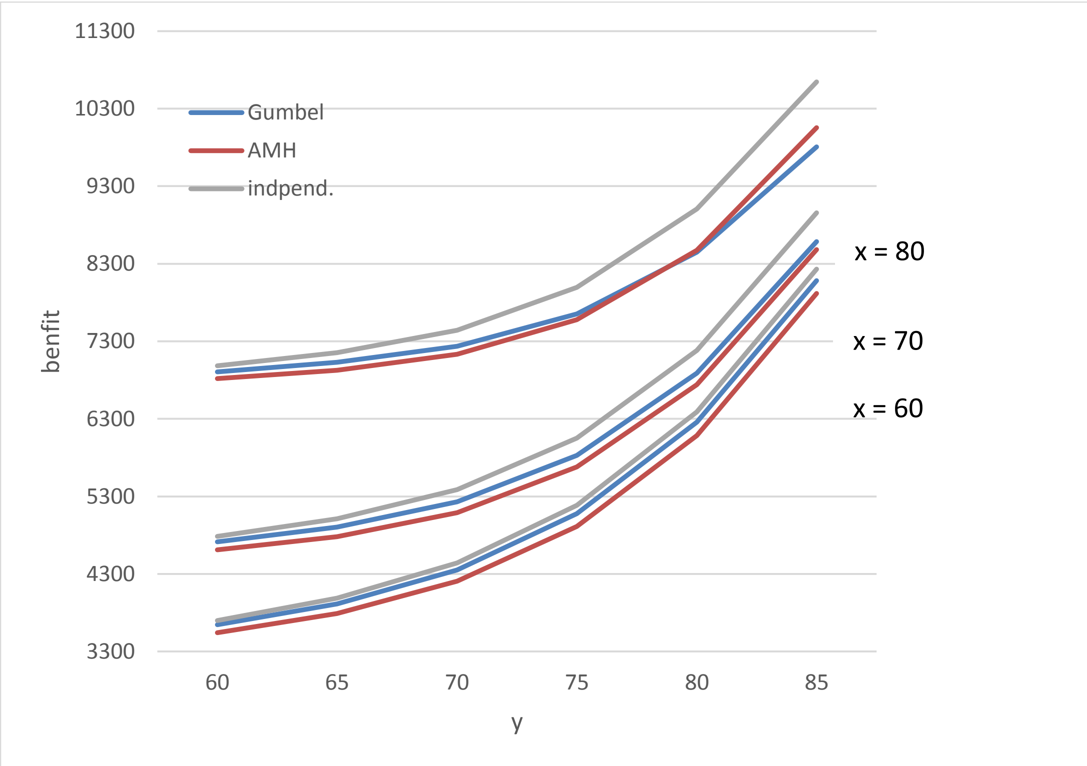


Figure 6: The benefit for man in the case JLS for *x* = {60,70,80}
Source: Own elaboration.

3. NUMERICAL EXAMPLES AND CONLCUSIONS

results - JLS

- The highest benefit is for independent future lifetimes of spouses.
- The benefit for AMH function is the lowest for younger people. For spouses at age over 80 years, this benefit is higher than for the Gumbel function and the relative increases are up to 12%.
- The relative increases of benefits for independent future lifetimes of spouses and for Gumbel function generally grow with the rise of the age of spouses (but for elderly people), what we can see in Figure 7. For younger men $(x \leq 70)$ the range of relative increase is from 2% to 4%, for older men are higher even about 14%.

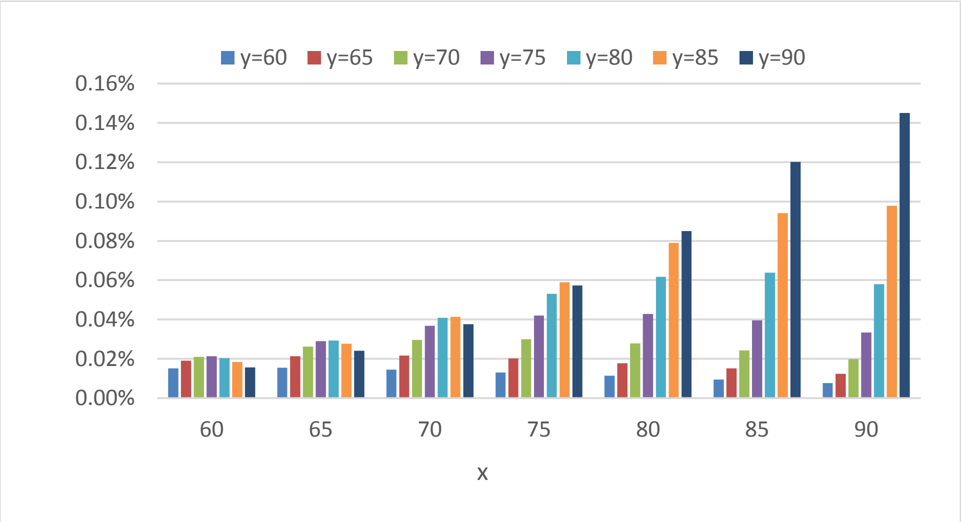


Figure 7: The relative increases of benefits for independent future lifetimes of spouses and for Gumbel function.
Source: Own elaboration.

3. NUMERICAL EXAMPLES AND CONLCUSIONS

results - JLS

- The relative increases of benefits for independent future lifetimes of spouses and for AMH function are more regular. First the relative increases grow and then increase. The range of relative increases is from 1.5% to about 6% (see the Figure 8).

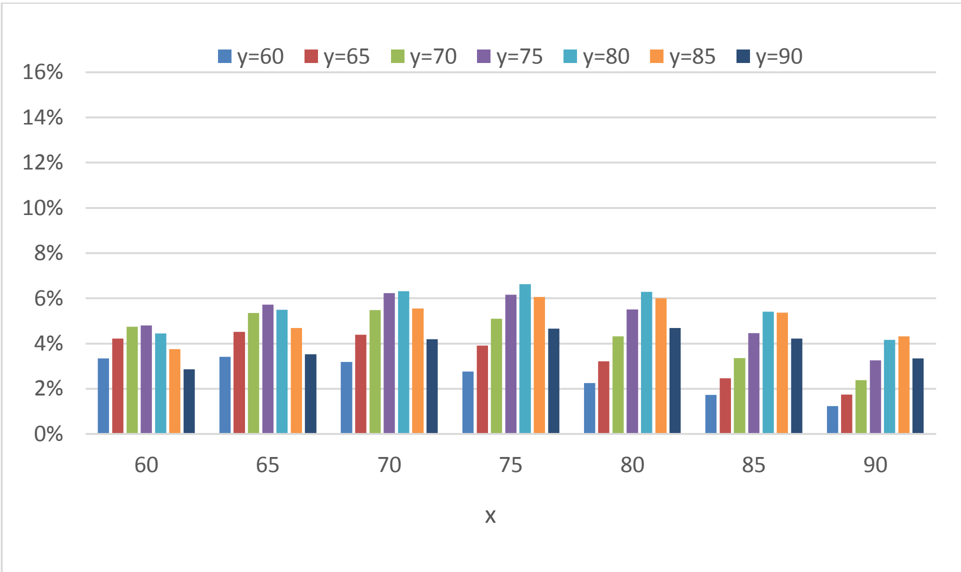


Figure 8: The relative increases of benefits for independent future lifetimes of spouses and for AMH function.
Source: Own elaboration.

3. NUMERICAL EXAMPLES AND CONLCUSIONS

results

- The conclusion is that the benefit is higher for independent future lifetimes of spouses than for copulas functions in the case of JLS, what is more profitable for customers.
- In the case of LSS the benefits for younger people is not higher in the case of independence. However for elderly people the result is similar.
- The benefit is much higher for JLS than for LSS, what is presented for example for $y = 70$ and $x \in \{60, 65, 70, 75, 80, 85, 90\}$ in Figure 9.

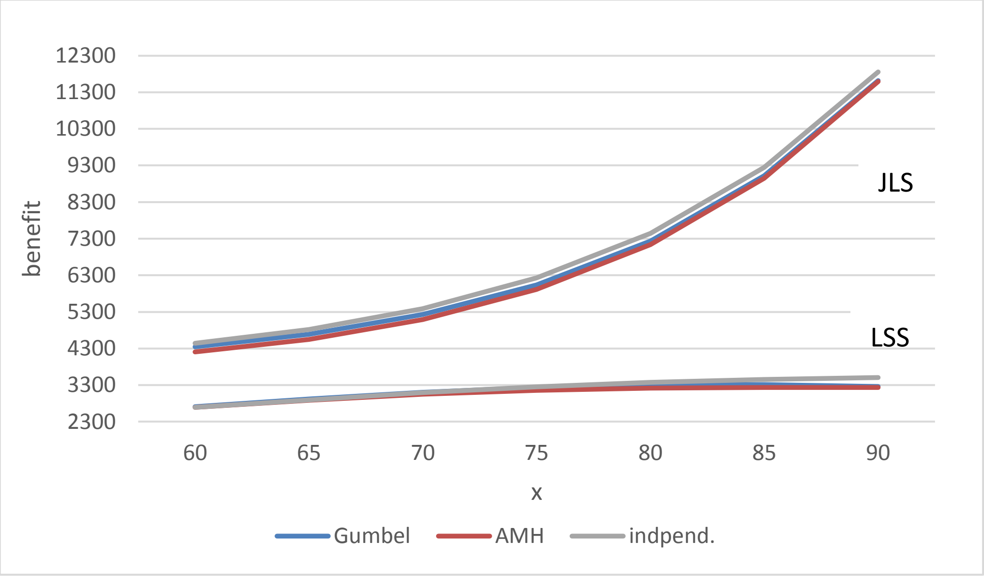


Figure 9: The benefit for LSS and JLS for $y = 70$
Source: Own elaboration.

4. LITERATURE

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