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Risk measure preserving piecewise linear approximation of empirical distributions

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Outline

- Motivation
- Risk measures and admissibility
- Approximation algorithm
- Discussion and conclusions

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Motivation

Complex stochastic models → Cannot be represented by analytic distributions → **Simulated samples**

- Review requirements
- Regulatory standards
- Interconnected internal systems

→

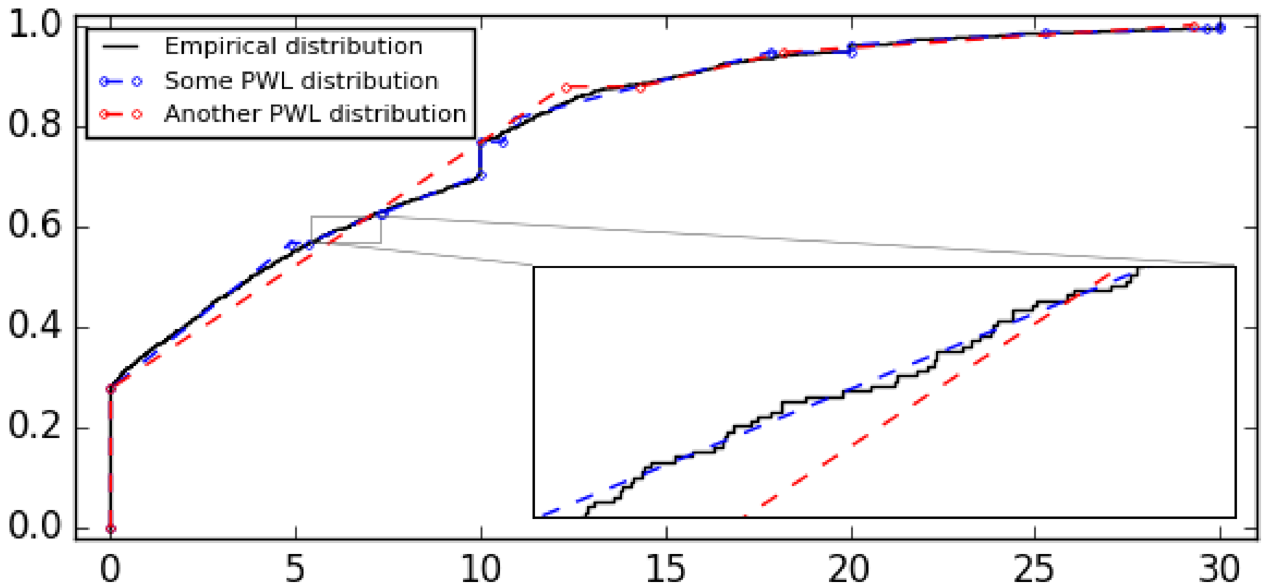
- **Storage**
- **Transfer**
- **Reuse**

Simulated data

Main idea

Problem

Approximate simulated distribution to reduce the amount of information to be storage



Solution

Approximate empirical distribution with piecewise linear distribution (PWL) that reflects its **riskiness**!

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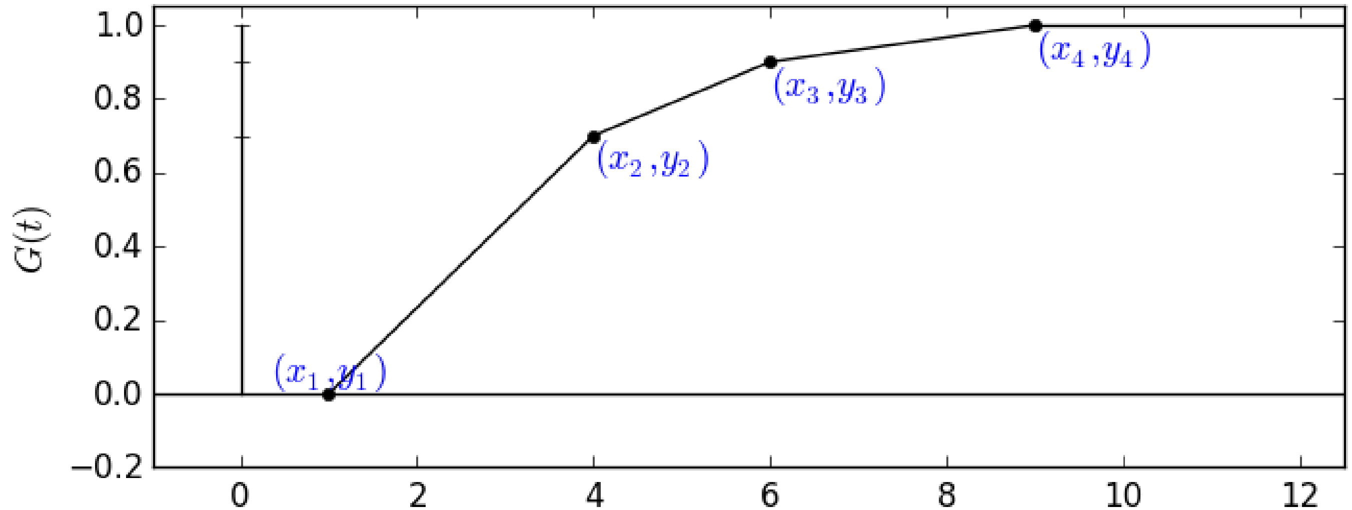
Approximation problem

F —simulation→ F_n —approximation→ G

- ***F*:** risk distribution of interest
- $F_n(t) = \frac{1}{n}\sum_{i=1}^{n} I_{\{X_i \le t\}}$: empirical sample distribution (size *n*)
- ***G***: piecewise linear approximation (*K* ordered pairs)

$$n \gg K$$

Piecewise linear distribution function



Two useful parametrizations:

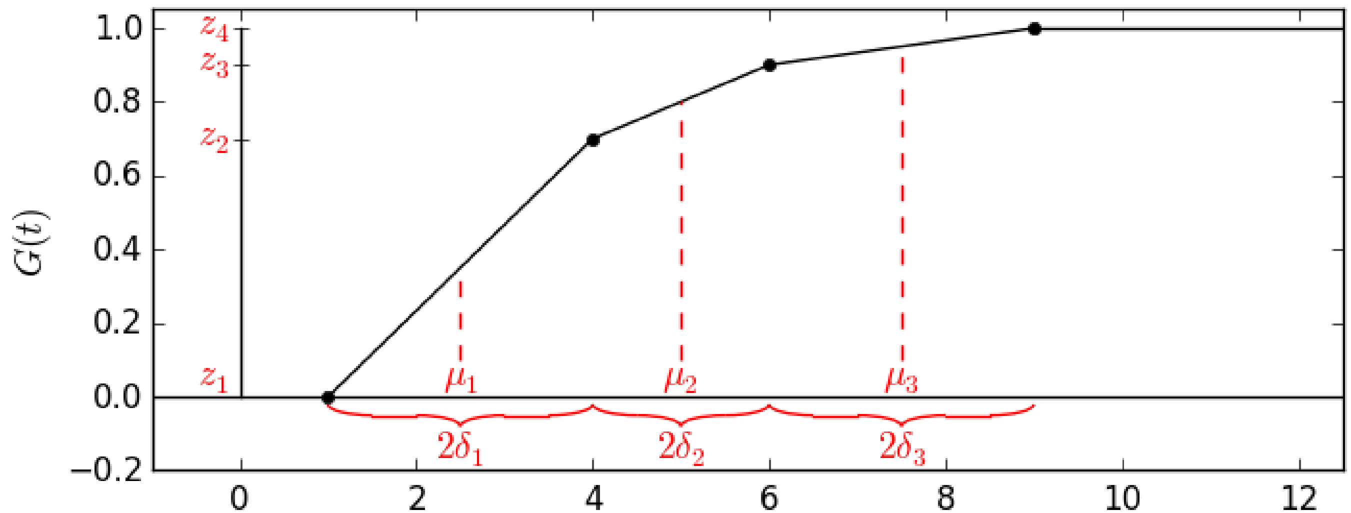
K **points:**

- x_k : abscissa of the point
- y_k : ordinate of the point

K = 4

- **x**=(1, 4, 6, 9)
- **y**=(0, 0.7, 0.9, 1)

Piecewise linear distribution function



Two useful parametrizations:

***S* segments** partition of [0,1]:

- $(z_s, z_{s+1}]$: initial and end points
- $\mu_s = G^{-1}\left(\frac{z_s+z_{s+1}}{2}\right)$ local segment mean
- δ_s : slope parameter

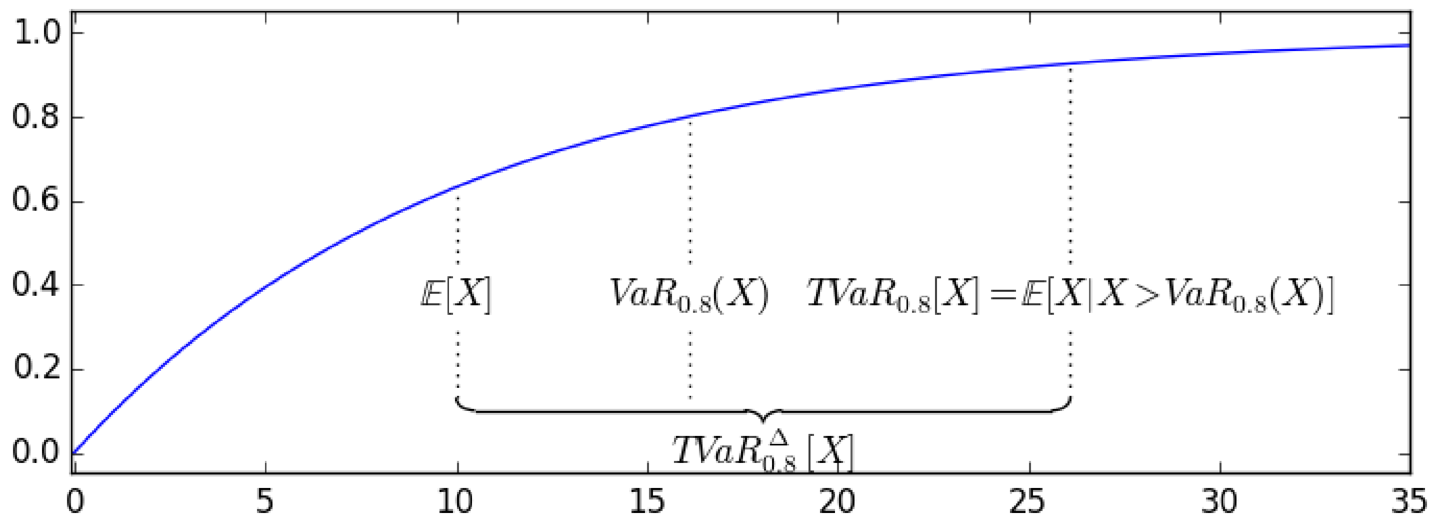
$S = 3$

- $\boldsymbol{z} = (0, 0.7, 0.9, 1)$
- $\boldsymbol{\mu} = (2.5, 5, 7.5)$
- $\boldsymbol{\delta} = (1.5, 1, 1.5)$

Tail value-at-risk (TVaR) and its deviation

For a random variable X and $\alpha \in (0,1)$:

- $\mathrm{TVa}R_\alpha[X] = E\big[X|X > \mathrm{Va}R_\alpha[X]\big]$
- $\mathrm{TVa}R^{\Delta}{}_\alpha[X] = \mathrm{TVa}R_\alpha[X] - E[X]$



Admissible approximation

G~*PWL*(**x**,**y**) is an admissible approximation of F_n with **accuracy** $\boldsymbol{\epsilon > 0}$ if:

- $E[F_n] = E[G]$

- $\dfrac{|\mathrm{TVa}R^{\Delta}{}_{\alpha}[G] - \mathrm{TVa}R^{\Delta}{}_{\alpha}[F_n]|}{\mathrm{TVa}R^{\Delta}{}_{\alpha}[F_n]} \leq \epsilon \qquad \forall \quad 0 \leq \alpha < 1$

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Approximation algorithm

Divide-and-conquer algorithm:

1. Fix $\epsilon > 0$ and initialize **z**=*(0, 1)* (or predefined quantiles **z** = *(0;* $z_1, \ldots, z_R$*,* 1).

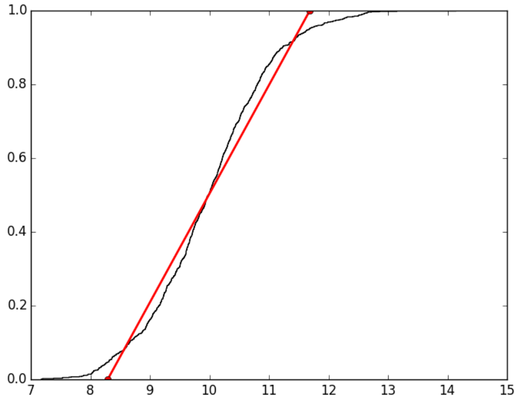
2. For every segment $(z_s, z_{s+1}] \in \boldsymbol{z}$, set $G^{-1}(p) = \mu_s + \delta_s\left(2\frac{t - z_s}{z_{s+1} - z_s} - 1\right)$, $G \sim PWL$:
   - $\mu_s = \frac{1}{n(z_{s+1} - z_s)} \sum_{i=nz_s+1}^{nz_{s+1}} X_{(i)}$
   - $\delta_s = \mathrm{argmin} \int_{z_s}^{z_{s+1}} \left(G^{-1}(p) - F_n^{\ -1}(p)\right)^2 dp$

3. In case $\boldsymbol{G}$ **is not admissible** in segment $(z_s, z_{s+1}]$:
   Bisect the segment $(z_s, z_{s+1}]$ at the point $\tilde{z}$ and insert it in vector **z**, *where*

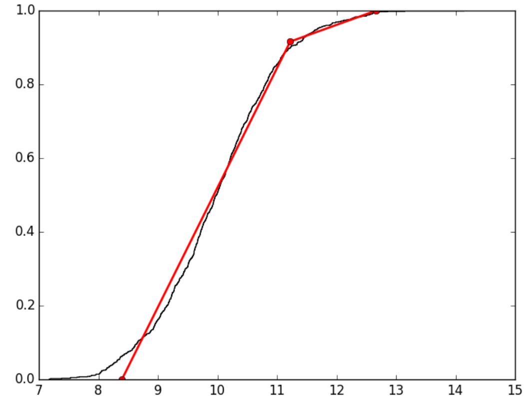
$$\tilde{z} = \underset{z_s \leq \alpha \leq z_{s+1}}{\mathrm{argmax}} \left|TVaR^{\Delta}{}_{\alpha}(G) - TVaR^{\Delta}{}_{\alpha}(F_n)\right| (1 - \alpha)$$

   return to Point **2.**

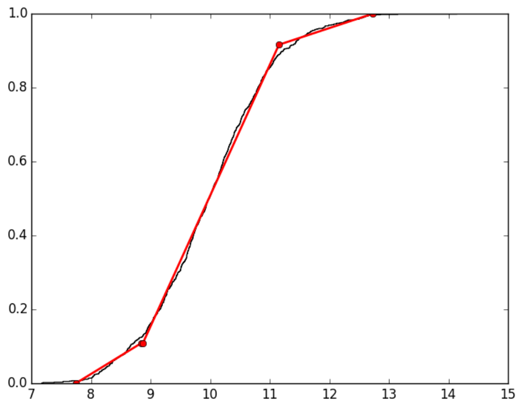
Algorithm illustration



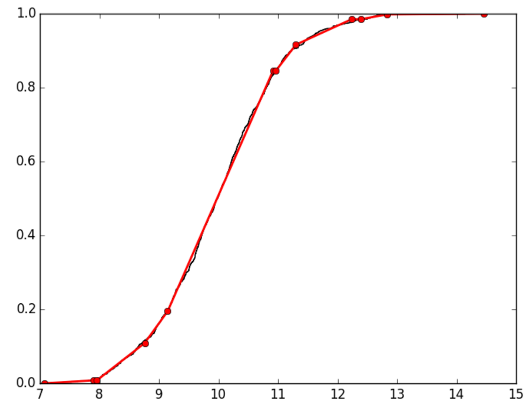
$K = 2$



$K = 3$



$K = 4$



$K = 12$ $\varepsilon = 0.1$

Performance of the algorithm

Excess-of-Loss reinsurance: Limit 10, deductible 12, aggregate limit 30

Loss distribution: Poisson($\lambda = 1$) frequency, Pareto($x_0 = 10, \alpha = 1.5$) severity

	Number of ordered pairs K			Run time in milliseconds		
	$n = 10^4$	$n = 10^5$	$n = 10^6$	$n = 10^4$	$n = 10^5$	$n = 10^6$
$\epsilon = 0.1$	11	11	11	2	14	155
$\epsilon = 0.01$	19	17	17	4	19	171
$\epsilon = 0.001$	31	27	28	5	25	192
$\epsilon = 0.0001$	62	48	45	7	35	219

Overall numerical complexity of the algorithm is $\boldsymbol{O(n\log(n/\epsilon))}$

PWL approximation experience at SCOR

- PWL approximation (with different mathematical foundation) is used in **SCOR (and Converium)** pricing system **(since Hummel 2005)**
- **Memory efficiency gains:** 6GB instead of 150TB over last 10 years

150 TB vs 6 GB

(compression ratio 25’000)

- **Infrastructure efficiency gains:** Simulations done on actuaries computers and approximations transferred to server later.

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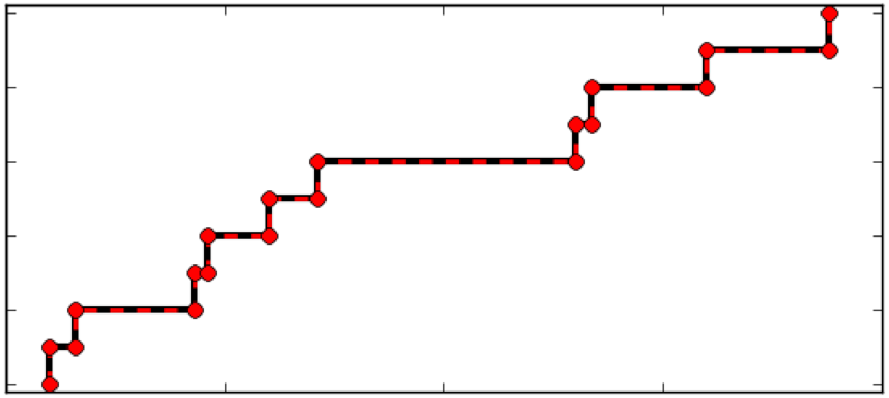
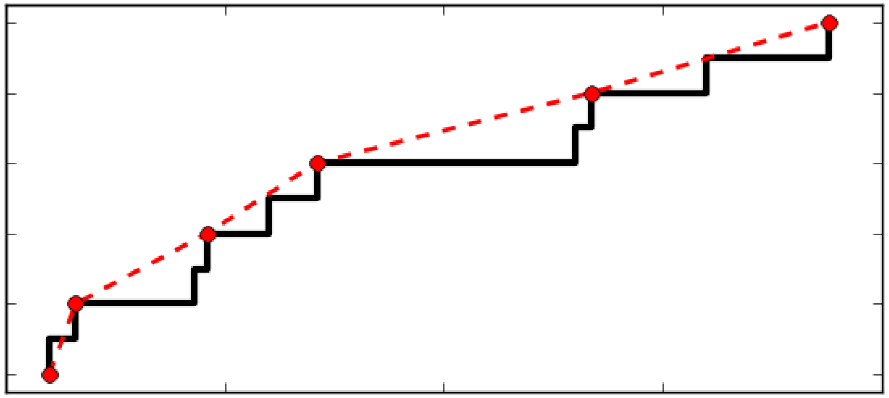
Alternative strategies

Store full sample



Store fixed set of quantiles



Store key statistics

- $\mathbb{E}[X]$
- $std(X)$
- $VaR_{99\%}(X)$

Store code and random number generator

```
rng = initialize_PRNG(seed = 1234)
sample = simulate(rng, parameters)
```

Comparison with alternative strategies

Strategy	Shape preserving	IT system independent	Memory & bandwidth efficient	Mean and risk preserving
Store full sample	✓	✓	✗	✓
Store fixed quantiles	✓	✓	✓	✗
Store key statistics	✗	✓	✓	✗
Store Code and RNG	✓	✗	✗	✓
PWL approximation	✓	✓	✓	✓

Conclusions

Piecewise linear approximation algorithm

- Approximation has the **same mean** and **preserves the shape** of the empirical distribution.
- **Relative error** of the approximation for $\mathbf{TVaR}^{\Delta}{}_{\alpha}$ **is uniformly bounded** (extendible to a set of **spectral risk measures**).
- Algorithm is efficient in terms of **run time**, **storage memory**, **and bandwidth requirements**.
- **Open source implementation** (C++, Python, R) is available.

References

- Acerbi, C. (2002). Spectral measures of risk: A coherent representation of subjective risk aversion. *Journal of Banking & Finance*, 26(7):1505-1518.
- Arbenz, P. and Guevara-Alarcón W. (2016). Risk measure preserving piecewise linear approximation of empirical distributions. *European Actuarial Journal*.
- Rockafellar, R. T. and Boyd, S. (2009). Generalized deviations in risk analysis. *Financial Stochastics*, 10(1):51-74.

Thank you!!!