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and Management

Basics of Capital Allocation Principles as Compositional Data

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Forewords

Joint work with Montserrat Guillén and Miguel Santolino from the research group Riskcenter of the University of Barcelona.

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Outline

Compositional
data

Mathematical
tools for
working with
compositions

Capital
allocation
problems and
principles

Outline

Relationships frequently used in geology, for instance

Compositional data

Mathematical tools for working with compositions

Capital allocation problems and principles

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Mathematical tools for working with compositions

Capital allocation problems and principles

Our aim: looking for a different perspective for analysing actuarial capital allocation

Compositions

- Idea

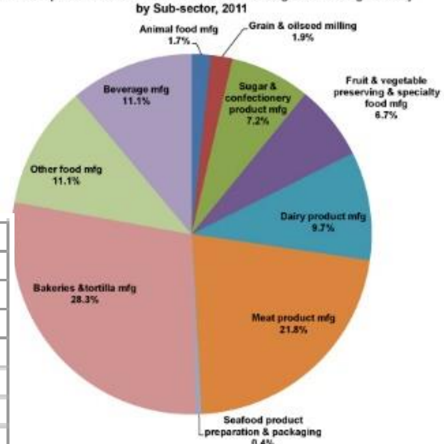
Quantitative descriptions of the components of a whole, where relative information is more relevant than absolute values.

For instance: $\vec{x} = (25\%, 30\%, 15\%, 10\%, 20\%)$

Ingredient	% of food	Ingredient	% of food
Corn	51.240	Vitamin E	0.200
Poultry By-Product Meal	18.210	Vitamin Premix	0.126
Soybean Meal	15.000	Taurine	0.100
Chicken Fat	8.953	Mineral Mix	0.040
Pal Enhancer	2.000	Manganese Sulfate	0.023
Soybean Oil	1.000	L-Tryptophan	0.017
Fish Oil	1.000		
DL-Methionine	0.894		
Non-Iodized Salt	0.642		
Choline Chloride	0.285		
L-Carnitine	0.270		

ELEMENT (symbol)	CRUST		HYDROSPHERE	TROPOSPHERE
	Percent by mass	Percent by volume	Percent by volume	Percent by volume
Oxygen (O)	46.10	94.04	33.0	21.0
Silicon (Si)	28.20	0.88		
Aluminum (Al)	8.23	0.48		
Iron (Fe)	5.63	0.49		
Calcium (Ca)	4.15	1.18		
Sodium (Na)	2.36	1.11		
Magnesium (Mg)	2.33	0.33		
Potassium (K)	2.09	1.42		
Nitrogen (N)				78.0
Hydrogen (H)			66.0	
Other	0.91	0.07	1.0	1.0

Figure 2: Composition of the Toronto Food and Beverage Processing Industry by Sub-sector, 2011



Compositions to ease descriptions

- Soil descriptions

A vector space structure for the simplex

- Simplex

$$S^n = \left\{ \vec{z} \in \mathbb{R}^n \mid z_i \geq 0, i = 1, \dots, n, \sum_{i=1}^{n} z_i = 1 \right\}$$

- Perturbation (addition)

$$\vec{x} \oplus \vec{y} = \left(\frac{x_1 \cdot y_1}{\sum_{i=1}^{n} x_i \cdot y_i}, \frac{x_2 \cdot y_2}{\sum_{i=1}^{n} x_i \cdot y_i}, \dots, \frac{x_n \cdot y_n}{\sum_{i=1}^{n} x_i \cdot y_i} \right)$$

- Powering (scalar product)

$$\lambda \odot \vec{x} = \left(\frac{x_1^{\lambda}}{\sum_{i=1}^{n} x_i^{\lambda}}, \frac{x_2^{\lambda}}{\sum_{i=1}^{n} x_i^{\lambda}}, \dots, \frac{x_n^{\lambda}}{\sum_{i=1}^{n} x_i^{\lambda}} \right)$$

We follow the notation introduced in **Aitchison and Egozcue (2005)**.

- Neutral element

$$\vec{0} = \left(\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n} \right)$$

A vector space structure for the simplex

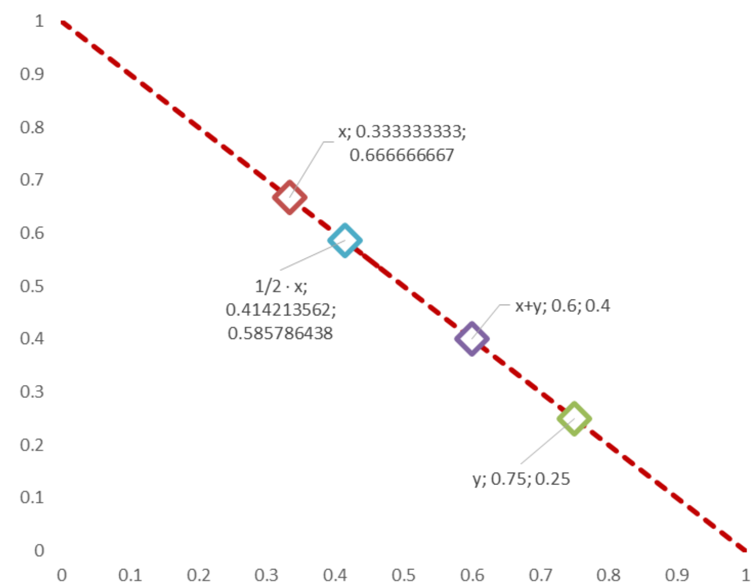
- For example, consider in S^2

$$\vec{x} = \left(\frac{1}{3}, \frac{2}{3}\right), \vec{y} = \left(\frac{3}{4}, \frac{1}{4}\right) \text{ and } \lambda = \frac{1}{2}$$

Then

$$\vec{x} \oplus \vec{y} = \left(\frac{1}{4} : \frac{5}{12}, \frac{1}{6} : \frac{5}{12}\right) = \left(\frac{3}{5}, \frac{2}{5}\right)$$

$$\lambda \odot \vec{x} = \left(\frac{\sqrt{1/3}}{\sqrt{1/3} + \sqrt{2/3}}, \frac{\sqrt{2/3}}{\sqrt{1/3} + \sqrt{2/3}}\right)$$



- Neutral element

$\vec{0} = \left(\frac{1}{2}, \frac{1}{2}\right)$. Note that $\vec{x} \oplus \vec{0} = \left(\frac{1}{6} : \frac{1}{2}, \frac{1}{3} : \frac{1}{2}\right) = \left(\frac{1}{3}, \frac{2}{3}\right) = \vec{x}$

A metric space structure for the simplex

- In a vector space it is possible to define distances in order to transform it in a metric space.

An example of distance defined in S^n is the simplicial distance (**Aitchison, 1983**)

$$\Delta(\vec{x},\vec{y}) = \sqrt{\sum_{i=1}^{n}\left[ln\left(\frac{x_i}{GM(\vec{x})}\right) - ln\left(\frac{y_i}{GM(\vec{y})}\right)\right]^2} \quad \text{where} \quad GM(\vec{z}) = \left(\prod_{i=1}^{n} z_i\right)^{1/n}$$

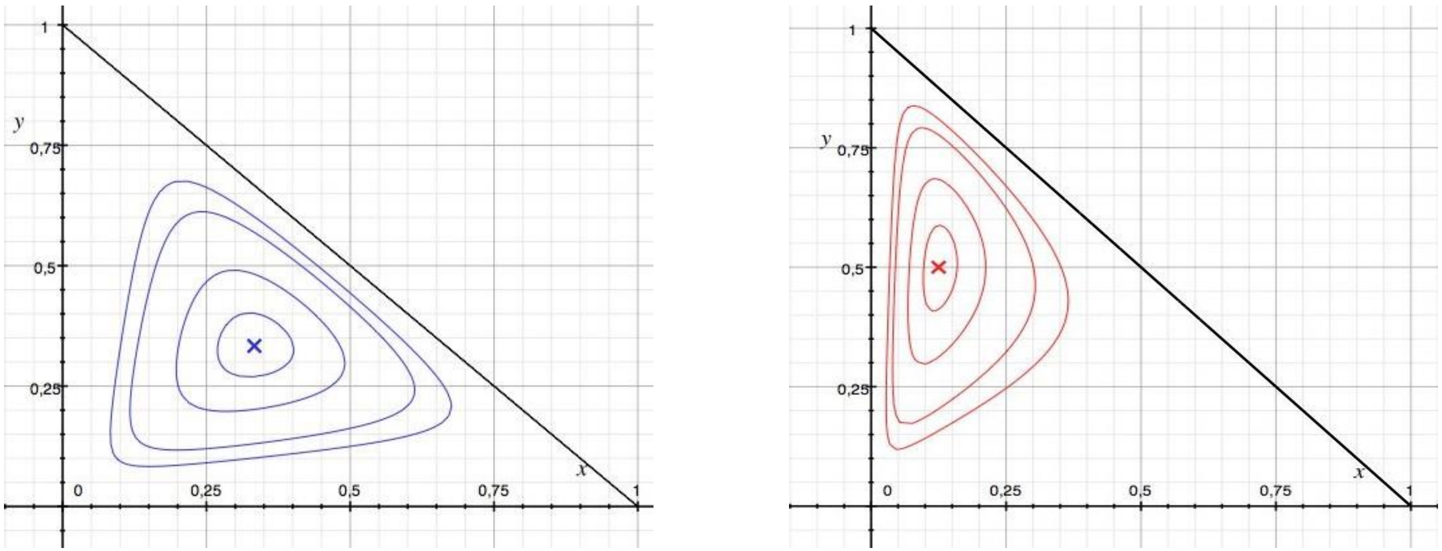
The simplicial metric is linked to a norm and to an inner product in a usual way:

$$\Delta(\vec{x},\vec{y}) = \|\vec{x} \ominus \vec{y}\|_\Delta = \sqrt{\langle \vec{x} \ominus \vec{y}, \vec{x} \ominus \vec{y}\rangle_\Delta}$$

where $\vec{x} \ominus \vec{y} = \vec{x} \oplus [(-1) \odot \vec{y}]$ and $\langle \vec{u},\vec{v}\rangle_\Delta = \frac{1}{2n} \cdot \sum_{i=1}^{n}\sum_{j=1}^{n}\left[ln\left(\frac{u_i}{u_j}\right) \cdot ln\left(\frac{v_i}{v_j}\right)\right]$

Level curves in S^3

- Once distances are defined, we can explore –for instance- the geometrical locus of all those elements in the simplex with the same distance to a given element in the simplex.



Level curves: projections of the geometrical locus in S^3 of elements which distances to point P are equal to d, being d=0.2, d=0.45, d=0.8 and d=1.0.

Left figure: P = (1/3, 1/3, 1/3) (neutral element) Right figure: P=(1/8, 1/2, 3/8)

Why using the simplicial metric

As it is shown in **De Baets (2013)**, if the simplicial arithmetic mean of a set of m compositions is defined by

$$AM_{\Delta}(\overrightarrow{x_1}, \overrightarrow{x_2}, \dots, \overrightarrow{x_m}) = \frac{1}{m} \odot [\overrightarrow{x_1} \oplus \overrightarrow{x_2} \oplus \cdots \oplus \overrightarrow{x_m}]$$

then

$$\frac{1}{m} \odot [\overrightarrow{x_1} \oplus \overrightarrow{x_2} \oplus \cdots \oplus \overrightarrow{x_m}] = argmin_{\vec{z}} \sum_{k=1}^{m} \|\vec{z} \ominus \overrightarrow{x_k}\|_{\Delta}^{2}$$

This latter expression clearly reminds the one of the arithmetic mean of m real numbers

$$\frac{1}{m} \cdot \sum_{k=1}^{n} u_k = argmin_{z} \sum_{k=1}^{m} \|z - u_k\|_2^2$$

Capital allocation principles

- Capital allocation **principles = solutions** to capital allocation problems, which may be defined in the following way:

'A positive amount K has to be distributed across n agents in such a way that the full allocation condition is satisfied, that is, all K units are distributed among the agents.'

It is possible to find out different solutions to a given capital allocation problem. We like to enumerate the elements related to the problem in such this way:

- The capital **K > 0** to be distributed;
- The agents, indexed by **i** = 1,…,n;
- A random variable linked to each agent, $\mathbf{X_i}$;
- A distribution criterion (proportional OR non-proportional);
- A function $\mathbf{f_i}$ that concentrates the information of X_i (in a stand-alone OR in a marginal way);
- The capital $\mathbf{K_i}$ assigned to each agent as a solution to the problem;
- The goal pursued by decision-makers with the allocation principle.

Capital allocation principles

- Given the previous notation:

A proportional principle may be represented by

$$K_i = K \cdot \frac{f_i(X_i)}{\sum_{j=1}^{n} f_j(X_j)} \quad \forall i = 1, \dots, n$$

Or, a non-proportional principle obtained using the quadratic optimization criterion explained in **Dhaene et al. 2012** may be represented by

$$K_i = \rho_i(X_i) + v_i \cdot \left(K - \sum_{j=1}^{n} \rho_j(X_j) \right) \quad \forall i = 1, \dots, n$$

where $f_i = \rho_i$ are risk measures for all i=1,…,n and v_i are weights that add up to 1 and that satisfy certain conditions.

Illustration of allocation principles

- Key elements of the problem

We follow the example of **van Gulick et al. (2012)** of an insurance company offering three types of life insurance portfolios:

- a (deferred) single life annuity that yields a yearly payment in every year that the insured is alive and older than 65;
- a survivor annuity that yields a yearly payment in every year that the spouse outlives the insured, if the insured dies before age 65;
- a death benefit insurance that yields a single payment in the year the insured dies, if the insured dies before age 65;

with 45,000 insured males, 15,000 insured males and 15,000 insured males, respectively.

An amount $K=TVaR_{99\%}(S)$, $S=X_{sl}+X_{surv}+X_{db}$, must be allocated to X_{sl}, X_{surv} and X_{dl}

Illustration of allocation principles

- Some of the solutions proposed

We show a table with some of the solutions proposed in the previous reference. The amount $K=TVaR_{99\%}(S)=376{,}356$.

	X_{sl}	X_{surv}	X_{db}
Proportional principle based on st.dev (σ)	335,734	24,725	15,907
Gradient allocation principle (∇)	364,477	7,979	3,900
Excess based allocation principle (EBA)	360,324	10,495	5,537

Where are we

Compositional data

Mathematical tools for working with compositions

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Where are we going

We are going to connect all these elements

Steps

Transform each allocation principle $\{K_1,K_2,\ldots,K_n\}$ of an amount K into a **relative allocation principle** dividing each K_i by K.

We have moved to the realm of compositions: $(x_1,x_2,\ldots,x_n)=(K_1/K, K_2/K,\ldots,K_n/K)$ **is a composition** and can be understood as belonging to S^n.

Given that we are aware that S^n is a metric space, once we have more than one relative allocation principle we may ask ourselves, for instance:

- Could a **ranking** between them be established based on distances between them?
- Could we **average** them?

Illustration

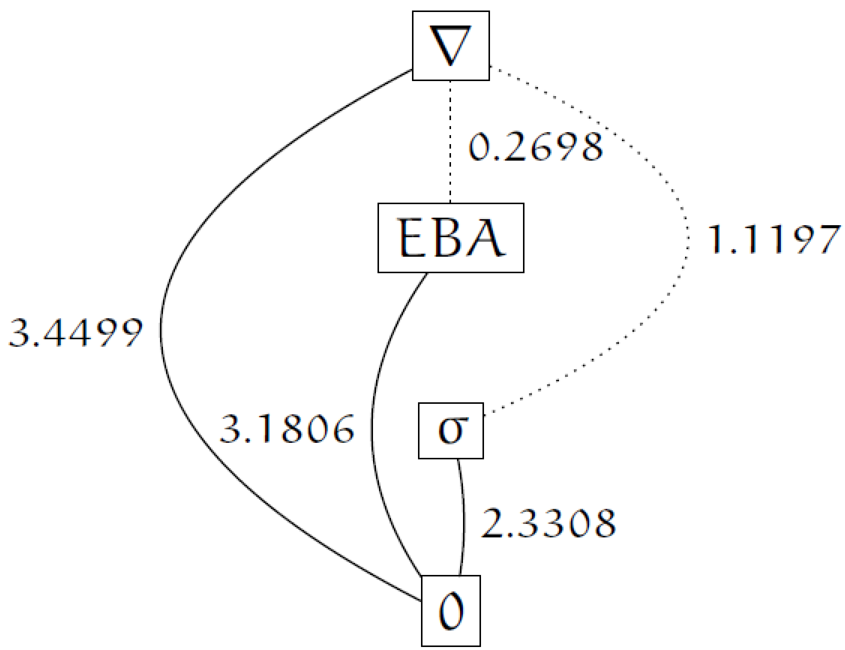
- Recall the previous example.

We now show in the table the relative allocation principles associated to the absolute ones shown before.

	X_{sl}	X_{surv}	X_{db}
Proportional principle based on st.dev (σ)	89.20%	6.57%	4.23%
Gradient allocation principle (∇)	96.84%	2.12%	1.04%
Excess based allocation principle (EBA)	95.74%	2.79%	1.47%

Illustration

The previous relative allocation principles may be ranked using the simplicial distance in the following way:



$$\nabla = (96.84\%, 2.12\%, 1.04\%)$$

$$\text{EBA} = (95.74\%, 2.79\%, 1.47\%)$$

$$\sigma = (89.20\%, 6.57\%, 4.23\%)$$

Illustration

The previous relative allocation principles may be properly averaged (**using the simplicial arithmetic mean** instead of averaging the components):

	X_{sl}	X_{surv}	X_{db}
Simplicial average of principles (∇, σ and EBA)	94.71%	3.42%	1.88%
Absolute principle linked to the simplicial average	356,431	12,859	7,066

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Thank you