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Survival Probability in the Classical Risk Model with a Franchise or a Liability Limit: Exponentially Distributed Claim Sizes

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1 Introduction

We consider the classical risk model where an insurance company has an opportunity to apply a franchise and a liability limit. Let $(\Omega, \mathfrak{F}, \mathbb{P})$ be a probability space satisfying the usual conditions, and let all the stochastic objects we use be defined on it.

In the classical risk model (see, e.g., [1–4, 7, 8]) claim sizes form a sequence $(Y_i)_{i\geq 1}$ of nonnegative i.i.d. r.v.'s with c.d.f. $F(y) = \mathbb{P}[Y_i \leq y]$ and finite expectation $\mu > 0$. The number of claims on the time interval $[0, t]$ is a Poisson process $(N_t)_{t\geq 0}$ with constant intensity $\lambda > 0$. The r.v.'s Y_i, $i \geq 1$, and the process $(N_t)_{t\geq 0}$ are mutually independent. Thus, the total claims on $[0, t]$ equal $\sum_{i=1}^{N_t} Y_i$. We set $\sum_{i=1}^{0} Y_i = 0$ if $N_t = 0$.

The insurance company has a nonnegative initial surplus x and receives premiums with constant intensity $c > 0$. In what follows, let the net profit condition hold, i.e. $c > \lambda\mu$. Moreover, we assume that the insurance company uses the expected value principle for premium calculation, which means that $c = \lambda\mu(1 + \theta)$, where $\theta > 0$ is a safety loading.

Let $X_t(x)$ be the surplus of the insurance company at time t provided that its initial surplus is x. Then the surplus process $\big(X_t(x)\big)_{t\geq 0}$ follows

$$X_t(x) = x + ct - \sum_{i=1}^{N_t} Y_i, \quad t \geq 0. \tag{1}$$

The infinite-horizon ruin probability is given by

$$\psi(x) = \mathbb{P}\big[\inf_{t\geq 0}\ X_t(x) < 0\big],$$

and the corresponding infinite-horizon survival probability equals

$$\varphi(x) = 1 - \psi(x).$$

It is well known that $\varphi(x)$ is a solution to the integro-differential equation

$$c\varphi'_+(x) = \lambda\varphi(x) - \lambda \int_0^x \varphi(x - y)\,\mathrm{d}F(y), \tag{2}$$

where $\varphi'_+(x)$ is the right derivative of $\varphi(x)$. The question concerning the differentiability of $\varphi(x)$ is investigated in [5] and [7, pp. 162–163] in different ways. Equation (2) has a unique solution such that $\lim_{x\to+\infty} \varphi(x) = 1$ provided that the net profit condition holds, otherwise ruin on an infinite horizon occurs with probability 1. Moreover, it is known that the condition $\lim_{x\to+\infty} \varphi(x) = 1$ is true for $\varphi(x)$ if and only if the condition $\varphi(0) = 1 - \lambda\mu/c$, which is equivalent to $\varphi(0) = \theta/(1 + \theta)$, holds. So if we find any solution to (2) with $\varphi(0) = \theta/(1 + \theta)$, then we can be sure that this solution is the survival probability.

If the claim sizes are exponentially distributed, then the closed form solution to (2) can be found and

$$\varphi(x) = 1 - \frac{1}{1 + \theta} \exp\left(-\frac{\theta x}{\mu(1 + \theta)}\right). \tag{3}$$

In this paper we deal with the classical risk model under the additional assumption that the insurance company uses a franchise and a liability limit. A franchise is a provision in an insurance policy whereby an insurer does not pay unless damage exceeds the franchise amount. It is applied to prevent a large number of trivial claims. A liability limit determines the maximum amount that is paid by an insurer. It is used to restrict the insurer's liability to the insured.

Let d and L be franchise and liability limit amounts, respectively. We choose them at the initial time and do not change them later. The problem of optimal control by the franchise amount is solved in [6] in the dynamic setting from viewpoint of survival probability maximization. We make the following natural assumption concerning these amounts: $0 \le d < L \le +\infty$. In particular, if $d = 0$, then a franchise is not used; if $L = +\infty$, then a liability limit is not used. Let $Y_i^{(d,\,L)}$, $i \ge 1$, denote an insurance compensation for the ith claim provided that the franchise and liability limit amounts are d and L. We let $F^{(d,\,L)}(y)$ stand for the c.d.f. of $Y_i^{(d,\,L)}$.

Normally, a franchise and a liability limit also imply reduction of insurance premiums. We suppose that the safety loading $\theta > 0$ is constant. Thus, the premium intensity is given by

$$c^{(d,\,L)} = \lambda(1+\theta)\,\mathbb{E}\big[Y_i^{(d,\,L)}\big]$$

provided that the insurance company uses the expected value principle for premium calculation.

Let $X_t^{(d,\,L)}(x)$ be the surplus of the insurance company at time t provided that its initial surplus is x, and the franchise and liability limit amounts are d and L, respectively. Then (1) for the surplus process $\big(X_t^{(d,\,L)}(x)\big)_{t\ge 0}$ can be rewritten as follows

$$X_t^{(d,\,L)}(x) = x + c^{(d,\,L)}t - \sum_{i=1}^{N_t} Y_i^{(d,\,L)}, \quad t \ge 0. \tag{4}$$

Let $\varphi^{(d,\,L)}(x)$ denote the corresponding infinite-horizon survival probability.

In what follows, we deal only with exponentially distributed claim sizes. This is one more of few cases where one succeeds in finding an analytic expression for the survival probability. It is easily seen that the c.d.f. of the insurance compensation is a sum of absolutely continuous and discrete components when the claim sizes are exponentially distributed and the insurance company uses the franchise and liability limit. That is why analytic expressions for the survival probability turn out different on certain intervals as we will show later. To get them, we apply results of [5] (see also [7, pp. 162–163]). Furthermore, we investigate how a franchise and a liability limit change the survival probability for small and large enough initial surpluses.

The rest of the paper is organized as follows. In Section 2 we consider the case where the insurance company establishes a franchise only. In Section 3 we suppose that the insurance company applies a liability limit only.

2 Survival Probability in the Classical Risk Model with a Franchise

If the insurance company establishes a franchise only and the claim sizes are exponentially distributed, then equation (2) for $\varphi^{(d,\,+\infty)}(x)$ can be written as

$$c^{(d,\,+\infty)}\big(\varphi^{(d,\,+\infty)}(x)\big)'_+ = \lambda\varphi^{(d,\,+\infty)}(x) - \lambda\int_0^x \varphi^{(d,\,+\infty)}(x-y)\,\mathrm{d}F^{(d,\,+\infty)}(y), \tag{5}$$

where

$$F^{(d,\,+\infty)}(y) = \begin{cases} 0 & \text{if} \quad y<0, \\ 1-e^{-d/\mu} & \text{if} \quad 0 \le y < d, \\ 1-e^{-y/\mu} & \text{if} \quad y \ge d, \end{cases}$$

and

$$c^{(d,\,+\infty)} = \lambda(1+\theta)\,\mathbb{E}\big[Y_i^{(d,\,+\infty)}\big] = \lambda(1+\theta)\int_d^{+\infty} \frac{y\,e^{-y/\mu}}{\mu}\,\mathrm{d}y = \lambda(1+\theta)(\mu+d)\,e^{-d/\mu}.$$

In Section 2.1 we derive analytic expressions for $\varphi^{(d,\,+\infty)}(x)$ in the case of exponentially distributed claim sizes. In Section 2.2 we investigate how a franchise changes the survival probability for small and large enough initial surpluses. Note that throughout this paper all sums equal 0 provided that their lower summation indices are greater than the upper ones.

2.1 Analytic Expression for the Survival Probability

To formulate the next theorem, introduce the constants

$$\begin{aligned} &\gamma = (1+\theta)(\mu+d), \\ &C_{1,\,1} = \frac{\theta}{1+\theta}, \\ &A_{2,\,0} = -\frac{\theta}{(1+\theta)(\gamma+\mu)}\,e^{-d/\gamma}, \\ &C_{2,\,1} = \frac{\theta}{1+\theta}\left(1+\frac{\gamma\mu+d(\gamma+\mu)}{(\gamma+\mu)^2}\,e^{-d/\gamma}\right), \\ &C_{2,\,2} = -\frac{\theta\gamma\mu}{(1+\theta)(\gamma+\mu)^2}\,e^{d/\mu}. \end{aligned}$$

Moreover, let the constants $A_{n+1,\,j}$, $0 \le j \le n-1$, be given in a recurrent way by formulas

$$A_{n+1,\,n-1} = -\frac{A_{n,\,n-2}}{n(\gamma+\mu)}\,e^{-d/\gamma}, \quad n \ge 2, \tag{6}$$

$$A_{n+1,\,j} = -\frac{1}{\gamma+\mu}\left[(j+2)\gamma\mu A_{n+1,\,j+1} + \frac{1}{j+1}\left(\sum_{i=j-1}^{n-2} A_{n,\,i}\,(-d)^{i-j+1}\binom{i+1}{j}\right)e^{-d/\gamma}\right],$$
$$1 \le j \le n-2, \quad n \ge 3, \tag{7}$$

$$A_{n+1,\,0} = -\frac{1}{\gamma+\mu}\left[2\gamma\mu A_{n+1,\,1} + \left(C_{n,\,1} + \sum_{i=0}^{n-2} A_{n,\,i}\,(-d)^{i+1}\right)e^{-d/\gamma}\right], \quad n \geq 2. \tag{8}$$

Next, let the constants $B_{n+1,\,j}$, $0 \leq j \leq n-2$, be given in a recurrent way by formulas

$$B_{3,\,0} = \frac{C_{2,\,2}}{\gamma+\mu}\,e^{d/\mu},$$

$$B_{n+1,\,n-2} = \frac{B_{n,\,n-3}}{(n-1)(\gamma+\mu)}\,e^{d/\mu}, \quad n \geq 3, \tag{9}$$

$$B_{n+1,\,j} = \frac{1}{\gamma+\mu}\left[(j+2)\gamma\mu B_{n+1,\,j+1} + \frac{1}{j+1}\left(\sum_{i=j-1}^{n-3} B_{n,\,i}\,(-d)^{i-j+1}\binom{i+1}{j}\right)e^{d/\mu}\right], \tag{10}$$
$$1 \leq j \leq n-3, \quad n \geq 4,$$

$$B_{n+1,\,0} = \frac{1}{\gamma+\mu}\left[2\gamma\mu B_{n+1,\,1} + \left(C_{n,\,2} + \sum_{i=0}^{n-3} B_{n,\,i}\,(-d)^{i+1}\right)e^{d/\mu}\right], \quad n \geq 3. \tag{11}$$

Finally, let the constants $C_{n+1,\,1}$ and $C_{n+1,\,2}$ be given by formulas

$$\begin{aligned} C_{n+1,\,1} = C_{n,\,1} &+ \frac{\gamma\mu(A_{n,\,0} - A_{n+1,\,0})}{\gamma+\mu} \\ &+ \sum_{i=0}^{n-3}\left(A_{n,\,i} - A_{n+1,\,i} + \frac{(i+2)\gamma\mu(A_{n,\,i+1} - A_{n+1,\,i+1})}{\gamma+\mu}\right)(nd)^{i+1} \\ &+ \left(A_{n,\,n-2} - A_{n+1,\,n-2} - \frac{n\gamma\mu A_{n+1,\,n-1}}{\gamma+\mu}\right)(nd)^{n-1} - A_{n+1,\,n-1}\,(nd)^{n} \\ &+ \frac{\gamma\mu}{\gamma+\mu}\left(\sum_{i=0}^{n-3}(i+1)(B_{n,\,i} - B_{n+1,\,i})(nd)^{i} - (n-1)B_{n+1,\,n-2}\,(nd)^{n-2}\right) \\ &\times \exp\left(-nd\,\frac{\gamma+\mu}{\gamma\mu}\right), \quad n \geq 2, \end{aligned} \tag{12}$$

$$\begin{aligned} C_{3,\,2} &= C_{2,\,2} + \frac{\gamma\mu B_{3,\,0}}{\gamma+\mu} - 2dB_{3,\,0} + \frac{\gamma\mu(A_{3,\,0} - A_{2,\,0} + 4dA_{3,\,1})}{\gamma+\mu}\,\exp\left(2d\,\frac{\gamma+\mu}{\gamma\mu}\right), \\ C_{n+1,\,2} = C_{n,\,2} &+ \frac{\gamma\mu(B_{n+1,\,0} - B_{n,\,0})}{\gamma+\mu} \\ &+ \sum_{i=0}^{n-4}\left(B_{n,\,i} - B_{n+1,\,i} + \frac{(i+2)\gamma\mu(B_{n+1,\,i+1} - B_{n,\,i+1})}{\gamma+\mu}\right)(nd)^{i+1} \\ &+ \left(B_{n,\,n-3} - B_{n+1,\,n-3} + \frac{(n-1)\gamma\mu B_{n+1,\,n-2}}{\gamma+\mu}\right)(nd)^{n-2} - B_{n+1,\,n-2}\,(nd)^{n-1} \\ &+ \frac{\gamma\mu}{\gamma+\mu}\left(\sum_{i=0}^{n-2}(i+1)(A_{n+1,\,i} - A_{n,\,i})(nd)^{i} + nA_{n+1,\,n-1}\,(nd)^{n-1}\right) \\ &\times \exp\left(nd\,\frac{\gamma+\mu}{\gamma\mu}\right), \quad n \geq 3. \end{aligned} \tag{13}$$

Note that to compute the constants by the formulas above for any $n \geq 2$, we have to know all the constants for $n-1$. Moreover, for any fixed $n \geq 2$, we start from the computation of $A_{n+1,j}$ for j from $n-1$ to 0, $B_{n+1,j}$ for j from $n-2$ to 0, and after that we can compute $C_{n+1,1}$ and $C_{n+1,2}$. We introduced all the constants only to formulate the next theorem. We will get them in the proof of this theorem.

Theorem 1. *Let the surplus process* $\big(X_t^{(d,+\infty)}(x)\big)_{t\geq 0}$ *follow* (4) *under the above assumptions with* $0 < d < +\infty$ *and* $L = +\infty$, *and the claim sizes be exponentially distributed with mean* μ. *Then*

$$\varphi^{(d,+\infty)}(x) = \varphi_{n+1}^{(d,+\infty)}(x) \quad \textit{for all} \quad x \in [nd, (n+1)d), \quad n \geq 0,$$

where

$$\varphi_1^{(d,+\infty)}(x) = C_{1,1}\, e^{x/\gamma}, \tag{14}$$

$$\varphi_2^{(d,+\infty)}(x) = \big(C_{2,1} + A_{2,0}\, x\big)\, e^{x/\gamma} + C_{2,2}\, e^{-x/\mu}, \tag{15}$$

$$\begin{aligned} \varphi_{n+1}^{(d,+\infty)}(x) = &\left(C_{n+1,1} + \sum_{j=0}^{n-1} A_{n+1,j}\, x^{j+1}\right) e^{x/\gamma} \\ &+ \left(C_{n+1,2} + \sum_{j=0}^{n-2} B_{n+1,j}\, x^{j+1}\right) e^{-x/\mu}, \quad n \geq 2. \end{aligned} \tag{16}$$

Proof. Substituting $c^{(d,+\infty)}$ and $F^{(d,+\infty)}(y)$ into (5) yields

$$(1+\theta)(\mu+d)\big(\varphi^{(d,+\infty)}(x)\big)'_+ = \varphi^{(d,+\infty)}(x), \quad x \in [0,d), \tag{17}$$

and

$$\begin{aligned} &\mu(1+\theta)(\mu+d)\big(\varphi^{(d,+\infty)}(x)\big)'_+ \\ &\quad = \mu\varphi^{(d,+\infty)}(x) - e^{d/\mu} \int_d^x \varphi^{(d,+\infty)}(x-y)\, e^{-y/\mu}\, \mathrm{d}y\,, \quad x \in [d,+\infty). \end{aligned} \tag{18}$$

By the results of [5], $\varphi^{(d,+\infty)}(x)$ is continuously differentiable on $\mathbb{R}_+$ (see also [7, pp. 162–163]). Let us introduce the functions $\varphi_{n+1}^{(d,+\infty)}(x)$, $n \geq 0$, in the following way: $\varphi_{n+1}^{(d,+\infty)}(x)$ is defined on $[nd, (n+1)d)$ and coincides with $\varphi^{(d,+\infty)}(x)$ on this interval.

Set $\gamma = (1+\theta)(\mu+d)$. Solving (17) gives (14), where the constant $C_{1,1}$ can be found from $\varphi_1^{(d,+\infty)}(0) = \theta/(1+\theta)$, which guarantees us that the solution is the survival probability. Thus, $C_{1,1} = \theta/(1+\theta)$.

By (18), the function $\varphi^{(d,+\infty)}(x)$ is easily seen to have the second derivative, which is continuous. Moreover, integro-differential equation (18) can be reduced to the differential one

$$\begin{aligned} &\gamma\mu\big(\varphi^{(d,+\infty)}(x)\big)'' + (\gamma-\mu)\big(\varphi^{(d,+\infty)}(x)\big)' - \varphi^{(d,+\infty)}(x) \\ &\quad = -\varphi^{(d,+\infty)}(x-d), \quad x \in [d,+\infty), \end{aligned} \tag{19}$$

in a standard way (see, e.g., [1–4, 7]).

If $x \in [d, 2d)$, then $0 \le x - d < d$ and the right-hand side of (19) has already been found. So we only need to solve the linear differential equation

$$\gamma\mu\big(\varphi_2^{(d,+\infty)}(x)\big)'' + (\gamma-\mu)\big(\varphi_2^{(d,+\infty)}(x)\big)' - \varphi_2^{(d,+\infty)}(x) = -\frac{\theta}{1+\theta}\, e^{(x-d)/\gamma} \tag{20}$$

on $[d, 2d)$ by standard techniques. Furthermore, since $\varphi^{(d,+\infty)}(x)$ and its first derivative are continuous at the point $x = d$, the following conditions must hold to guarantee that the solution is the survival probability:

$$\begin{cases} \lim\limits_{x\uparrow d} \varphi_1^{(d,+\infty)}(x) = \varphi_2^{(d,+\infty)}(d), \\ \lim\limits_{x\uparrow d} \big(\varphi_1^{(d,+\infty)}(x)\big)' = \big(\varphi_2^{(d,+\infty)}(d)\big)'. \end{cases} \tag{21}$$

The solution to (20) can be found by writing

$$\varphi_2^{(d,+\infty)}(x) = \varphi_{2,\,\mathrm{gen}}^{(d,+\infty)}(x) + \varphi_{2,\,\mathrm{part1}}^{(d,+\infty)}(x),$$

where $\varphi_{2,\,\mathrm{gen}}^{(d,+\infty)}(x)$ is a general solution to the linear homogeneous equation corresponding to (20), and $\varphi_{2,\,\mathrm{part1}}^{(d,+\infty)}(x)$ is a particular solution to linear heterogeneous equation (20).

Note that $\varphi_{2,\,\mathrm{part1}}^{(d,+\infty)}(x)$ can be written as

$$\varphi_{2,\,\mathrm{part1}}^{(d,+\infty)}(x) = A_{2,0}\, x\, e^{x/\gamma},$$

where $A_{2,0}$ is a constant. Substituting $\varphi_{2,\,\mathrm{part1}}^{(d,+\infty)}(x)$ into (20) and applying the method of undetermined coefficients yield $A_{2,0}$. Moreover, we can write $\varphi_{2,\,\mathrm{gen}}^{(d,+\infty)}(x)$ as

$$\varphi_{2,\,\mathrm{gen}}^{(d,+\infty)}(x) = C_{2,1}\, e^{x/\gamma} + C_{2,2}\, e^{-x/\mu},$$

where the constants $C_{2,1}$ and $C_{2,2}$ are such that $\varphi_2^{(d,+\infty)}(x)$ satisfies the conditions (21).

Thus, $\varphi_2^{(d,+\infty)}(x)$ is defined by (15) and the constants $A_{2,0}$, $C_{2,1}$, and $C_{2,2}$ are given before the assertion of the theorem. It is easy to check that the second classical derivative of $\varphi^{(d,+\infty)}(x)$ does not exist at the point $x = d$.

If $x \in [2d, 3d)$, then $d \le x - d < 2d$ and the right-hand side of (19) has already been found. Therefore, $\varphi_3^{(d,+\infty)}(x)$ is a solution to

$$\begin{aligned} &\gamma\mu\big(\varphi_3^{(d,+\infty)}(x)\big)'' + (\gamma-\mu)\big(\varphi_3^{(d,+\infty)}(x)\big)' - \varphi_3^{(d,+\infty)}(x) \\ &\quad = -\frac{\theta}{1+\theta}\left(1 + \frac{\gamma\mu + d(\gamma+\mu)}{(\gamma+\mu)^2}\, e^{-d/\gamma} - \frac{1}{\gamma+\mu}\, e^{-d/\gamma}\,(x-d)\right) e^{(x-d)/\gamma} \\ &\qquad + \frac{\theta\mu(\mu+d)}{(\gamma+\mu)^2}\, e^{2d/\mu}\, e^{-x/\mu}. \end{aligned} \tag{22}$$

Furthermore, the continuity of $\varphi^{(d,+\infty)}(x)$ and its derivative at the point $x = 2d$ implies

$$\begin{cases} \lim\limits_{x\uparrow 2d} \varphi_2^{(d,+\infty)}(x) = \varphi_3^{(d,+\infty)}(2d), \\ \lim\limits_{x\uparrow 2d} \big(\varphi_2^{(d,+\infty)}(x)\big)' = \big(\varphi_3^{(d,+\infty)}(2d)\big)'. \end{cases} \tag{23}$$

We can write $\varphi_3^{(d,+\infty)}(x)$ as

$$\varphi_3^{(d,+\infty)}(x) = \varphi_{3,\,\mathrm{gen}}^{(d,+\infty)}(x) + \varphi_{3,\,\mathrm{part1}}^{(d,+\infty)}(x) + \varphi_{3,\,\mathrm{part2}}^{(d,+\infty)}(x),$$

where $\varphi_{3,\,\mathrm{gen}}^{(d,+\infty)}(x)$ is a general solution to the linear homogeneous equation corresponding to (22), and $\varphi_{3,\,\mathrm{part1}}^{(d,+\infty)}(x)$ and $\varphi_{3,\,\mathrm{part2}}^{(d,+\infty)}(x)$ are particular solutions to the equations

$$\begin{aligned}&\gamma\mu\big(\varphi_3^{(d,+\infty)}(x)\big)'' + (\gamma-\mu)\big(\varphi_3^{(d,+\infty)}(x)\big)' - \varphi_3^{(d,+\infty)}(x)\\ &\quad = -\frac{\theta}{1+\theta}\left(1 + \frac{\gamma\mu + d(\gamma+\mu)}{(\gamma+\mu)^2}\, e^{-d/\gamma} - \frac{1}{\gamma+\mu}\, e^{-d/\gamma}\,(x-d)\right) e^{(x-d)/\gamma}\end{aligned} \tag{24}$$

and

$$\begin{aligned}&\gamma\mu\big(\varphi_3^{(d,+\infty)}(x)\big)'' + (\gamma-\mu)\big(\varphi_3^{(d,+\infty)}(x)\big)' - \varphi_3^{(d,+\infty)}(x)\\ &\quad = \frac{\theta\mu(\mu+d)}{(\gamma+\mu)^2}\, e^{2d/\mu}\, e^{-x/\mu},\end{aligned} \tag{25}$$

respectively.

Next, $\varphi_{3,\,\mathrm{part1}}^{(d,+\infty)}(x)$ and $\varphi_{3,\,\mathrm{part2}}^{(d,+\infty)}(x)$ can be written as

$$\varphi_{3,\,\mathrm{part1}}^{(d,+\infty)}(x) = (A_{3,0} + A_{3,1}\,x)\,x\,e^{x/\gamma}$$

and

$$\varphi_{3,\,\mathrm{part2}}^{(d,+\infty)}(x) = B_{3,0}\,x\,e^{-x/\mu},$$

where $A_{3,0}$, $A_{3,1}$, and $B_{3,0}$ are constants. Substituting $\varphi_{3,\,\mathrm{part1}}^{(d,+\infty)}(x)$ and $\varphi_{3,\,\mathrm{part2}}^{(d,+\infty)}(x)$ into (24) and (25), respectively, and applying the method of undetermined coefficients yield $A_{3,0}$, $A_{3,1}$, and $B_{3,0}$. Moreover, we can write $\varphi_{3,\,\mathrm{gen}}^{(d,+\infty)}(x)$ as

$$\varphi_{3,\,\mathrm{gen}}^{(d,+\infty)}(x) = C_{3,1}\,e^{x/\gamma} + C_{3,2}\,e^{-x/\mu},$$

where the constants $C_{3,1}$ and $C_{3,2}$ are such that $\varphi_3^{(d,+\infty)}(x)$ satisfies the conditions (23), which guarantees us that the solution is the survival probability.

All the constants are given before the assertion of the theorem. In particular, $A_{3,1}$, $A_{3,0}$, and $C_{3,1}$ are given by (6), (8), and (12), respectively, with $n = 2$.

In the general case, if we know $\varphi_n^{(d,+\infty)}(x)$, $n \ge 2$, we can find $\varphi_{n+1}^{(d,+\infty)}(x)$ solving the equation

$$\gamma\mu\big(\varphi_{n+1}^{(d,+\infty)}(x)\big)'' + (\gamma-\mu)\big(\varphi_{n+1}^{(d,+\infty)}(x)\big)' - \varphi_{n+1}^{(d,+\infty)}(x) = -\varphi_n^{(d,+\infty)}(x-d). \tag{26}$$

By the above, applying an induction argument yields

$$\varphi_n^{(d,+\infty)}(x) = \varphi_{n,\,\mathrm{gen}}^{(d,+\infty)}(x) + \varphi_{n,\,\mathrm{part1}}^{(d,+\infty)}(x) + \varphi_{n,\,\mathrm{part2}}^{(d,+\infty)}(x), \tag{27}$$

where

$$\varphi_{n,\,\mathrm{gen}}^{(d,+\infty)}(x) = C_{n,1}\,e^{x/\gamma} + C_{n,2}\,e^{-x/\mu},$$

$$\varphi_{n,\,\mathrm{part1}}^{(d,\,+\infty)}(x) = \left(\sum_{i=0}^{n-2} A_{n,\,i}\, x^i\right) x\, e^{x/\gamma},$$

$$\varphi_{n,\,\mathrm{part2}}^{(d,\,+\infty)}(x) = \left(\sum_{i=0}^{n-3} B_{n,\,i}\, x^i\right) x\, e^{-x/\mu}.$$

Thus, (27) can be rewritten as

$$\varphi_n^{(d,\,+\infty)}(x) = \left(C_{n,\,1} + \sum_{i=0}^{n-2} A_{n,\,i}\, x^{i+1}\right) e^{x/\gamma} + \left(C_{n,\,2} + \sum_{i=0}^{n-3} B_{n,\,i}\, x^{i+1}\right) e^{-x/\mu}. \tag{28}$$

Therefore, we get

$$\varphi_{n+1}^{(d,\,+\infty)}(x) = \varphi_{n+1,\,\mathrm{gen}}^{(d,\,+\infty)}(x) + \varphi_{n+1,\,\mathrm{part1}}^{(d,\,+\infty)}(x) + \varphi_{n+1,\,\mathrm{part2}}^{(d,\,+\infty)}(x),$$

where

$$\varphi_{n+1,\,\mathrm{gen}}^{(d,\,+\infty)}(x) = C_{n+1,\,1}\, e^{x/\gamma} + C_{n+1,\,2}\, e^{-x/\mu},$$

$$\varphi_{n+1,\,\mathrm{part1}}^{(d,\,+\infty)}(x) = \left(\sum_{i=0}^{n-1} A_{n+1,\,i}\, x^i\right) x\, e^{x/\gamma}, \tag{29}$$

$$\varphi_{n+1,\,\mathrm{part2}}^{(d,\,+\infty)}(x) = \left(\sum_{i=0}^{n-2} B_{n+1,\,i}\, x^i\right) x\, e^{-x/\mu}, \tag{30}$$

which gives (16).

We now assume that we have already found $\varphi_n^{(d,\,+\infty)}(x)$, $n \geq 2$, which means that we know all the constants in (28). We will now derive formulas to find the constants in (16).

If $x \in [nd, (n+1)d)$, then $(n-1)d \leq x - d < nd$ and

$$\begin{aligned} -\varphi_n^{(d,\,+\infty)}(x-d) = &-\left(C_{n,\,1} + \sum_{i=0}^{n-2} A_{n,\,i}\, (x-d)^{i+1}\right) e^{(x-d)/\gamma} \\ &-\left(C_{n,\,2} + \sum_{i=0}^{n-3} B_{n,\,i}\, (x-d)^{i+1}\right) e^{-(x-d)/\mu}. \end{aligned} \tag{31}$$

Applying the binomial theorem in (31) yields

$$\begin{aligned} &-\varphi_n^{(d,\,+\infty)}(x-d) \\ &\quad = -\left[C_{n,\,1} + \sum_{i=0}^{n-2} A_{n,\,i} \left(\sum_{j=0}^{i+1} \binom{i+1}{j} x^j\, (-d)^{i-j+1}\right)\right] e^{(x-d)/\gamma} \\ &\qquad -\left[C_{n,\,2} + \sum_{i=0}^{n-3} B_{n,\,i} \left(\sum_{j=0}^{i+1} \binom{i+1}{j} x^j\, (-d)^{i-j+1}\right)\right] e^{-(x-d)/\mu}. \end{aligned} \tag{32}$$

Interchanging the order of summation in (32) gives

$$
\begin{aligned}
&-\varphi_n^{(d,+\infty)}(x-d)\\
&= -\left[\left(C_{n,1}+\sum_{i=0}^{n-2} A_{n,i}\,(-d)^{i+1}\right)+\sum_{j=1}^{n-1}\left(\sum_{i=j-1}^{n-2} A_{n,i}\,(-d)^{i-j+1}\binom{i+1}{j}\right)x^j\right]e^{(x-d)/\gamma}\\
&\quad-\left[\left(C_{n,2}+\sum_{i=0}^{n-3} B_{n,i}\,(-d)^{i+1}\right)+\sum_{j=1}^{n-2}\left(\sum_{i=j-1}^{n-3} B_{n,i}\,(-d)^{i-j+1}\binom{i+1}{j}\right)x^j\right]e^{-(x-d)/\mu}.
\end{aligned}
\tag{33}
$$

To derive formulas for $A_{n+1,j}$, $0 \le j \le n-1$, we calculate $\left(\varphi_{n+1,\,\mathrm{part1}}^{(d,+\infty)}(x)\right)'$ and $\left(\varphi_{n+1,\,\mathrm{part1}}^{(d,+\infty)}(x)\right)''$ using (29), and substitute them, as well as (33), into (26). Thus, we get

$$
\begin{aligned}
&\mu\Bigg[(2A_{n+1,0}+2\gamma A_{n+1,1})\\
&\quad+\sum_{j=1}^{n-2}\left(\frac{A_{n+1,j-1}}{\gamma}+2(j+1)A_{n+1,j}+(j+2)(j+1)\gamma A_{n+1,j+1}\right)x^j\\
&\quad+\left(\frac{A_{n+1,n-2}}{\gamma}+2nA_{n+1,n-1}\right)x^{n-1}+\frac{A_{n+1,n-1}}{\gamma}\,x^n\Bigg]+(\gamma-\mu)\\
&\quad\times\left[A_{n+1,0}+\sum_{j=1}^{n-1}\left(\frac{A_{n+1,j-1}}{\gamma}+(j+1)A_{n+1,j}\right)x^j+\frac{A_{n+1,n-1}}{\gamma}\,x^n\right]\\
&\quad-\sum_{j=1}^{n} A_{n+1,j-1}\,x^j = -\Bigg[\left(C_{n,1}+\sum_{i=0}^{n-2} A_{n,i}\,(-d)^{i+1}\right)\\
&\quad+\sum_{j=1}^{n-1}\left(\sum_{i=j-1}^{n-2} A_{n,i}\,(-d)^{i-j+1}\binom{i+1}{j}\right)x^j\Bigg]e^{-d/\gamma}.
\end{aligned}
\tag{34}
$$

To find $A_{n+1,j}$, $0 \le j \le n-1$, from (34), we apply the method of undetermined coefficients. To be more precise, we equate the expressions of x^{n-1}, x^j, $1 \le j \le n-2$, and x^0 in (34), and get (6), (7), and (8), respectively.

Thus, to find $A_{n+1,n-1}$, it suffices to know $A_{n,n-2}$. The constants $A_{n+1,j}$, $1 \le j \le n-2$, are expressed in a recurrent way in terms of $A_{n+1,j+1}$ and $A_{n,i}$, $j-1 \le i \le n-2$. Finally, $A_{n+1,0}$ can be found in terms of $A_{n+1,1}$, $C_{n,1}$, and $A_{n,i}$, $0 \le i \le n-2$.

Similarly, to derive formulas for $B_{n+1,j}$, $0 \le j \le n-2$, we calculate $\left(\varphi_{n+1,\,\mathrm{part2}}^{(d,+\infty)}(x)\right)'$ and $\left(\varphi_{n+1,\,\mathrm{part2}}^{(d,+\infty)}(x)\right)''$ using (30), and substitute them, as well as (34), into (26). Thus, we obtain

$$
\begin{aligned}
&\gamma\Bigg[(-2B_{n+1,0}+2\mu B_{n+1,1})\\
&+\sum_{j=1}^{n-3}\left(\frac{B_{n+1,j-1}}{\mu}-2(j+1)B_{n+1,j}+(j+2)(j+1)\mu B_{n+1,j+1}\right)x^j\\
&+\left(\frac{B_{n+1,n-3}}{\mu}-2(n-1)B_{n+1,n-2}\right)x^{n-2}+\frac{B_{n+1,n-2}}{\mu}\,x^{n-1}\Bigg]+(\gamma-\mu)\\
&\times\left[B_{n+1,0}+\sum_{j=1}^{n-2}\left(-\frac{B_{n+1,j-1}}{\mu}+(j+1)B_{n+1,j}\right)x^j-\frac{B_{n+1,n-2}}{\mu}\,x^{n-1}\right]\\
&-\sum_{j=1}^{n-1}B_{n+1,j-1}\,x^j=-\Bigg[\left(C_{n,2}+\sum_{i=0}^{n-3}B_{n,i}\,(-d)^{i+1}\right)\\
&+\sum_{j=1}^{n-2}\left(\sum_{i=j-1}^{n-3}B_{n,i}\,(-d)^{i-j+1}\binom{i+1}{j}\right)x^j\Bigg]\,e^{d/\mu}.
\end{aligned}
\tag{35}
$$

Equating the expressions of x^{n-2}, x^j, $1\le j\le n-3$, and x^0 in (35) yields (9), (10), and (11), respectively.

Since the continuity of $\varphi^{(d,+\infty)}(x)$ and its derivative at the point $x=nd$ implies

$$
\begin{cases}
\lim\limits_{x\uparrow nd}\varphi_n^{(d,+\infty)}(x)=\varphi_{n+1}^{(d,+\infty)}(nd),\\
\lim\limits_{x\uparrow nd}\big(\varphi_n^{(d,+\infty)}(x)\big)'=\big(\varphi_{n+1}^{(d,+\infty)}(nd)\big)',
\end{cases}
$$

the constants $C_{n+1,1}$ and $C_{n+1,2}$ can be found by solving the system of linear equations

$$
\begin{aligned}
&\left(C_{n+1,1}+\sum_{i=0}^{n-1}A_{n+1,i}\,(nd)^{i+1}\right)e^{nd/\gamma}+\left(C_{n+1,2}+\sum_{i=0}^{n-2}B_{n+1,i}\,(nd)^{i+1}\right)e^{-nd/\mu}\\
&=\left(C_{n,1}+\sum_{i=0}^{n-2}A_{n,i}\,(nd)^{i+1}\right)e^{nd/\gamma}+\left(C_{n,2}+\sum_{i=0}^{n-3}B_{n,i}\,(nd)^{i+1}\right)e^{-nd/\mu}
\end{aligned}
$$

and

$$\begin{aligned}&\Bigg[\frac{C_{n+1,\,1}}{\gamma}+A_{n+1,\,0}+\sum_{i=0}^{n-2}\bigg(\frac{A_{n+1,\,i}}{\gamma}+(i+2)A_{n+1,\,i+1}\bigg)(nd)^{i+1}\\&\quad+\frac{A_{n+1,\,n-1}}{\gamma}(nd)^{n}\Bigg]e^{nd/\gamma}+\Bigg[-\frac{C_{n+1,\,2}}{\mu}+B_{n+1,\,0}\\&\quad+\sum_{i=0}^{n-3}\bigg(-\frac{B_{n+1,\,i}}{\mu}+(i+2)B_{n+1,\,i+1}\bigg)(nd)^{i+1}-\frac{B_{n+1,\,n-2}}{\mu}(nd)^{n-1}\Bigg]e^{-nd/\mu}\\&=\Bigg[\frac{C_{n,\,1}}{\gamma}+A_{n,\,0}+\sum_{i=0}^{n-3}\bigg(\frac{A_{n,\,i}}{\gamma}+(i+2)A_{n,\,i+1}\bigg)(nd)^{i+1}\\&\quad+\frac{A_{n,\,n-2}}{\gamma}(nd)^{n-1}\Bigg]e^{nd/\gamma}+\Bigg[-\frac{C_{n,\,2}}{\mu}+B_{n,\,0}\\&\quad+\sum_{i=0}^{n-4}\bigg(-\frac{B_{n,\,i}}{\mu}+(i+2)B_{n,\,i+1}\bigg)(nd)^{i+1}-\frac{B_{n,\,n-3}}{\mu}(nd)^{n-2}\Bigg]e^{-nd/\mu},\end{aligned}$$

which guarantees us that the solution is the survival probability. Thus, $C_{n+1,\,1}$ and $C_{n+1,\,2}$ are given by (12) and (13), which completes the proof. □

Remark 1. Theorem 1 gives the analytic expression for $\varphi^{(d,\,+\infty)}(x)$ for all $x \geq 0$. However, the computations become too tedious for large initial surpluses. So this theorem is useful when the initial surpluses are not too large, otherwise it is reasonable to use the Cramér-Lundberg approximation (see, e.g., [1–4, 7, 8]).

2.2 Case of Small and Large Enough Initial Surpluses

We now investigate how a franchise changes the survival probability for small and large enough initial surpluses.

Theorem 2. *Let the surplus processes $\big(X_t(x)\big)_{t\geq 0}$ and $\big(X_t^{(d,\,+\infty)}(x)\big)_{t\geq 0}$ follow* (1) *and* (4)*, respectively, under the above assumptions with $0 < d < +\infty$ and $L = +\infty$. Moreover, let $\varphi(x)$ and $\varphi^{(d,\,+\infty)}(x)$ be the corresponding survival probabilities, and the claim sizes be exponentially distributed with mean μ.*

(i) If

$$x \in \left[0,\ \min\left\{\frac{\mu(1+\theta)}{\theta}\ln\left(1+\frac{\theta d}{\mu(1+\theta)}\right),\ d\right\}\right], \tag{36}$$

then $\varphi^{(d,\,+\infty)}(x) < \varphi(x)$ for any $0 < d < +\infty$.

(ii) For

$$d \in \left(0,\ \frac{\mu(1+\theta)\ \ln(1+\theta)}{\theta}\right) \tag{37}$$

and large enough initial surpluses, we have $\varphi^{(d,\,+\infty)}(x) > \varphi(x)$.

Proof. We now prove assertion (i) of the theorem. Introduce the function

$$g_1(x) = \frac{\varphi(x)}{\varphi^{(d,\,+\infty)}(x)} \tag{38}$$

for $x \in [0, d]$. Substituting (3) and (14) into (38) yields

$$g_1(x) = \frac{1 + \theta - \exp\left(-\frac{\theta x}{\mu(1+\theta)}\right)}{\theta} \exp\left(-\frac{x}{(1+\theta)(\mu+d)}\right).$$

Taking the derivative gives

$$g_1'(x) = \frac{1}{\theta(\mu+d)} \exp\left(-\frac{x}{(1+\theta)(\mu+d)}\right) \times \left(\frac{\mu(1+\theta)+\theta d}{\mu(1+\theta)} \exp\left(-\frac{\theta x}{\mu(1+\theta)}\right) - 1\right).$$

Since

$$\frac{\mu(1+\theta)+\theta d}{\mu(1+\theta)} \exp\left(-\frac{\theta x}{\mu(1+\theta)}\right) - 1 > 0$$

for all x given by (36), we have $g_1'(x) > 0$ for these x. Furthermore, $g_1(0) = 1$, which gives $g_1(x) > 1$ for these x. Thus, assertion (i) of the theorem follows, and we now prove assertion (ii).

By the Cramér-Lundberg approximation (see, e.g., [1–4, 7, 8]), we have

$$\varphi^{(d,\,+\infty)}(x) \sim 1 - \frac{\theta}{(1+\theta)\, R^{(d,\,+\infty)}\, \mu^{(d,\,+\infty)}}\, e^{-R^{(d,\,+\infty)}\, x}, \tag{39}$$

where $R^{(d,\,+\infty)}$ is a unique positive solution (if it exists) to

$$\int_0^{+\infty} e^{R^{(d,\,+\infty)}\, y} \left(1 - F^{(d,\,+\infty)}(y)\right) \mathrm{d}y = (1+\theta)(\mu+d)\, e^{-d/\mu}, \tag{40}$$

and

$$\mu^{(d,\,+\infty)} = \frac{1}{(1+\theta)(\mu+d)\, e^{-d/\mu}} \int_0^{+\infty} y\, e^{R^{(d,\,+\infty)}\, y} \left(1 - F^{(d,\,+\infty)}(y)\right) \mathrm{d}y, \tag{41}$$

if the improper integral in the right-hand side of (41) is finite.

First, we show that there is a unique positive solution $R^{(d,\,+\infty)}$ to (40) such that the integral in the right-hand side of (41) is finite. If $R^{(d,\,+\infty)} < 1/\mu$, then substituting $F^{(d,\,+\infty)}(y)$ into (40) and doing elementary computations yield

$$e^{dR^{(d,\,+\infty)}} = (1+\theta)(\mu+d)\, R^{(d,\,+\infty)} + 2 + \frac{1}{\mu R^{(d,\,+\infty)} - 1}. \tag{42}$$

If $R^{(d,\,+\infty)} \geq 1/\mu$, then the integral in the left-hand side of (40) is infinite. So in what follows, we consider the case $R^{(d,\,+\infty)} < 1/\mu$ only.

For $R \in [0, 1/\mu)$, introduce the functions

$$g_2(R) = e^{dR}$$

and

$$g_3(R) = (1+\theta)(\mu+d)\, R + 2 + \frac{1}{\mu R - 1}.$$

Note that $g_2(0) = 1$ and $g_3(0) = 1$. Moreover, we have

$$g_2(R) \approx 1 + dR$$

and

$$g_3(R) \approx 1 + g_3'(0)\, R \approx 1 + (d + \theta d + \theta\mu)\, R$$

for small enough $R > 0$. Consequently, $g_3(R) > g_2(R)$ in some right semi-neighbourhood of the point $R = 0$.

The function $g_2(R)$ is increasing on $[0, 1/\mu)$. The function $g_3(R)$ is increasing on $[0, R^*)$ and decreasing on $(R^*, 1/\mu)$, where

$$R^* = \frac{1}{\mu}\left(1 - \sqrt{\frac{\mu}{(1+\theta)(\mu+d)}}\right).$$

Thus, (42) has the unique solution $R^{(d,+\infty)}$ on $(0, 1/\mu)$. Note that $R^{(d,+\infty)}$ is the unique positive solution to (40). It is evident that $\mu^{(d,+\infty)}$ is finite in this case.

Next, from (3) and (39) we conclude that $\varphi^{(d,+\infty)}(x) > \varphi(x)$ for large enough initial surpluses provided that

$$R^{(d,+\infty)} > \frac{\theta}{\mu(1+\theta)}.$$

Let $g_4(R) = g_2(R) - g_3(R)$ on $[0, 1/\mu)$. The function $g_4(R)$ is negative on $(0, R^{(d,+\infty)})$ and positive on $(R^{(d,+\infty)}, 1/\mu)$. Moreover, $g_4(0) = g_4(R^{(d,+\infty)}) = 0$ and

$$g_4\left(\frac{\theta}{\mu(1+\theta)}\right) = \exp\left(\frac{\theta d}{\mu(1+\theta)}\right) - \frac{\theta d}{\mu} - 1. \tag{43}$$

Therefore, $\varphi^{(d,+\infty)}(x) > \varphi(x)$ for large enough initial surpluses provided that the expression in the right-hand side of (43) is negative.

Let

$$g_5(d) = \exp\left(\frac{\theta d}{\mu(1+\theta)}\right) - \frac{\theta d}{\mu} - 1.$$

Taking the derivative yields

$$g_5'(d) = \frac{\theta}{\mu}\left(\frac{1}{1+\theta}\exp\left(\frac{\theta d}{\mu(1+\theta)}\right) - 1\right).$$

Since $g_5(0) = 0$ and $g_5'(d) < 0$ for d given by (37), we prove assertion (ii) of the theorem. □

3 Survival Probability in the Classical Risk Model with a Liability Limit

If the insurance company establishes a liability limit only and the claim sizes are exponentially distributed, then equation (2) for $\varphi^{(0,\,L)}(x)$ can be written as

$$c^{(0,\,L)}\big(\varphi^{(0,\,L)}(x)\big)'_{+} = \lambda\varphi^{(0,\,L)}(x) - \lambda\int_0^x \varphi^{(0,\,L)}(x-y)\,\mathrm{d}F^{(0,\,L)}(y), \tag{44}$$

where

$$F^{(0,\,L)}(y) = \begin{cases} 0 & \text{if} \quad y < 0, \\ 1 - e^{-y/\mu} & \text{if} \quad 0 \le y < L, \\ 1 & \text{if} \quad y \ge L, \end{cases}$$

and

$$c^{(0,\,L)} = \lambda(1+\theta)\,\mathbb{E}\big[Y_i^{(0,\,L)}\big] = \lambda(1+\theta)\bigg(\int_0^L \frac{y\,e^{-y/\mu}}{\mu}\,\mathrm{d}y + Le^{-L/\mu}\bigg) = \lambda\mu(1+\theta)\big(1-e^{-L/\mu}\big).$$

In Section 3.1 we derive analytic expressions for $\varphi^{(0,\,L)}(x)$ in the case of exponentially distributed claim sizes. In Section 3.2 we investigate how a liability limit changes the survival probability for small enough and large enough initial surpluses.

3.1 Analytic Expression for the Survival Probability

To formulate the next theorem, introduce the constants

$$\begin{aligned}
\bar{\gamma}_1 &= 1 - (1+\theta)\big(1 - e^{-L/\mu}\big), \\
\bar{\gamma}_2 &= \mu(1+\theta)\big(1 - e^{-L/\mu}\big), \\
\bar{C}_{1,\,1} &= -\frac{\theta\big(1 - e^{-L/\mu}\big)}{\bar{\gamma}_1}, \\
\bar{C}_{1,\,2} &= \frac{\theta}{\bar{\gamma}_1(1+\theta)}, \\
\bar{A}_{2,\,0} &= -\frac{\theta}{\bar{\gamma}_1\bar{\gamma}_2(1+\theta)}\exp\bigg(-L\bigg(\frac{1}{\mu} + \frac{\bar{\gamma}_1}{\bar{\gamma}_2}\bigg)\bigg), \\
\bar{C}_{2,\,1} &= \frac{\theta}{\bar{\gamma}_1(1+\theta)}\bigg(\bigg(1 - \frac{1}{\bar{\gamma}_1}\bigg)e^{-L/\mu} - (1+\theta)\big(1 - e^{-L/\mu}\big)\bigg), \\
\bar{C}_{2,\,2} &= \frac{\theta}{\bar{\gamma}_1(1+\theta)}\bigg(1 + \bigg(\frac{1}{\bar{\gamma}_1} + \frac{L}{\bar{\gamma}_2} - 1\bigg)\exp\bigg(-L\bigg(\frac{1}{\mu} + \frac{\bar{\gamma}_1}{\bar{\gamma}_2}\bigg)\bigg)\bigg).
\end{aligned}$$

Moreover, let the constants $\bar{A}_{n+1,\,j}$, $0 \le j \le n-1$, be given in a recurrent way by formulas

$$\bar{A}_{n+1,\,n-1} = -\frac{\bar{A}_{n,\,n-2}}{n\bar{\gamma}_2}\exp\bigg(-L\bigg(\frac{1}{\mu} + \frac{\bar{\gamma}_1}{\bar{\gamma}_2}\bigg)\bigg), \quad n \ge 2, \tag{45}$$

$$
\begin{aligned}
\bar{A}_{n+1,\,j} = &-\frac{(j+2)\bar{\gamma}_2 \bar{A}_{n+1,\,j+1}}{\bar{\gamma}_1} - \frac{1}{j+1}\,\exp\biggl(-L\Bigl(\frac{1}{\mu}+\frac{\bar{\gamma}_1}{\bar{\gamma}_2}\Bigr)\biggr) \\
&\times \Biggl[\sum_{i=j-1}^{n-3}\biggl(\frac{\bar{A}_{n,\,i}}{\bar{\gamma}_2}+\frac{(i+2)\bar{A}_{n,\,i+1}}{\bar{\gamma}_1}\biggr)(-L)^{i-j+1}\binom{i+1}{j} \\
&+\frac{\bar{A}_{n,\,n-2}}{\bar{\gamma}_2}\,(-L)^{n-j+1}\binom{n-1}{j}\Biggr], \quad 1 \le j \le n-2, \quad n \ge 3,
\end{aligned} \tag{46}
$$

$$
\begin{aligned}
\bar{A}_{n+1,\,0} = &-\frac{2\bar{\gamma}_2 \bar{A}_{n+1,\,1}}{\bar{\gamma}_1} - \exp\biggl(-L\Bigl(\frac{1}{\mu}+\frac{\bar{\gamma}_1}{\bar{\gamma}_2}\Bigr)\biggr) \\
&\times \Biggl[\frac{\bar{C}_{n,\,2}}{\bar{\gamma}_2}+\frac{\bar{A}_{n,\,0}}{\bar{\gamma}_1}+\sum_{i=0}^{n-3}\biggl(\frac{\bar{A}_{n,\,i}}{\bar{\gamma}_2}+\frac{(i+2)\bar{A}_{n,\,i+1}}{\bar{\gamma}_1}\biggr)(-L)^{i+1} \\
&+\frac{\bar{A}_{n,\,n-2}}{\bar{\gamma}_2}\,(-L)^{n-1}\Biggr], \quad n \ge 2.
\end{aligned} \tag{47}
$$

Finally, let the constants $\bar{C}_{n+1,\,1}$ and $\bar{C}_{n+1,\,2}$ be given by formulas

$$
\begin{aligned}
\bar{C}_{n+1,\,1} = \bar{C}_{n,\,1} + \frac{\bar{\gamma}_2}{\bar{\gamma}_1}\Biggl(&\sum_{i=0}^{n-2}(i+1)(\bar{A}_{n+1,\,i}-\bar{A}_{n,\,i})(nL)^i \\
&+ n\bar{A}_{n+1,\,n-1}(nL)^{n-1}\Biggr)\,e^{nL\bar{\gamma}_1/\bar{\gamma}_2}, \quad n \ge 2,
\end{aligned} \tag{48}
$$

$$
\begin{aligned}
\bar{C}_{n+1,\,2} = \bar{C}_{n,\,2} &+ \frac{\bar{\gamma}_2(\bar{A}_{n,\,0}-\bar{A}_{n+1,\,0})}{\bar{\gamma}_1} \\
&+ \sum_{i=1}^{n-2}\biggl(\bar{A}_{n,\,i-1}-\bar{A}_{n+1,\,i-1}+\frac{(i+1)\bar{\gamma}_2(\bar{A}_{n,\,i}-\bar{A}_{n+1,\,i})}{\bar{\gamma}_1}\biggr)(nL)^i \\
&+ \biggl(\bar{A}_{n,\,n-2}-\bar{A}_{n+1,\,n-2}-\frac{n\bar{\gamma}_2\bar{A}_{n+1,\,n-1}}{\bar{\gamma}_1}\biggr)(nL)^{n-1} \\
&- \bar{A}_{n+1,\,n-1}(nL)^n, \quad n \ge 2.
\end{aligned} \tag{49}
$$

Note that to compute the constants by the formulas above for any $n \ge 2$, we have to know all the constants for $n-1$. Moreover, for any fixed $n \ge 2$, we start from the computation of $\bar{A}_{n+1,\,j}$ for j from $n-1$ to 0, and after that we can compute $\bar{C}_{n+1,\,1}$ and $\bar{C}_{n+1,\,2}$. We introduced all the constants only to formulate the next theorem and will get them in the proof.

Theorem 3. *Let the surplus process $\bigl(X_t^{(0,\,L)}(x)\bigr)_{t\ge 0}$ follow* (4) *under the above assumptions with $d=0$ and $0<L<+\infty$, and the claim sizes be exponentially distributed with mean μ. Then*

$$
\varphi^{(0,\,L)}(x) = \varphi_{n+1}^{(0,\,L)}(x) \quad \textit{for all} \quad x \in [nL,(n+1)L), \quad n \ge 0,
$$

where

$$
\varphi_1^{(0,\,L)}(x) = \bar{C}_{1,\,1} + \bar{C}_{1,\,2}\,e^{\bar{\gamma}_1 x/\bar{\gamma}_2}, \tag{50}
$$

$$\varphi_2^{(0,\,L)}(x) = \bar{C}_{2,\,1} + \big(\bar{C}_{2,\,2} + \bar{A}_{2,\,0}\; x\big)\, e^{\bar{\gamma}_1 x/\bar{\gamma}_2}, \tag{51}$$

$$\varphi_{n+1}^{(0,\,L)}(x) = \bar{C}_{n+1,\,1} + \left(\bar{C}_{n+1,\,2} + \sum_{j=0}^{n-1} \bar{A}_{n+1,\,j}\; x^{j+1}\right) e^{\bar{\gamma}_1 x/\bar{\gamma}_2}, \quad n \geq 2. \tag{52}$$

Proof. By the results of [5], $\varphi^{(0,\,L)}(x)$ is continuous on $\mathbb{R}_+$ and continuously differentiable on this interval, except at the point $x = L$, where there are only one-sided derivatives (see also [7, pp. 162–163]).

Let us introduce the functions $\varphi_{n+1}^{(0,\,L)}(x)$, $n \geq 0$, in the following way: $\varphi_{n+1}^{(0,\,L)}(x)$ is defined on $[nL, (n+1)L)$ and coincides with $\varphi^{(0,\,L)}(x)$ on this interval.

If $x \in [0, L)$, then substituting $c^{(0,\,L)}$ and $F^{(0,\,L)}(y)$ into (44) yields

$$\mu(1+\theta)\big(1 - e^{-L/\mu}\big)\big(\varphi_1^{(0,\,L)}(x)\big)' = \varphi_1^{(0,\,L)}(x) - \frac{1}{\mu}\int_0^x \varphi_1^{(0,\,L)}(x-y)\, e^{-y/\mu}\, \mathrm{d}y. \tag{53}$$

This gives that $\varphi_1^{(0,\,L)}(x)$ has the continuous second derivative on $[0, L)$. So integro-differential equation (53) can be reduced to the differential one

$$\mu(1+\theta)\big(1 - e^{-L/\mu}\big)\big(\varphi_1^{(0,\,L)}(x)\big)'' - \big(1 - (1+\theta)\big(1 - e^{-L/\mu}\big)\big)\big(\varphi_1^{(0,\,L)}(x)\big)' = 0 \tag{54}$$

in a standard way (see, e.g., [1–4, 7]).

Let $\bar{\gamma}_1 = 1-(1+\theta)\big(1-e^{-L/\mu}\big)$ and $\bar{\gamma}_2 = \mu(1+\theta)\big(1-e^{-L/\mu}\big)$. Solving (54) yields (50), where the constants $\bar{C}_{1,\,1}$ and $\bar{C}_{1,\,2}$ can be found from

$$\begin{cases} \varphi_1^{(0,\,L)}(0) = \dfrac{\theta}{1+\theta}, \\ \big(\varphi_1^{(0,\,L)}(0)\big)' = \dfrac{\varphi_1^{(0,\,L)}(0)}{\bar{\gamma}_2}. \end{cases} \tag{55}$$

The first condition in (55) guarantees us that the solution is the survival probability, and the second one follows from (53). Thus, the constants $\bar{C}_{1,\,1}$ and $\bar{C}_{1,\,2}$ are given before the assertion of the theorem.

If $x \in [L, +\infty)$, then substituting $c^{(0,\,L)}$ and $F^{(0,\,L)}(y)$ into (44) yields

$$\begin{aligned} &\mu(1+\theta)\big(1 - e^{-L/\mu}\big)\big(\varphi^{(0,\,L)}(x)\big)' \\ &\quad = \varphi^{(0,\,L)}(x) - \frac{e^{-x/\mu}}{\mu}\int_{x-L}^x \varphi^{(0,\,L)}(y)\, e^{y/\mu}\, \mathrm{d}y - e^{-L/\mu}\varphi^{(0,\,L)}(x-L). \end{aligned} \tag{56}$$

Note that we imply the right derivative of $\varphi^{(0,\,L)}(x)$ at the point $x = L$. It is easily seen from (56) that the second classical derivative of $\varphi^{(0,\,L)}(x)$ exists on $[L, +\infty)$ except at the point $x = 2L$. So integro-differential equation (56) can be reduced to the differential one

$$\bar{\gamma}_2\big(\varphi^{(0,\,L)}(x)\big)'' - \bar{\gamma}_1\big(\varphi^{(0,\,L)}(x)\big)' = -e^{-L/\mu}\big(\varphi^{(0,\,L)}(x-L)\big)', \quad x \in [L, +\infty), \tag{57}$$

where we imply the right second derivative of $\varphi^{(0,\,L)}(x)$ at the point $x = 2L$.

If $x \in [L, 2L)$, then (57) can be rewritten as

$$\bar{\gamma}_2 \big(\varphi_2^{(0,\,L)}(x)\big)'' - \bar{\gamma}_1 \big(\varphi_2^{(0,\,L)}(x)\big)' = -\frac{\theta}{\bar{\gamma}_2(1+\theta)} \exp\left(-\frac{L}{\mu} + \frac{\bar{\gamma}_1(x-L)}{\bar{\gamma}_2}\right). \tag{58}$$

By the results of [5], the following conditions must hold to guarantee that the solution is the survival probability:

$$\begin{cases} \lim\limits_{x \uparrow L} \varphi_1^{(0,\,L)}(x) = \varphi_2^{(0,\,L)}(L), \\ \lim\limits_{x \uparrow L} \big(\varphi_1^{(0,\,L)}(x)\big)' = \big(\varphi_2^{(0,\,L)}(L)\big)' + \dfrac{\theta e^{-L/\mu}}{\bar{\gamma}_2(1+\theta)}, \end{cases} \tag{59}$$

(see also [7, pp. 162–163]).

The solution to (58) can be found by writing

$$\varphi_2^{(0,\,L)}(x) = \varphi_{2,\,\mathrm{gen}}^{(0,\,L)}(x) + \varphi_{2,\,\mathrm{part}}^{(0,\,L)}(x),$$

where $\varphi_{2,\,\mathrm{gen}}^{(0,\,L)}(x)$ is a general solution to the linear homogeneous equation corresponding to (58), and $\varphi_{2,\,\mathrm{part}}^{(0,\,L)}(x)$ is a particular solution to linear heterogeneous equation (58).

Since $\varphi_{2,\,\mathrm{part}}^{(0,\,L)}(x)$ can be written as

$$\varphi_{2,\,\mathrm{part}}^{(0,\,L)}(x) = \bar{A}_{2,\,0}\, x\, e^{\bar{\gamma}_1 x/\bar{\gamma}_2},$$

where $\bar{A}_{2,\,0}$ is a constant, substituting $\varphi_{2,\,\mathrm{part}}^{(0,\,L)}(x)$ into (58) and applying the method of undetermined coefficients yield $\bar{A}_{2,\,0}$. Furthermore, we can write $\varphi_{2,\,\mathrm{gen}}^{(0,\,L)}(x)$ as

$$\varphi_{2,\,\mathrm{gen}}^{(0,\,L)}(x) = \bar{C}_{2,\,1} + \bar{C}_{2,\,2}\, e^{\bar{\gamma}_1 x/\bar{\gamma}_2},$$

where the constants $\bar{C}_{2,\,1}$ and $\bar{C}_{2,\,2}$ are such that $\varphi_2^{(0,\,L)}(x)$ satisfies conditions (59).

Thus, $\varphi_2^{(0,\,L)}(x)$ is defined by (51) and the constants $\bar{A}_{2,\,0}$, $\bar{C}_{2,\,1}$, and $\bar{C}_{2,\,2}$ are given before the assertion of the theorem.

In the general case, if we know $\varphi_n^{(0,\,L)}(x)$, $n \geq 2$, we can find $\varphi_{n+1}^{(0,\,L)}(x)$ applying considerations similar to those in the proof of Theorem 1. Since

$$\bar{\gamma}_2 \big(\varphi_{n+1}^{(0,\,L)}(x)\big)'' - \bar{\gamma}_1 \big(\varphi_{n+1}^{(0,\,L)}(x)\big)' = -e^{-L/\mu} \big(\varphi_n^{(0,\,L)}(x-L)\big)',$$

we get (52) by induction on n.

Applying the method of undetermined coefficients to

$$\begin{aligned}
&\bar{\gamma}_2\Bigg[\bigg(\frac{2\bar{\gamma}_1\bar{A}_{n+1,\,0}}{\bar{\gamma}_2}+2\bar{A}_{n+1,\,1}\bigg)\\
&\quad+\sum_{j=1}^{n-2}\bigg(\frac{\bar{\gamma}_1^2\bar{A}_{n+1,\,j-1}}{\bar{\gamma}_2^2}+\frac{2(j+1)\bar{\gamma}_1\bar{A}_{n+1,\,j}}{\bar{\gamma}_2}+(j+2)(j+1)\bar{A}_{n+1,\,j+1}\bigg)\,x^j\\
&\quad+\bigg(\frac{\bar{\gamma}_1^2\bar{A}_{n+1,\,n-2}}{\bar{\gamma}_2^2}+\frac{2n\bar{\gamma}_1\bar{A}_{n+1,\,n-1}}{\bar{\gamma}_2}\bigg)\,x^{n-1}+\frac{\bar{\gamma}_1^2\bar{A}_{n+1,\,n-1}}{\bar{\gamma}_2^2}\,x^n\Bigg]\\
&\quad-\bar{\gamma}_1\Bigg[\bar{A}_{n+1,\,0}+\sum_{j=1}^{n-1}\bigg(\frac{\bar{\gamma}_1\bar{A}_{n+1,\,j-1}}{\bar{\gamma}_2}+(j+1)\bar{A}_{n+1,\,j}\bigg)\,x^j+\frac{\bar{\gamma}_1\bar{A}_{n+1,\,n-1}}{\bar{\gamma}_2}\,x^n\Bigg]\\
&\quad=-\Bigg[\bigg(\frac{\bar{\gamma}_1\bar{C}_{n,\,2}}{\bar{\gamma}_2}+\bar{A}_{n,\,0}+\sum_{i=0}^{n-3}\bigg(\frac{\bar{\gamma}_1\bar{A}_{n,\,i}}{\bar{\gamma}_2}+(i+2)\bar{A}_{n,\,i+1}\bigg)\,(-L)^{i+1}\\
&\quad+\frac{\bar{\gamma}_1\bar{A}_{n,\,n-2}}{\bar{\gamma}_2}\,(-L)^{n-1}\bigg)+\sum_{j=1}^{n-1}\bigg(\sum_{i=j-1}^{n-3}\bigg(\frac{\bar{\gamma}_1\bar{A}_{n,\,i}}{\bar{\gamma}_2}+(i+2)\bar{A}_{n,\,i+1}\bigg)\,(-L)^{i-j+1}\binom{i+1}{j}\\
&\quad+\frac{\bar{\gamma}_1\bar{A}_{n,\,n-2}}{\bar{\gamma}_2}\,(-L)^{n-j+1}\binom{n-1}{j}\bigg)\,x^j\Bigg]\,\exp\bigg(-L\bigg(\frac{1}{\mu}+\frac{\bar{\gamma}_1}{\bar{\gamma}_2}\bigg)\bigg)
\end{aligned}$$

yields the constants $\bar{A}_{n+1,\,j}$, $0 \le j \le n-1$ in (52). Thus, they are given by (45), (46), and (47).

To find the constants $\bar{C}_{n+1,\,1}$ and $\bar{C}_{n+1,\,2}$ in (52), we use the continuity of $\varphi^{(0,\,L)}(x)$ and its derivative at the point $x = nL$, which implies

$$\begin{cases} \lim\limits_{x\uparrow nL}\varphi_n^{(0,\,L)}(x)=\varphi_{n+1}^{(0,\,L)}(nL),\\ \lim\limits_{x\uparrow nL}\big(\varphi_n^{(0,\,L)}(x)\big)'=\big(\varphi_{n+1}^{(0,\,L)}(nL)\big)', \end{cases}$$

and guarantees that the solution is the survival probability. Thus, $\bar{C}_{n+1,\,1}$ and $\bar{C}_{n+1,\,2}$ are given by (48) and (49), respectively. The proof is complete. □

3.2 Case of Small Enough and Large Enough Initial Surpluses

We now investigate how a liability limit changes the survival probability for small enough and large enough initial surpluses.

Theorem 4. *Let the surplus processes $\big(X_t(x)\big)_{t\ge0}$ and $\big(X_t^{(0,\,L)}(x)\big)_{t\ge0}$ follow (1) and (4), respectively, under the above assumptions with $d = 0$ and $0 < L < +\infty$. Moreover, let $\varphi(x)$ and $\varphi^{(0,\,L)}(x)$ be the corresponding survival probabilities, and the claim sizes be exponentially distributed with mean μ. Then $\varphi^{(0,\,L)}(x) > \varphi(x)$ for any $0 < L < +\infty$ and for small enough and large enough initial surpluses.*

Proof. By (3) and (50), we have

$$\varphi(x)\approx\frac{\theta}{1+\theta}+\frac{\theta x}{\mu(1+\theta)^2}$$

and

$$\varphi_1^{(0,\,L)}(x) \approx \frac{\theta}{1+\theta} + \frac{\theta x}{\mu(1+\theta)^2\big(1-e^{-L/\mu}\big)}$$

for small enough initial surpluses. This gives the assertion of the theorem for such initial surpluses.

By the Cramér-Lundberg approximation (see, e.g., [1–4, 7, 8]), we have

$$\varphi^{(0,\,L)}(x) \sim 1 - \frac{\theta}{R^{(0,\,L)}\mu^{(0,\,L)}(1+\theta)}\, e^{-R^{(0,\,L)}x}, \tag{60}$$

for large enough initial surpluses, where $R^{(0,\,L)}$ is a unique positive solution (if it exists) to

$$\int_0^{+\infty} e^{R^{(0,\,L)}y}\,\big(1 - F^{(0,\,L)}(y)\big)\,\mathrm{d}y = \mu(1+\theta)\big(1-e^{-L/\mu}\big), \tag{61}$$

and

$$\mu^{(0,\,L)} = \frac{1}{\mu(1+\theta)\big(1-e^{-L/\mu}\big)} \int_0^{+\infty} y\, e^{R^{(0,\,L)}y}\,\big(1 - F^{(0,\,L)}(y)\big)\,\mathrm{d}y. \tag{62}$$

It is obvious that the improper integral in the right-hand side of (62) is finite.

If $R^{(0,\,L)} = 1/\mu$ is a solution to (61), then

$$L = \mu(1+\theta)\big(1-e^{-L/\mu}\big) \tag{63}$$

must hold.

If $R^{(0,\,L)} \neq 1/\mu$, then substituting $F^{(0,\,L)}(y)$ into (61) yields

$$e^{LR^{(0,\,L)}} = (1+\theta)\big(e^{L/\mu}-1\big)\big(\mu R^{(0,\,L)}-1\big) + e^{L/\mu}.$$

Consider the function

$$\bar{g}(R) = e^{LR} - (1+\theta)\big(e^{L/\mu}-1\big)\big(\mu R - 1\big) - e^{L/\mu}$$

on $\mathbb{R}_+$. It is decreasing on $[0, \bar{R}^*)$ and increasing on $(\bar{R}^*, +\infty)$, where

$$\bar{R}^* = \frac{1}{L}\ln\bigg(\frac{\mu(1+\theta)\big(e^{L/\mu}-1\big)}{L}\bigg).$$

Moreover, $\bar{g}(0) = \theta\big(e^{L/\mu}-1\big) > 0$, $\bar{g}(1/\mu) = 0$, and $\lim_{R\to+\infty}\bar{g}(R) = +\infty$. Note that $\bar{R}^* = 1/\mu$ if and only if (63) holds.

Thus, we conclude that $R = 1/\mu$ is a unique zero of $\bar{g}(R)$ if (63) is true. This means that (61) has the unique solution $R^{(0,\,L)} = 1/\mu$. Otherwise, if (63) is not true, then $R = R^{(0,\,L)}$ and $R = 1/\mu$ are positive zeros of $\bar{g}(R)$, which means that $R^{(0,\,L)}$ is a unique positive solution to (61) such that $R^{(0,\,L)} \neq 1/\mu$. Furthermore, in this case $\bar{g}(R)$ is negative between its zeros $R = R^{(0,\,L)}$ and $R = 1/\mu$, otherwise it is positive.

Since

$$\bar{g}\bigg(\frac{\theta}{\mu(1+\theta)}\bigg) = \exp\bigg(\frac{\theta L}{\mu(1+\theta)}\bigg) - 1 > 0$$

and

$$\frac{\theta}{\mu(1+\theta)} < \frac{1}{\mu},$$

we get

$$\frac{\theta}{\mu(1+\theta)} < R^{(0,\,L)}.$$

Consequently, by (3) and (60), we have the assertion of the theorem for large enough initial surpluses. □

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