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A matrix approach to pricing marriage insurance with mortality dependence *

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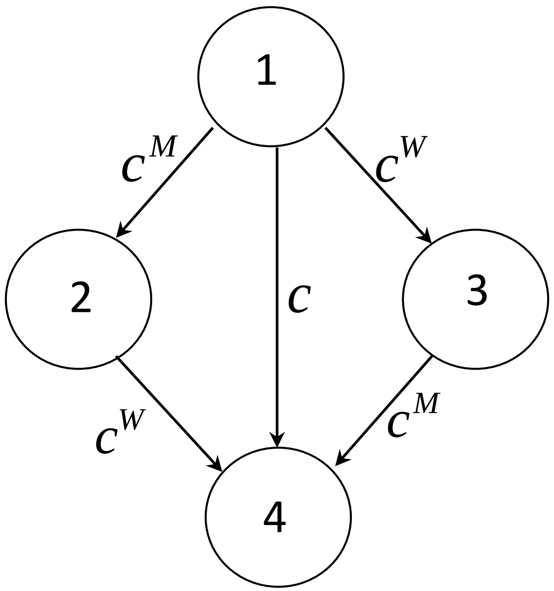
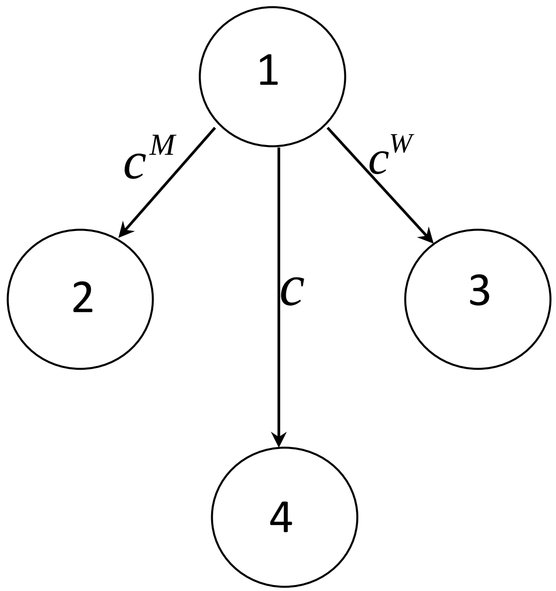
I. MULTIPLE STATE MODEL FOR MARRIAGE INSURANCE

LSS –Last Surviving Status



JLS – Joint-life Status



State space:

1 - both spouses are alive
2 - husband is dead
3 - wife is dead
4 - both spouses are dead

$$(S,T) = (\{1,2,3,4\}, \{(1,2),(1,3),(1,4),(2,4),(3,4)\})$$

$$(S,T) = (\{1,2,3,4\}, \{(1,2),(1,3),(1,4)\})$$

II. Probabilistic structure of the model

Decrements

Maximum duration of life according to Life Tables

x - *age at a policy issue* of husband
y - *age at a policy issue* of wife

T_x^M - the *remaining lifetimes* of x-year-old man (husband); $T_x^M \in [0, w_x^M]$ and $w_x^M = \omega - x$
T_y^W - the *remaining lifetimes* of y-year-old woman (wife); $T_x^W \in [0, w_x^W]$ and $w_x^W = \omega - y$

n – *insurance period*

$$n = \max\{w_x^M, w_y^W\} \text{ for LSS}$$

$$n = \min\{w_x^M, w_y^W\} \text{ for JLS}$$

II. Probabilistic structure of the model

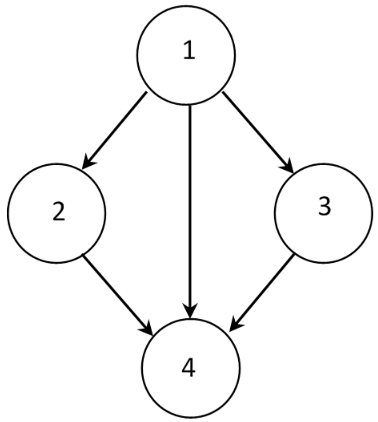
$\{X(t); t \in \mathrm{T}\}$ - stochastic process with values in finite set *S*.

ASSUMPTION: $\mathrm{T} = \{0,1,2,\ldots\}$ - discrete, $\{X(t): t = 0,1,2,\ldots\}$ - *Markov chain*

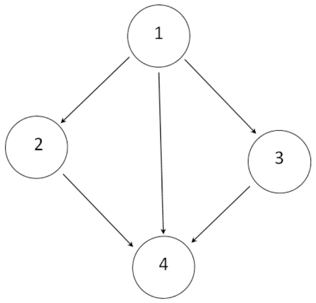
$$\{\mathbf{Q}(k)\}_{k=0}^{n-1}, \text{ where } \mathbf{Q}(k) = (q_{ij}(k))_{i,j=1}^{N} \text{ and } q_{ij}(k) = P(X(k+1) = j \mid X(k) = i).$$

For marriage insurance with LSS

$$\mathbf{Q}(k) = \begin{pmatrix} q_{11}(k) & q_{12}(k) & q_{13}(k) & q_{14}(k) \\ 0 & q_{22}(k) & 0 & q_{24}(k) \\ 0 & 0 & q_{33}(k) & q_{34}(k) \\ 0 & 0 & 0 & 1 \end{pmatrix}$$



T_x^M, T_y^W - **INDEPENDENT** ⇨ $F(w,z) = P(T_x^M \le w, T_y^W \le z) = F^M(w) \cdot F^W(z)$



$$\boldsymbol{Q}(k) = \boldsymbol{Q}^X(k) \circ \boldsymbol{Q}^Y(k)$$

↑ *Hadamard (or Schur) product*

$$\boldsymbol{Q}^Y(k) = \begin{pmatrix} p_{y+k} & q_{y+k} & p_{y+k} & q_{y+k} \\ 0 & 1 & 0 & 1 \\ 0 & 0 & p_{y+k} & q_{y+k} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\boldsymbol{Q}^X(k) = \begin{pmatrix} p_{x+k} & p_{x+k} & q_{x+k} & q_{x+k} \\ 0 & p_{x+k} & 0 & q_{x+k} \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

T_x^M, T_y^W - **DEPENDENT** ⇨ $F(w,z) = P(T_x^M \le w, T_y^W \le z) = C\left(F^M(w), F^W(z)\right)$

- $F^M(w) = P(T_{x_0}^M \le w)$ i $F^W(z) = P(T_{y_0}^W \le z)$
- *copula* $C(w, z)$ describes the structure of dependence between $T_{x_0}^M$ and $T_{y_0}^W$
- based on the correlation coefficient τ - Kendall for $T_{x_0}^M$ and $T_{y_0}^W$ (men at age $[x_0, \omega)$ and women at age $[y_0, \omega)$) the copula type is selected
- if $T_{x_0}^M$ and $T_{y_0}^W$ are independent, then $C(w, z)=w \cdot z$

x_0, y_0 - reference age for men and women

- $S^M(w) = P\left(T_{x_0}^M > w\right)$, $S^W(z) = P\left(T_{y_0}^W > z\right)$ - *survival functions*
- *joint survival function* : $S(w,z) = P(T_{x_0}^M > w, T_{y_0}^W > z)$.
- *survival copula* $C^*(w,z)$: $S(w,z) = C^*\left(S^M(w), S^W(z)\right)$

$$C^*(w,z) = w + z - 1 + C(1-w, 1-z)$$

THEOREM

For $(S,T)=(\{1,2,3,4\},\{(1,2),(1,3),(1,4),(2,4),(3,4)\})$ and $x = x_0 + t$, $y = y_0 + s$ ($t \geq 0$ and $s \geq 0$) elements of transition matrices $\{\mathbf{Q}(k)\}_{k=0}^{n-1}$ have the following form:

- for $k \in \{0,1,2,\ldots n-1\}$ and $x+k<\omega$ and $y+k<\omega$

$$q_{11}(k)=\frac{C^*\left(S_{x_0}^M(t+k+1),S_{y_0}^W(s+k+1)\right)}{C^*\left(S_{x_0}^M(t+k),S_{y_0}^W(s+k)\right)}$$

$$q_{12}(k)=\frac{C^*\left(S_{x_0}^M(t+k),S_{y_0}^W(s+k+1)\right)-C^*\left(S_{x_0}^M(t+k+1),S_{y_0}^W(s+k+1)\right)}{C^*\left(S_{x_0}^M(t+k),S_{y_0}^W(y+k)\right)}$$

$$q_{13}(k)=\frac{C^*\left(S_{x_0}^M(t+k+1),S_{y_0}^W(s+k)\right)-C^*\left(S_{x_0}^M(t+k+1),S_{y_0}^W(s+k+1)\right)}{C^*\left(S_{x_0}^M(t+k),S_{y_0}^W(s+k)\right)}$$

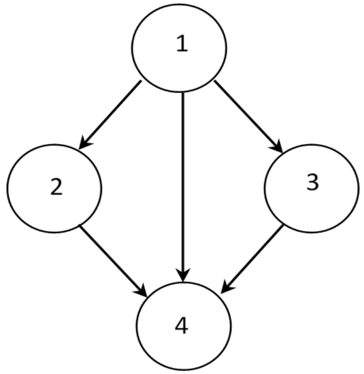
$$q_{14}(k)=1-\sum_{j=1}^{3}q_{1j}(k)$$

$$q_{22}(k)=\frac{C^*\left(S_{x_0}^M(t),S_{y_0}^W(s+k+1)\right)-C^*\left(S_{x_0}^M(t+k),S_{y_0}^W(s+k+1)\right)}{C^*\left(S_{x_0}^M(t),S_{y_0}^W(s+k)\right)-C^*\left(S_{x_0}^M(t+k),S_{y_0}^W(s+k)\right)}$$

$$q_{24}(k)=1-q_{22}(k)$$

$$q_{33}(k)=\frac{C^*\left(S_{x_0}^M(t+k+1),S_{y_0}^W(s)\right)-C^*\left(S_{x_0}^M(t+k+1),S_{y_0}^W(s+k)\right)}{C^*\left(S_{x_0}^M(t+k),S_{y_0}^W(s)\right)-C^*\left(S_{x_0}^M(t+k),S_{y_0}^W(s+k)\right)},$$

$$q_{34}(k)=1-q_{33}(k)$$



THEOREM

For $(S,T) = (\{1,2,3,4\}, \{(1,2),(1,3),(1,4),(2,4),(3,4)\})$ and $x = x_0 + t$, $y = y_0 + s$ ($t \geq 0$ and $s \geq 0$)

elements of transition matrices $\{\mathbf{Q}(k)\}_{k=0}^{n-1}$ have the following form:

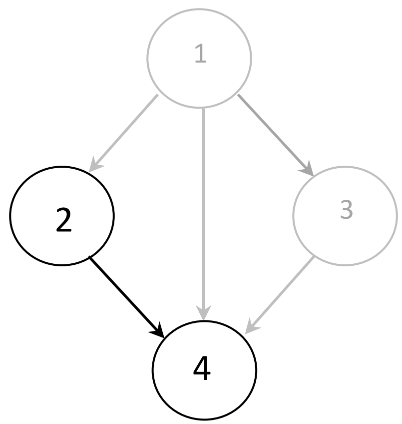
- for $k \in \{0,1,2,\ldots n-1\}$ and $x + k \geq \omega$ and $y + k < \omega$ (husband exceeded age ω)

$$q_{11}(k) = q_{12}(k) = q_{13}(k) = q_{14}(k) = 0,$$

$$q_{22}(k) = \frac{C^*\left(S_{x_0}^M(t), S_{y_0}^W(s+k+1)\right) - C^*\left(S_{x_0}^M(t+k), S_{y_0}^W(s+k+1)\right)}{C^*\left(S_{x_0}^M(t), S_{y_0}^W(s+k)\right) - C^*\left(S_{x_0}^M(t+k), S_{y_0}^W(s+k)\right)},$$

$$q_{24}(k) = 1 - q_{22}(k),$$

$$q_{33}(k) = q_{34}(k) = 0,$$



THEOREM

For $(S,T)=(\{1,2,3,4\},\{(1,2),(1,3),(1,4),(2,4),(3,4)\})$ and $x = x_0 + t$, $y = y_0 + s$ ($t \geq 0$ and $s \geq 0$)

elements of transition matrices $\{\mathbf{Q}(k)\}_{k=0}^{n-1}$ have the following form:

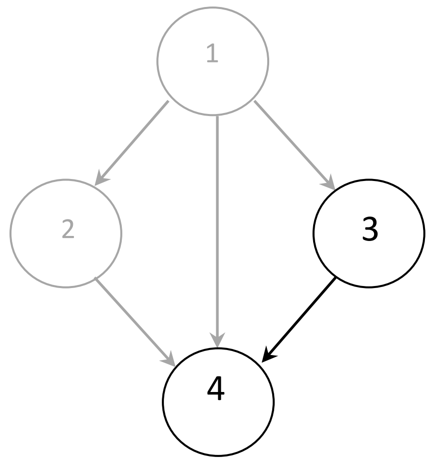
- for $k \in \{0,1,2,\ldots n-1\}$ and $x+k<\omega$ and $y+k \geq \omega$ (wife exceeded age ω)

$$q_{11}(k)=q_{12}(k)=q_{13}(k)=q_{14}(k)=0,$$

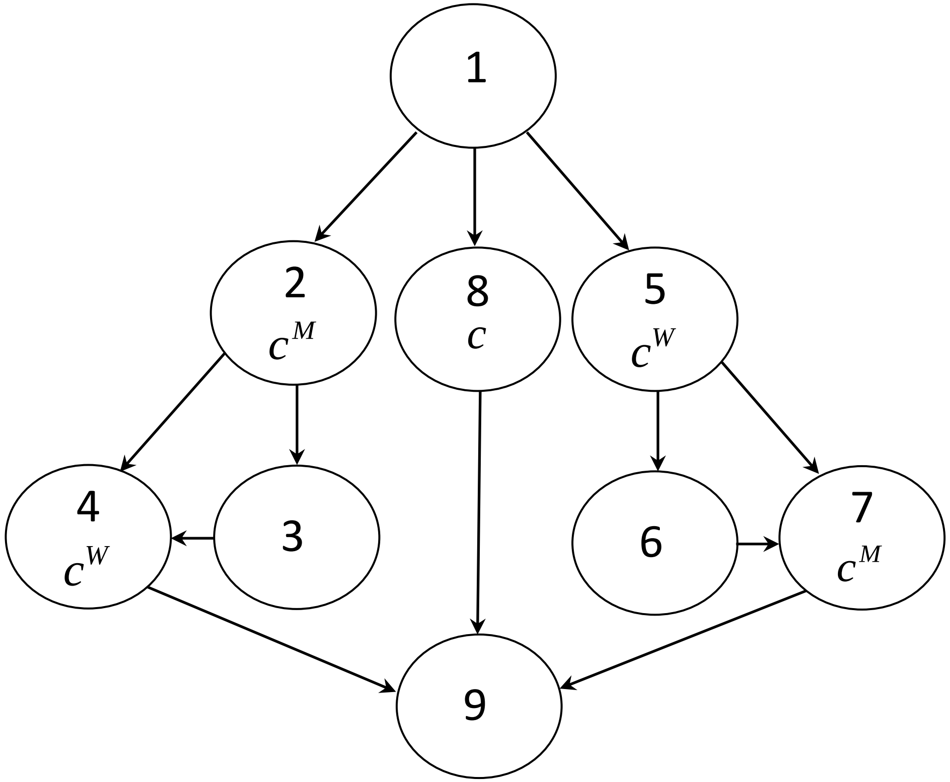
$$q_{22}(k)=q_{24}(k)=0,$$

$$q_{33}(k)=\frac{C^*\left(S_{x_0}^M(t+k+1), S_{y_0}^W(s)\right)-C^*\left(S_{x_0}^M(t+k+1), S_{y_0}^W(s+k)\right)}{C^*\left(S_{x_0}^M(t+k), S_{y_0}^W(s)\right)-C^*\left(S_{x_0}^M(t+k), S_{y_0}^W(s+k)\right)},$$

$$q_{34}(k)=1-q_{33}(k).$$



III. MATRIX FORM OF ACTUARIAL VALUES

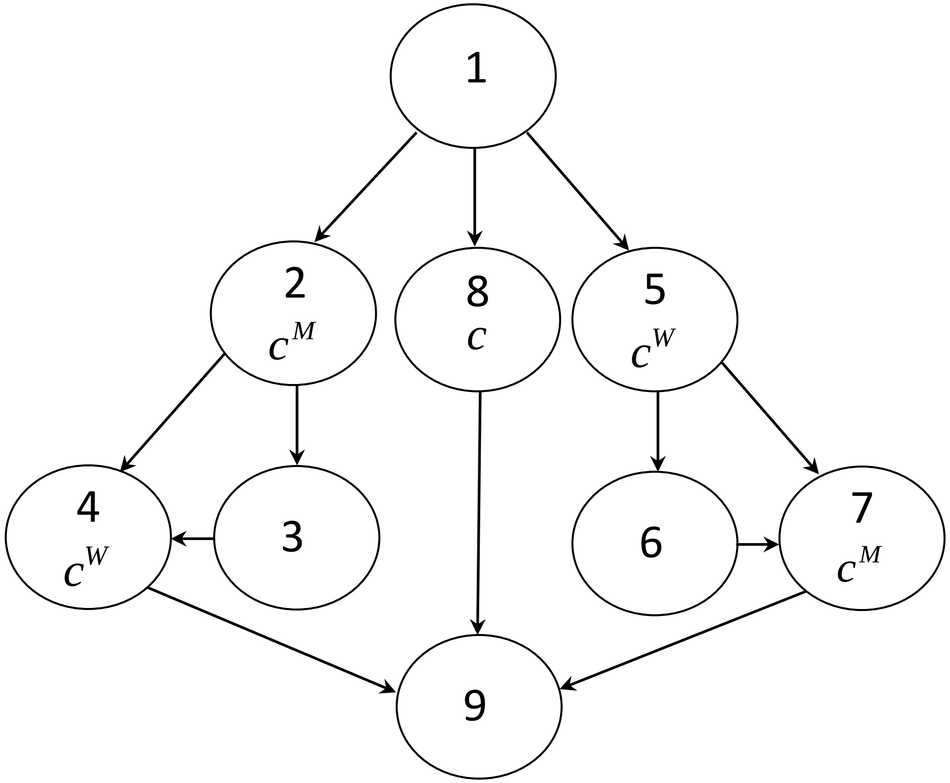


State space of *modified multiple state model* (S^*, T^*)

1 - both spouses are alive
2 - death of the husband, wife is alive
3 - husband has been dead for at least one year, wife is alive
4 - husband has been dead for at least one year, death of the wife
5 - death of the wife, husband is alive
6 - wife has been dead for at least one year, husband is alive
7 - wife has been dead for at least one year, death of the husband
8 - both spouses are dead
9 - both spouses have been dead for at least one

$$\mathbf{Q}^*(k) = \begin{pmatrix} q_{11}(k) & q_{12}(k) & 0 & 0 & q_{13}(k) & 0 & 0 & q_{14}(k) & 0 \\ 0 & 0 & q_{22}(k) & q_{24}(k) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q_{22}(k) & q_{24}(k) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & q_{33}(k) & q_{34}(k) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & q_{33}(k) & q_{34}(k) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

III. MATRIX FORM OF ACTUARIAL VALUES



<u>Matricies:</u>

D – matrix describing the probabilistic structure of the model (*based on* $\{\mathbf{Q}^*(k)\}_{k=0}^{n-1}$: $\mathbf{P}^T(t) = \mathbf{P}^T(0)\prod_{k=0}^{t-1}\mathbf{Q}^*(k)$)

$\mathbf{C_{in}}$ – benefits matrix

$\mathbf{C_{out}}$ – premiums matrix

$\mathbf{C} = \mathbf{C_{in}} + \mathbf{C_{out}}$ – cash flow matrix

Λ - matrix describing the interest rate (*constant, function depending on time or stochastic)*

$$\mathbf{S} = (1,1,\ldots,1)^T \in R^{N^*}$$

$$\mathbf{J}_i = \Big(0,0,\ldots,0,\underset{i}{1},0\ldots,0\Big)^T \in R^{N^*}$$

$$\mathbf{I}_{t+1} = \Big(0,0,\ldots,0,\ \underset{t+1}{1}\ ,0\ldots,0\Big)^T \in R^{n+1}$$

III. MATRIX FORM OF ACTUARIAL VALUES

- **net single premium $\boldsymbol{\pi}$**, paid in advance, when $X^*(0) = 1$

$$\pi = \mathbf{S}^T Diag(\mathbf{C}_{in}\mathbf{D}^T)\boldsymbol{\Lambda}\mathbf{I}_1$$

- **net period premium $\boldsymbol{p}$** payable in advance at the beginning of the time unit during the first m units ($\mathbf{1} \le \boldsymbol{m} \le \boldsymbol{n}^*$), if $X^*(t) = 1$

$$p = \frac{\mathbf{S}^T Diag(\mathbf{C}_{in}\mathbf{D}^T)\boldsymbol{\Lambda}\mathbf{I}_1}{\mathbf{I}_1^T\boldsymbol{\Lambda}^T[\mathbf{I} - \sum_{t=m+1}^{n+1}\mathbf{I}_t\mathbf{I}_t^T]\mathbf{D}\mathbf{I}_1}$$

- **vector of net prospectives reserves at moment $\boldsymbol{t}$**

$$\mathbf{V}(t) = \left(\mathbf{C}_{out}^T + \mathbf{C}_{in^-}^T + \mathbf{F}^T(t,\mathbf{C})\boldsymbol{\Lambda}\right)\mathbf{I}_{t+1}$$

where

$$\mathbf{F}^T(t,\mathbf{C}) = \sum_{k=t+1}^{n}\prod_{u=t}^{k-1}\mathbf{Q}(u)\mathbf{C}^T\mathbf{I}_{k+1}\mathbf{I}_{k+1}^T$$

IV. Numerical examples

Benefits: $c^M = c^W = 1$; $c = c^M + c^W = 2$

Probabilistic structure:

- independence: Lower Silesia Life Table 2011
- dependence: Main Statistical Office – data for Lower Silesia 2011; $x_0 = y_0 = 60$
  Kendall correlation coefficient between T_{60}^M and T_{60}^W : $\tau = 0.076$
  Gumbel copula: $C(u,v) = \exp\left(-\left((-\ln u)^\alpha + (-\ln v)^\alpha \right)^{\frac{1}{\alpha}} \right)$ where $\alpha = 1.0786$.

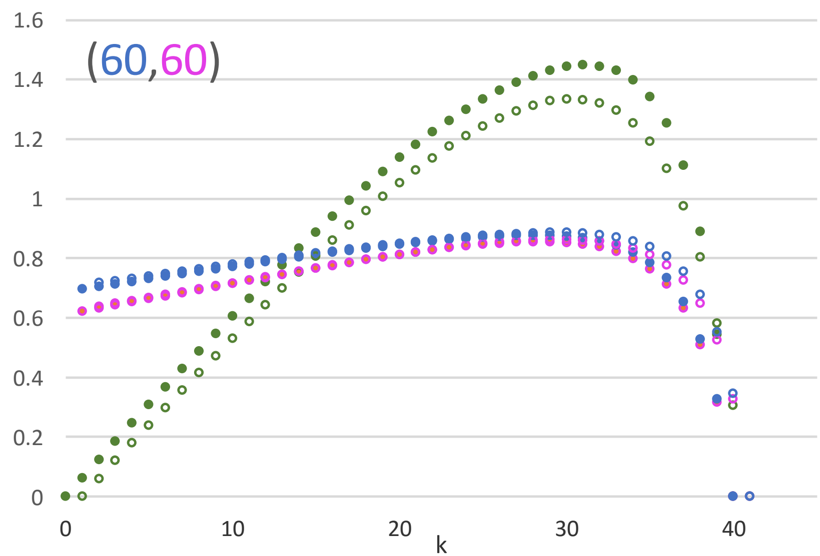
Interest rate:

- the real Polish market data related to the yield to maturity on fixed interest bonds from 03.03.2015
- the best fitted model is **Svensson model** => term interest rate is equal to **2.1%**

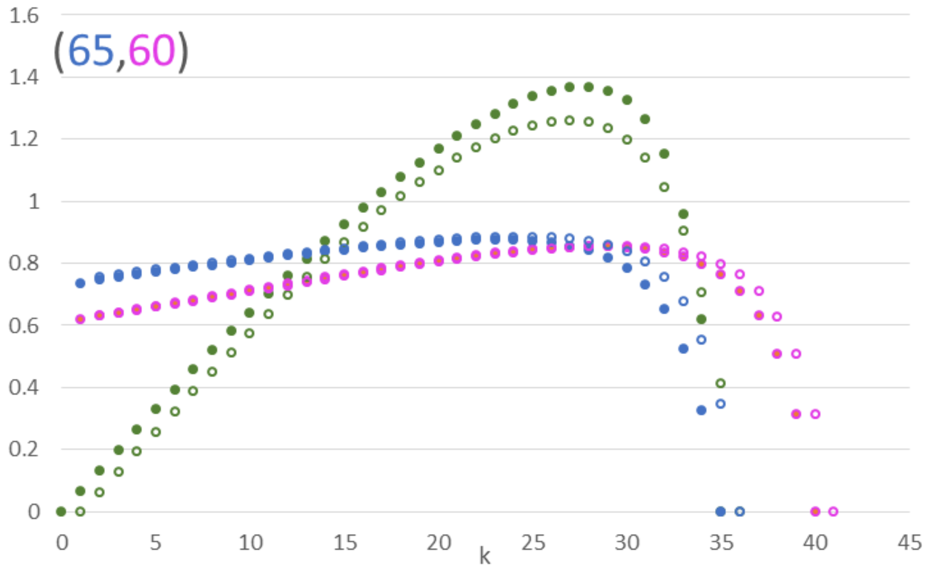
NET PREMIUMS

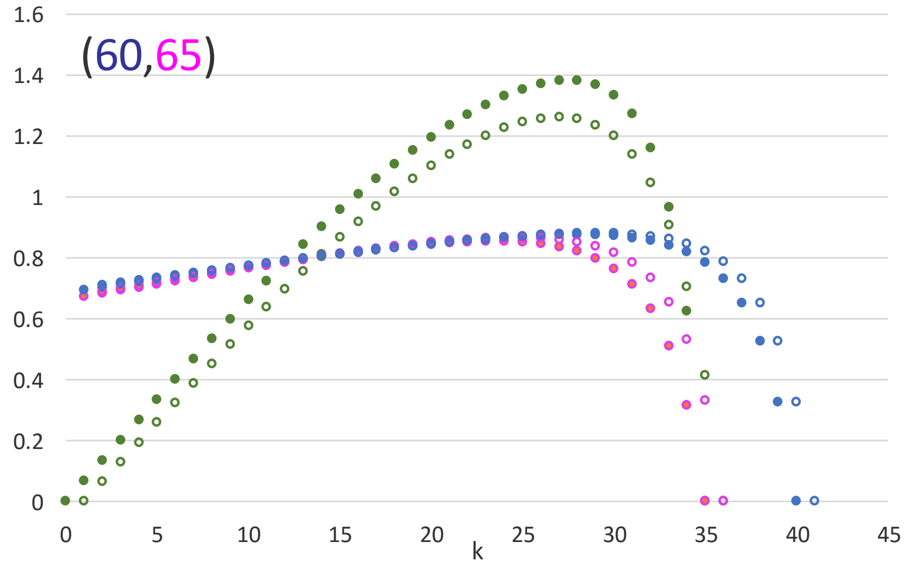
(*husband age*, *wife age*) = (x, y)	(60, 60)	(65, 60)	(60, 65)	(65 ,65)
The insurance period (n)	40	40	40	35
The period of premiums payment (m)	40	35	35	35
T_y^M , T_x^W **- INDEPENDENT**				
Net single premium - π	1.298378	1.341719	1.348619	1.392516
Net period premium - p	0.100589	0.116768	0.112917	0.128969
T_y^M , T_x^W **- DEPENDENT**				
Net single premium - π	1.298377	1.338764	1.346594	1.389498
Net period premium - p	0.099013	0.114646	0.110538	0.125823

NET PROSPECTIVE RESERVES



- both spouses are alive (independence)
- wife is alive (independence)
- husband is alive (independence)
- both spouses are alive (dependence)
- wife is alive (dependence)
- husband is alive (dependence)





AVERAGE PROFIT EXPECTED TO EMERGE AT THE END OF YEAR *k*

$$\mathbf{EPR} = \begin{pmatrix} EPR(1) \\ EPR(2) \\ \vdots \\ EPR(n) \end{pmatrix}$$

$$EPR(k) = \mathbf{I}_k^T \cdot \breve{\mathbf{D}} \cdot \left(\left(-\mathbf{C}_{out}^T \mathbf{I}_k \mathbf{I}_k^T \breve{\mathbf{\Lambda}} - \breve{\mathbf{Q}}(k-1)\mathbf{C}_{\underline{in}}^T \right) \mathbf{I}_{k+1} + \left(\mathbf{V}^{\mathbf{T}} \mathbf{I}_k \mathbf{I}_k^T \breve{\mathbf{\Lambda}} - \breve{\mathbf{Q}}(k-1)\mathbf{V}^{\mathbf{T}} \right) \mathbf{I}_{k+1} \right).$$

FIRST-ORDER BASIS (prudent)

T_x^M, T_x^W - independent

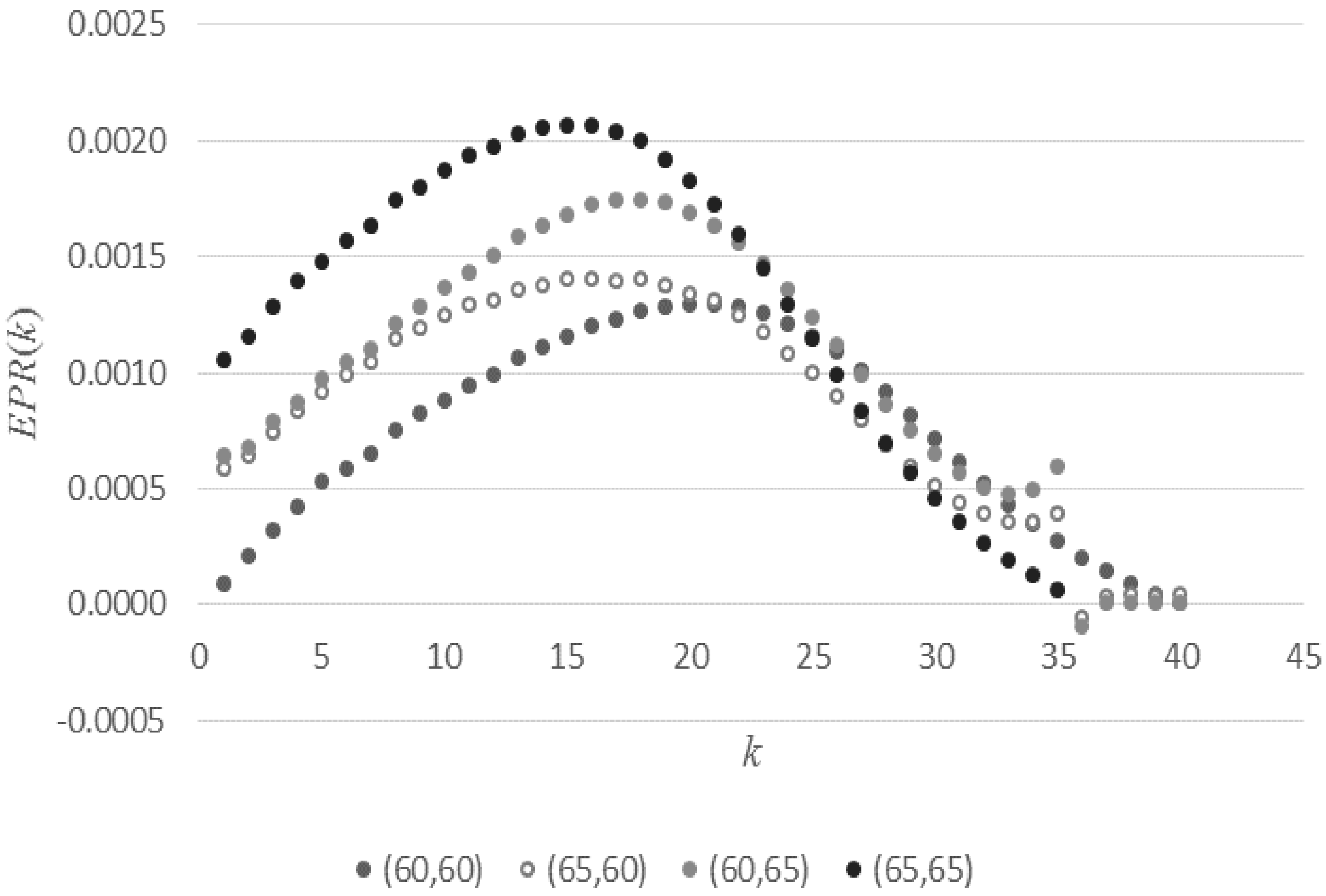
The interest rate determined based on safe financial instruments ($i = 2.1\%$)

SECOND-ORDER BASIS (real)

T_x^M, T_x^W - dependent

The real interest rate ($i = 3.1\%$)

AVERAGE PROFIT EXPECTED TO EMERGE AT THE END OF YEAR *k*



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