Chain-ladder method: dynamic run-off uncertainty analysis

Mario V. Wüthrich
RiskLab, ETH Zurich
Swiss Finance Institute Professor

joint work with
Michael Merz (University of Hamburg)

June 2, 2016
ASTIN Colloquium, Lisbon

Outline

- Chain-ladder method
- Claims development result
- Examples

Chain-ladder algorithm

accident year i	development year j 0	1	2	3	4	5	6	7	8	9
2004	1'216	1'347	1'786	2'281	2'656	2'909	3'283	3'587	3'754	3'921
2005	798	1'051	1'215	1'349	1'655	1'926	2'132	2'287	2'567	
2006	1'115	1'387	1'930	2'177	2'513	2'931	3'047	3'182		
2007	1'052	1'321	1'700	1'971	2'298	2'645	3'003			
2008	808	1'029	1'229	1'590	1'842	2'150				
2009	1'016	1'251	1'698	2'105	2'385					
2010	948	1'108	1'315	1'487						
2011	917	1'082	1'484							
2012	1'001	1'376					$C_{i,j}$	**to be predicted**		
2013	841									

▷ $C_{i,j}$ = cumulative claim of accident year i and development year j.

▷ $\mathcal{D}_t = \{C_{i,j};\ i + j \le t\}$ = observations at time t.

▷ Chain-ladder (CL) **algorithm** is based on the (regression) assumption

$$C_{i,j+1} \approx f_j \, C_{i,j},$$

for CL factors f_j **not** depending on accident year i.

Stochastic models underlying the CL algorithm

▷ CL algorithm is not based on a stochastic model (deterministic algorithm).

▷ We need a stochastic representation to quantify prediction uncertainty.

▷ Stochastic models introduced providing the CL reserves:

- ⋆ Mack's distribution-free CL model (1993)
- ⋆ Poisson and over-dispersed Poisson (ODP) model of Renshaw-Verrall (1998)
- ⋆ Bayesian CL models by Gisler (2006), Bühlmann et al. (2009)
- ⋆ Gamma-gamma Bayesian CL model by Merz-Wüthrich (2008, 2014)

Bayesian chain-ladder (BCL) model

Model assumptions (gamma-gamma BCL model).
Assume there are fixed given variance parameters $\sigma_0^2, \ldots, \sigma_{J-1}^2$.

- Conditionally, given CL parameters $\boldsymbol{F} = (F_0, \ldots, F_{J-1})$:
  - ⋆ $(C_{i,j})_{j=0,\ldots,J}$ independent (in i) and Markovian (in j) with gamma innovations
  - ⋆ with for all $1 \leq i \leq I$ and $0 \leq j \leq J-1$

$$\begin{aligned} \mathbb{E}\left[C_{i,j+1} \middle| C_{i,j}, \boldsymbol{F}\right] &= F_j \; C_{i,j}, \\ \mathrm{Var}\left(C_{i,j+1} \middle| C_{i,j}, \boldsymbol{F}\right) &= \sigma_j^2 F_j^2 \; C_{i,j}. \end{aligned}$$

- The components of $\boldsymbol{F}^{-1}$ are independent and gamma distributed.

▷ This model has the CL property: $C_{i,j+1} \approx F_j \; C_{i,j}$, for given CL factors F_j.

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  - ⋆ $(C_{i,j})_{j=0,\ldots,J}$ independent (in i) and Markovian (in j) with gamma innovations
  - ⋆ with for all $1 \leq i \leq I$ and $0 \leq j \leq J-1$

$$\mathbb{E}\left[C_{i,j+1} \middle| C_{i,j}, \boldsymbol{F}\right] = F_j \; C_{i,j},$$

$$\operatorname{Var}\left(C_{i,j+1} \middle| C_{i,j}, \boldsymbol{F}\right) = \sigma_j^2 F_j^2 \; C_{i,j}.$$

- The components of $\boldsymbol{F}^{-1}$ are independent and gamma distributed.

▷ This model has the CL property: $C_{i,j+1} \approx F_j \; C_{i,j}$, for given CL factors F_j.

BCL predictor

▷ Predictors can be calculated explicitly in the above model for observations $\mathcal{D}_t$.

▷ BCL predictor at time $t \geq I > J$ for non-informative priors

$$\widehat{C}_{i,J}^{(t)} \;=\; \mathbb{E}\left[C_{i,J}\middle|\,\mathcal{D}_t\right] \;=\; C_{i,t-i}\prod_{j=t-i}^{J-1}\widehat{f}_j^{(t)},$$

with CL factor estimators

$$\widehat{f}_j^{(t)} = \frac{\sum_{i=1}^{t-j-1} C_{i,j+1}}{\sum_{i=1}^{t-j-1} C_{i,j}}.$$

CL claims prediction

accident year i	development year j									
	0	1	2	3	4	5	6	7	8	9
2004	1'216	1'347	1'786	2'281	2'656	2'909	3'283	3'587	3'754	3'921
2005	798	1'051	1'215	1'349	1'655	1'926	2'132	2'287	2'567	*2'681*
2006	1'115	1'387	1'930	2'177	2'513	2'931	3'047	3'182	*3'424*	*3'577*
2007	1'052	1'321	1'700	1'971	2'298	2'645	3'003	*3'214*	*3'458*	*3'612*
2008	808	1'029	1'229	1'590	1'842	2'150	*2'368*	*2'534*	*2'727*	*2'848*
2009	1'016	1'251	1'698	2'105	2'385	*2'733*	*3'010*	*3'221*	*3'465*	*3'619*
2010	948	1'108	1'315	1'487	*1'731*	*1'983*	*2'184*	*2'337*	*2'514*	*2'626*
2011	917	1'082	1'484	*1'769*	*2'058*	*2'358*	*2'597*	*2'779*	*2'990*	*3'123*
2012	1'001	1'376	*1'776*	*2'116*	*2'462*	*2'821*	*3'106*	*3'324*	*3'577*	*3'736*
2013	841	*1'039*	*1'341*	*1'598*	*1'859*	*2'130*	*2'346*	*2'510*	*2'701*	*2'821*
$\widehat{f}_j^{(t)}$	1.2343	1.2904	1.1918	1.1635	1.1457	1.1013	1.0702	1.0760	1.0444	

▷ What about prediction uncertainty?

▷ Consider the conditional mean square error of prediction (MSEP)

$$\mathrm{msep}_{C_{i,J}|\mathcal{D}_t}\left(\widehat{C}_{i,J}^{(t)}\right) = \mathbb{E}\left[\left(C_{i,J} - \widehat{C}_{i,J}^{(t)}\right)^2 \middle| \mathcal{D}_t\right].$$

Conditional MSEP formula

▷ Conditional MSEP can be calculated explicitly and exactly in the above model.

Conditional MSEP for non-informative priors for single accident years i:

$$\mathrm{msep}_{C_{i,J}|\mathcal{D}_t}\left(\widehat{C}_{i,J}^{(t)}\right) = \left(\widehat{C}_{i,J}^{(t)}\right)^2 \left(\sum_{j=t-i}^{J-1}\left[\frac{\sigma_j^2}{\widehat{C}_{i,j}^{(t)}} + \frac{\sigma_j^2}{\sum_{\ell=1}^{t-j-1} C_{\ell,j}}\right] + o\left(\frac{\sigma_\ell^2}{C_{k,\ell}}\right)\right).$$

▷ This is identical to the famous Mack formula (1993) up to:

- a different variance parametrization, and
- and a correction term of order $o\left(\sigma_\ell^2/C_{k,\ell}\right)$.

▷ Aggregation over accident years i is similar.

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Claims development result (1/2)

▷ Conditional MSEP formula above considers the total prediction uncertainty over the entire run-off (static view).

▷ Solvency considerations require a dynamic view: possible changes in predictions over the next accounting year(s).

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end point of path (static view) $\Longleftrightarrow$ whole path behavior (dynamic view)

▷ Define the claims development result of accounting year $t+1 > I$ by

$$\mathrm{CDR}_i(t+1) \; = \; \widehat{C}_{i,J}^{(t+1)} - \widehat{C}_{i,J}^{(t)}.$$

Claims development result (2/2)

accident year i	development year j 0	1	2	3	4	5	6	7	8	9
2004	1'216	1'347	1'786	2'281	2'656	2'909	3'283	3'587	3'754	3'921
2005	798	1'051	1'215	1'349	1'655	1'926	2'132	2'287	2'567	*
2006	1'115	1'387	1'930	2'177	2'513	2'931	3'047	3'182	*	
2007	1'052	1'321	1'700	1'971	2'298	2'645	3'003	*		
2008	808	1'029	1'229	1'590	1'842	2'150	*			
2009	1'016	1'251	1'698	2'105	2'385	*				
2010	948	1'108	1'315	1'487	*					
2011	917	1'082	1'484	*						
2012	1'001	1'376	*							
2013	841	*								

▷ Martingale property of $(\widehat{C}_{i,J}^{(t)})_{t \geq I}$ implies

$$\mathbb{E}\left[\mathrm{CDR}_i(t+1) \middle| \mathcal{D}_t\right] = \mathbb{E}\left[\widehat{C}_{i,J}^{(t+1)} - \widehat{C}_{i,J}^{(t)} \middle| \mathcal{D}_t\right] = 0.$$

▷ Solvency: study the one-year uncertainty

$$\mathrm{msep}_{\mathrm{CDR}_i(t+1)|\mathcal{D}_t}(0) = \mathbb{E}\left[\left(\mathrm{CDR}_i(t+1) - 0\right)^2 \middle| \mathcal{D}_t\right].$$

One-year uncertainty formula

▷ Conditional MSEP can be calculated explicitly and exactly in the above model.

Conditional MSEP for non-informative priors for single accident years i:

$$\mathrm{msep}_{\mathrm{CDR}_i(t+1)|\mathcal{D}_t}(0) = \left(\widehat{C}_{i,J}^{(t)}\right)^2$$

$$\times \left(\left[\frac{\sigma_{t-i}^2}{C_{i,t-i}} + \frac{\sigma_{t-i}^2}{\sum_{\ell=1}^{i-1} C_{\ell,t-i}} + \sum_{j=t-i+1}^{J-1} \alpha_j^{(t)} \frac{\sigma_j^2}{\sum_{\ell=1}^{t-j-1} C_{\ell,j}} \right] + o\left(\frac{\sigma_\ell^2}{C_{k,\ell}}\right) \right),$$

with (credibility) weight

$$\alpha_j^{(t)} = \frac{C_{t-j,j}}{\sum_{\ell=1}^{t-j} C_{\ell,j}} \in (0,1].$$

▷ This is identical to Merz-Wüthrich formula (2008) up to the differences mentioned above.

Total uncertainty vs. one-year uncertainty

Total uncertainty:

$$\mathrm{msep}_{C_{i,J}|\mathcal{D}_t}\left(\widehat{C}_{i,J}^{(t)}\right) \approx \left(\widehat{C}_{i,J}^{(t)}\right)^2 \sum_{j=t-i}^{J-1}\left[\frac{\sigma_j^2}{\widehat{C}_{i,j}^{(t)}} + \frac{\sigma_j^2}{\sum_{\ell=1}^{t-j-1} C_{\ell,j}}\right].$$

One-year uncertainty:

$$\mathrm{msep}_{\mathrm{CDR}_i(t+1)|\mathcal{D}_t}(0) \approx \left(\widehat{C}_{i,J}^{(t)}\right)^2 \times \left[\frac{\sigma_{t-i}^2}{\widehat{C}_{i,t-i}^{(t)}} + \frac{\sigma_{t-i}^2}{\sum_{\ell=1}^{i-1} C_{\ell,t-i}} + \sum_{j=t-i+1}^{J-1} \alpha_j^{(t)} \frac{\sigma_j^2}{\sum_{\ell=1}^{t-j-1} C_{\ell,j}}\right].$$

Process uncertainty, parameter estimation uncertainty and its reduction in time.

Residual uncertainty for remaining accounting years

This suggests for accounting year $t+2$:

$$\mathbb{E}\left[\left.\operatorname{msep}_{\mathrm{CDR}_i(t+2)|\mathcal{D}_{t+1}}(0)\right|\mathcal{D}_t\right]$$

$$\approx \left(\widehat{C}_{i,J}^{(t)}\right)^2\left[\frac{\sigma_{t-i+1}^2}{\widehat{C}_{i,t-i+1}^{(t)}}+\left(1-\alpha_{t-i+1}^{(t)}\right)\frac{\sigma_{t-i+1}^2}{\sum_{\ell=1}^{i-2}C_{\ell,t-i+1}}\right]$$

$$+\left(\widehat{C}_{i,J}^{(t)}\right)^2\sum_{j=t-i+2}^{J-1}\left[\alpha_{j-1}^{(t)}\left(1-\alpha_j^{(t)}\right)\frac{\sigma_j^2}{\sum_{\ell=1}^{t-j-1}C_{\ell,j}}\right].$$

- ▷ This can be derived analytically and iterated!
- ▷ It allocates the total MSEP formula across different accounting periods, i.e., this provides a run-off of risk pattern.
- ▷ This was shown in Röhr (2013), Merz-Wüthrich (2014), Diers et al. (2016) and Gisler (2016).

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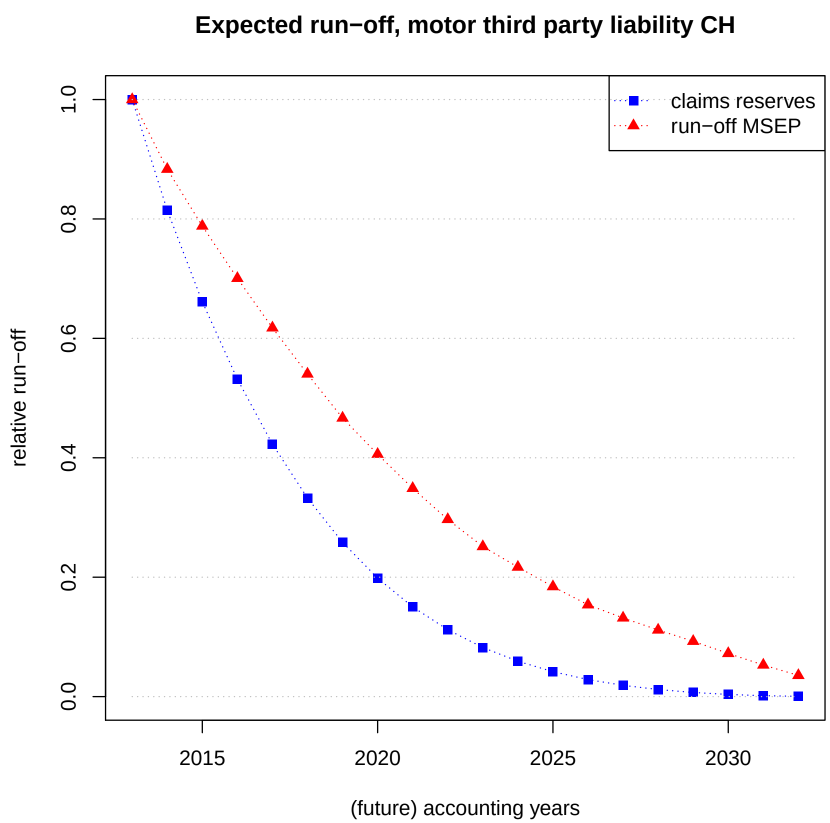
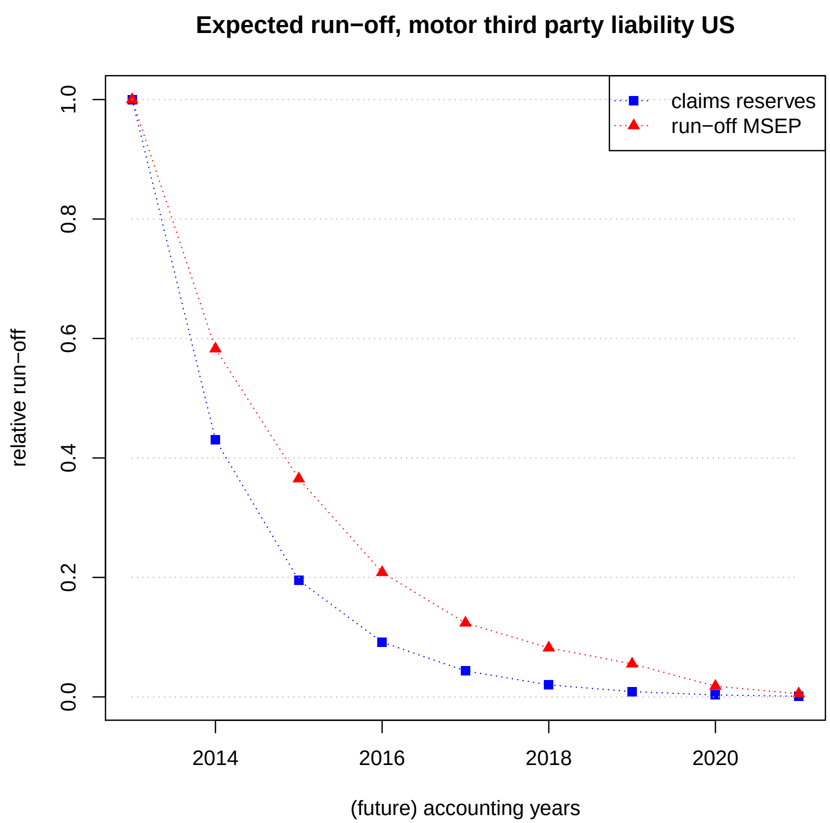
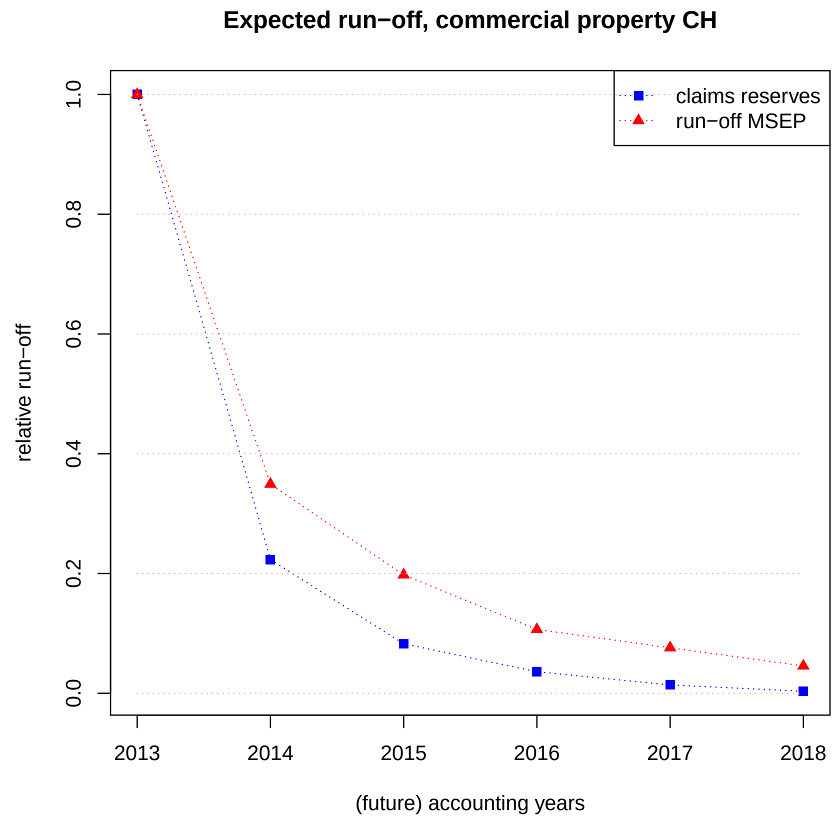
Motor third party liability: CH & US

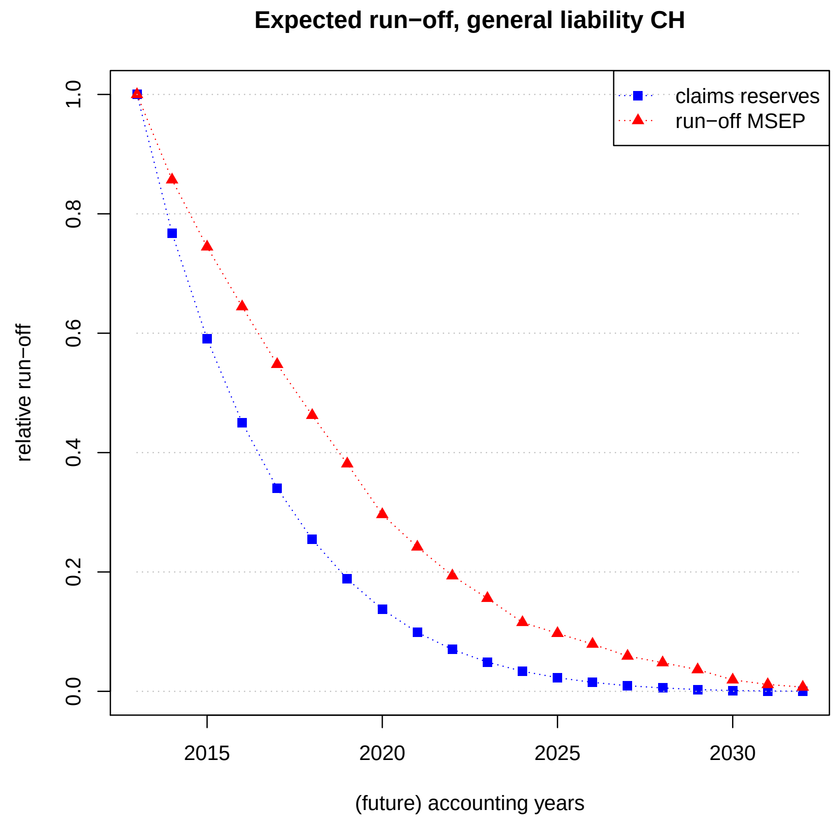




- ▷ Expected run-off of claims reserves is faster than the one of underlying risks.
- ▷ Legal environment is important for run-off.

Commercial property & general liability (CH)





▷ Different lines of business behave differently (short- and long-tailed business).

Conclusions and implementation

▷ The one-year uncertainty formula was generalized to arbitrary accounting years.

▷ This allocates the total uncertainty formula across accounting years.

▷ This improves risk margin calculations under Solvency II.

▷ Standard approximation techniques typically under-estimate run-off risk.

- CRAN R package: `ChainLadder`

⋆ Merz - Wüthrich (2014). Claims run-off uncertainty: the full picture.
SSRN Manuscript **ID 2524352**.

⋆ Wüthrich - Merz (2015). Stochastic claims reserving manual: advances in dynamic modeling.
SSRN Manuscript **ID 2649057**.