# ASSET ALLOCATION TO PREVENT UNEXPECTED LARGE LOSSES IN AN EXTREME VALUE THEORY FRAMEWORK

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# Outline

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2. Theory overview
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# Aim of the work

ASSET ALLOCATION PROBLEM:

The aim of this work is to examine the optimal portfolio selection problem for a risk-adverse investor who wants to prevent large losses:

- minimizing quantile-based risk measure (for large quantile)
- under the Extreme Value Theory (EVT) framework.

EXTREMAL DEPENDENCE ANALYSIS

We also analyse the role of extremal dependence in this problem.

# Risk-based approach

## Risk

- Several crises have overtaken the financial markets : **forecasting risk** and **risk management**, **has** thus **become a major concern** for both financial institutions and market regulators

## Extremal events

- Rare frequency but big impact in terms of losses caused. Shocks to the underlying financial system that can compromise an institution solvency.

## Risk models

- Combination of:
  - **probability distribution** model
  - **risk measure**

  It provides a measure of risk that could be employed in portfolio selection, risk management, derivatives pricing and so forth

## Risk measure

- Any functional (mapping) $\rho(X)$ that sets a real number to any random variable

# Optimal portfolio selection

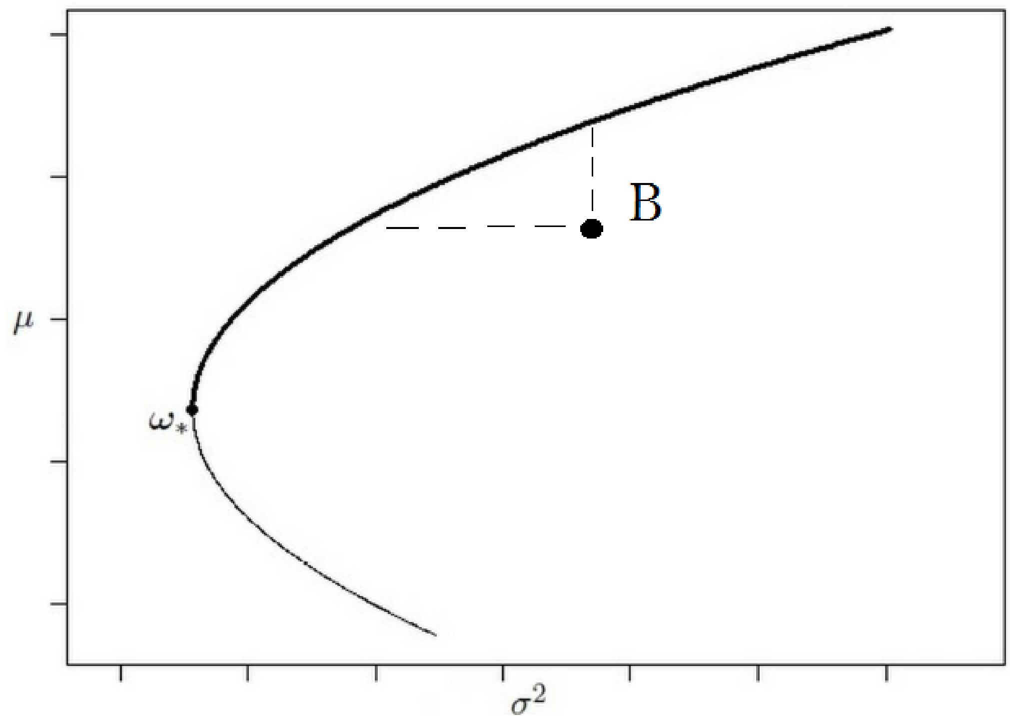
## Optimal Portfolio Choice

- **Choosing** the proportions of various **assets to minimize risk for a given level of expected return**, or equivalently maximize portfolio expected return for a given amount of portfolio risk.

- Firstly addressed by Markowitz (1952):
  - mean-variance model
  - **normally** distributed **returns**
  - **variance** as a risk measure.

$$\begin{cases} \min_{w \in R^n} \sigma_P^2(\boldsymbol{w}) \\ \textit{constraints :} \\ \mu_P(\boldsymbol{w}) = \mu_0 \\ \sum_{k=1}^{n} w_k = 1 \end{cases}$$

Efficient frontier



# Drawbacks of the mean-variance and our approach

## Asymmetry and heavy tails

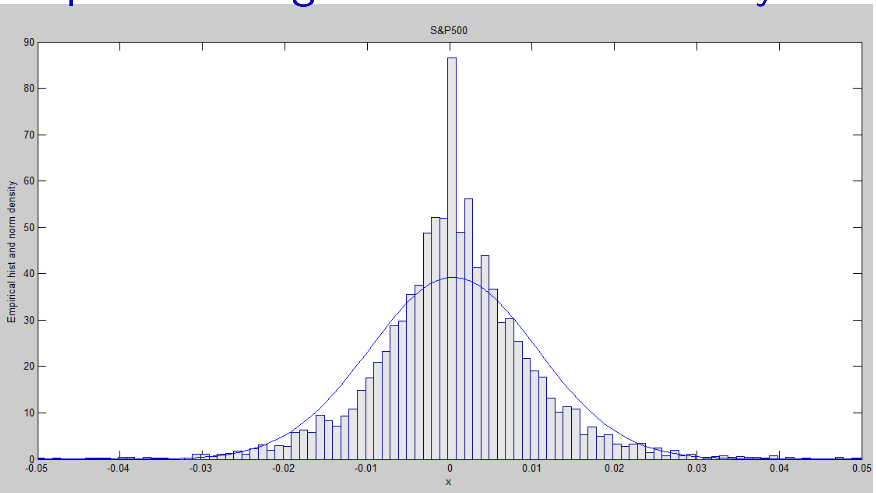
**Empirical evidence**:

- The distribution of **financial assets return** is actually **skewed and fat-tailed**
- **Normal** distributed return assumption **not reliable**

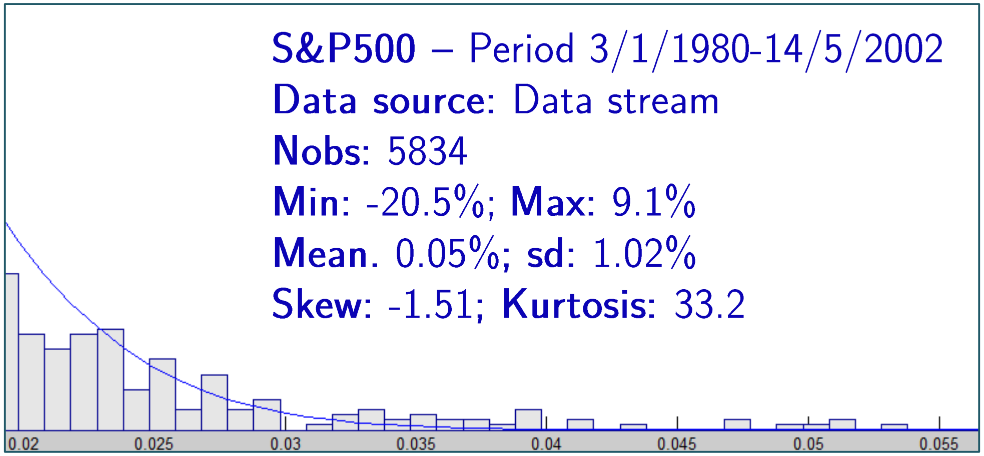
## Extreme value Theory

- **Concerned** with the **asymptotic distribution of extreme events**
- Models the tails, **without** making **assumptions on the underlying data distribution**

### Empirical histogram and normal density




### Focus on the right tail



# Drawbacks of the mean-variance and our approach

## Variance as a risk measure

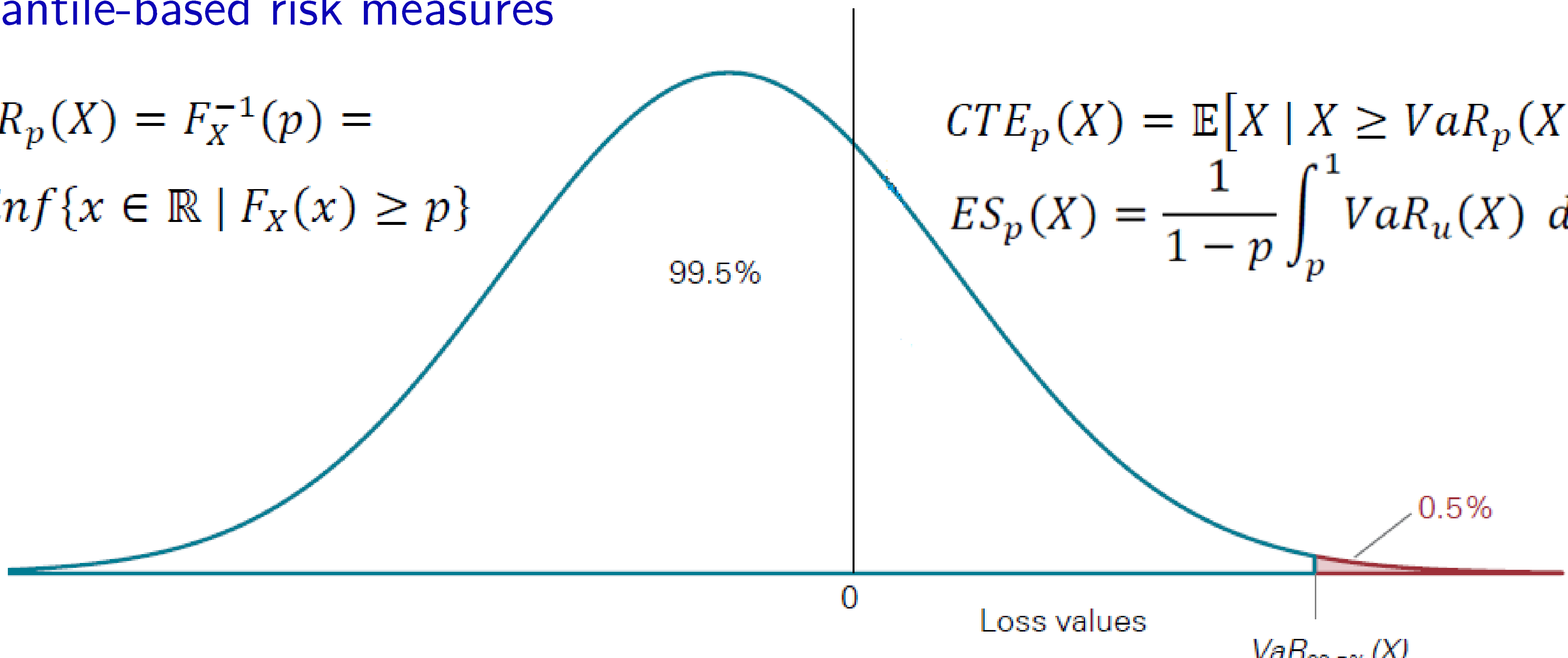
- It measures the spread of the distribution around the mean
- It assigns the same weight to gains as well as losses

⇒ **Underweights extreme events** and might lead to an **optimistic asset allocation**

## Quantile-based risk measures

$$VaR_p(X) = F_X^{-1}(p) =$$
$$= inf\{x \in \mathbb{R} \mid F_X(x) \geq p\}$$

$$CTE_p(X) = \mathbb{E}[X \mid X \geq VaR_p(X)]$$
$$ES_p(X) = \frac{1}{1-p}\int_p^1 VaR_u(X)\ du$$



# Safety first approach

Plausibility of considering the allocation problem faced by an extremely risk-averse institution

- Bensalah (2002, p. 5)

*"There is no final answer or optimal measure of risk"*

A given allocation is optimal conditional on the actual correspondence of its underlying assumptions to the risk preference of the investor.

A plausible profile of risk preference, alternative to that based on variance, is represented by the safety-first criterion, a concept introduced by Roy (1952) and developed by Arzac and Bawa (1977), which is based on a constraint limiting downside risk.

- practical circumstances lead to asymmetric treatment of upside and downside risk
- psychologically sensible, since a lot of experimental evidence for loss aversion is available.

# Literature review

- Embrechts et al., 1998, 1999; Longin, 1997a,b 2000; McNeil, 1997, 1998; Danielsson and de Vries, 1997b, 1997c, 2000; Danielsson et al., 1998.

  Focus is on estimating the **unconditional stationary distribution of asset returns**. Most of them are comparative studies of EVT with historical simulation, RiskMetrics, modelling with normal distribution or t distribution.
- Suggested by Diebold et al. (1998) and implemented by McNeil and Frey (2000).

  2 stage: focus on estimation of the **conditional Var filtering the data** through an AR(1)-GARCH(1,1) model and applying EVT to the residual.

Extending the use of EVT for VaR calculation to the problem of ptf selection

- **Portfolio selection with limited downside risk** by Jansen et al. (2000).
- Bensalah (2002), considers both the direct problem of maximizing return, with constraint on the VaR, and its dual plus a comparison between different ways of calculating VaR (historical simulation, normal VaR and EVT)
- *Selected benchmark papers: Bradley and Taqqu (2004a, 2004b)*

# Theory overview
# EVT Main results

# Asymptotic results

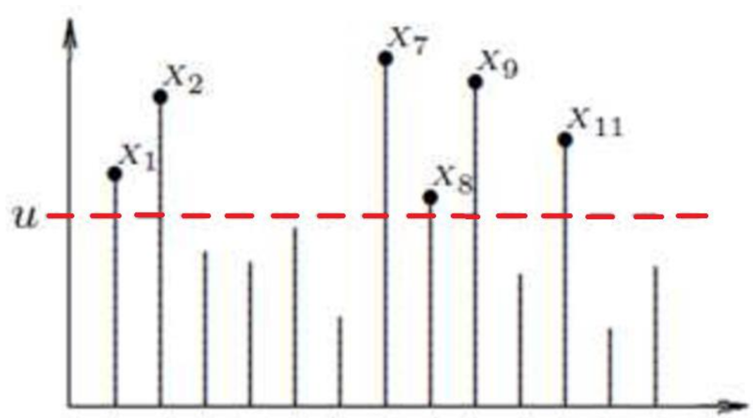
**Peak over threshold approach (POT)**
**F. Balkema and de Haan (1974) and Pickands (1975)**

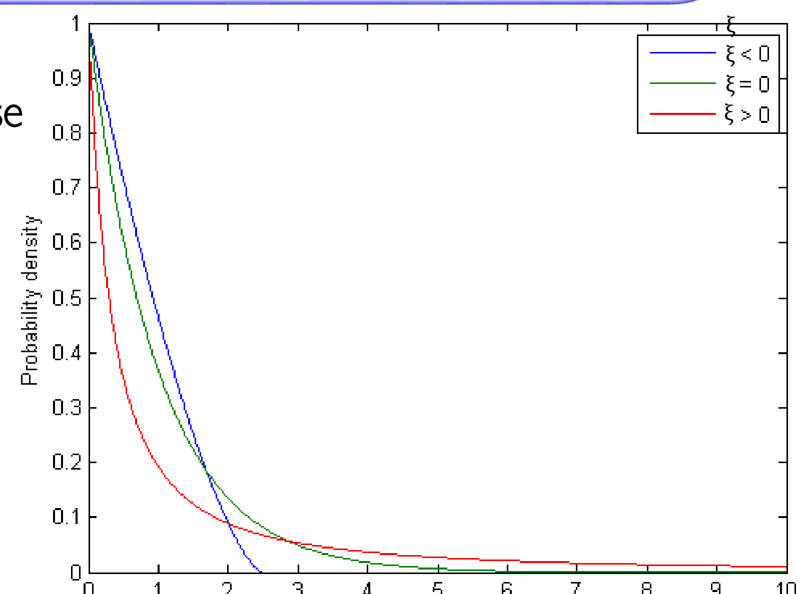
Let $Z_1, Z_2, \ldots$ *i.i.d.* r.v where $F$ belongs to a wide class of continuous distribution function, the distribution of all the amounts exceeding some large threshold $u$ is

$$F_u(z) = P\{Z - u \leq z \mid Z > u\}, 0 < z < z_F - u$$

where $z_F$ right end point of the support of the distribution.
It can be shown that is approximately a generalized Pareto distribution (GPD):

$$H_{\xi,\tilde{\sigma}}(z) = 1 - \left[1 + \xi\left(\frac{z}{\tilde{\sigma}}\right)\right]^{-\frac{1}{\xi}}, \begin{cases} z \geq 0 \text{ for } \xi \geq 0 \\ 0 \leq z \leq \tilde{\sigma}/\xi \text{ otherwise} \end{cases}$$



# $VaR_\alpha$ and $S_\alpha$ estimation

For a given high threshold u, consider the decomposition of the tail of F:

$$1 - F(z) = (1 - F(u))(1 - F_u(z)), z \geq u$$

GPD

We obtain for z>u and u large

$$F(z) = 1 - \lambda_u \left[1 + \xi\left(\frac{z-u}{\sigma}\right)\right]^{-\frac{1}{\xi}}, \lambda_u = P\{Z > u\}$$

**Quantile based risk measure under EVT parametric POT approach**

Setting $F(z_\alpha) = \alpha$ and solving for $z_\alpha$

$$VaR_\alpha(Z) = u + \frac{\sigma}{\xi}\left\{\left[\frac{1}{\lambda_u}(1-\alpha)\right]^{-\xi} - 1\right\}$$

Provided $VaR_\alpha(Z) > u$

$$S_\alpha(Z) = \frac{VaR_\alpha(Z)}{1-\xi} + \frac{\sigma - \xi u}{1-\xi} \; *$$

Estimation through MLE is regular if $\xi \geq -1/2$, the estimators are consistent and asymptotically normal; see Davison (1984b,a) and Smith (1985,1987).

* For references see for example Bradley and Taqqu pp. 1044-1045

# Extremal dependence

We use two complementary dependence measures from EVT proposed by Coles et all (1999) for bivariate *r.v.*

- Influence of the marginals removed by standardizing to have unit Frèchet distribution

$$P\{Y_2 > y | Y_1 > y\} \sim L(y) y^{1-\frac{1}{\eta}} \qquad \chi = \lim_{y \to \infty} P\{Y_2 > y | Y_1 > y\}$$

$$\begin{cases} \chi > 0 \text{ asymptotically dependent (only when } \eta = 1) \\ \chi = 0 \text{ asymptotically independent} \end{cases}$$

Obtained using the joint survivor expression by Ledford and Tawn (1998) where $L(y)$ slowly varying function, $\eta \in (0, 1]$ *coefficient of tail dependence.*

$$\bar{\chi} = \lim_{y \to \infty} \frac{2 log P\{Y_1 > y\}}{log P\{Y_1 > y, Y_2 > y\}} - 1 = 2\eta - 1$$

## Interpretation

❶ $\bar{\chi} = 1$ $\chi$ measures the strength of asymptotic dependence (through $L(y) \to c$)

❷ $\chi = 0$ $\bar{\chi}$ provides a measure of the strength of dependence

# EVT- Extremal dependence

## Estimation

❶ First estimate $\eta$ and then estimate $\bar{\chi}$.

❷ We test H$_o$: $\bar{\chi} = 1$ If non rejected, then one estimates $\chi$ (through $L(y)\to c$).

Following the methodology suggested by Poon et al. (2002) based on the Hill estimator where k number of exceedances of the threshold $y_0$*.

$$\widehat{\bar{\chi}} = 2\widehat{\eta} - 1 = 2\left(\frac{1}{k}\sum_{i=1}^{k}\log Y_{\min}^{(i)} - \log y_0\right) - 1 \quad \widehat{\mathrm{Var}}(\hat{\bar{\chi}}) = \frac{4\widehat{\eta}^2}{k} = \frac{(\widehat{\bar{\chi}}+1)^2}{k}$$

Under regularity conditions on the tail, $k = k(n)\to\infty$, $k/n \to 0$ as $n\to\infty$, the estimator $\hat{\eta}$ is asymptotically normal :

$$\sqrt{k(n)}(\hat{\eta} - \eta) \xrightarrow{d} \mathcal{N}(0, \eta^2)$$

If $H_o$ non then estimate $\chi$ under the condition that $\bar{\chi}=\eta=1$.

$$\widehat{\chi} = \widehat{c} = \frac{k}{n}y_0 \quad \widehat{\mathrm{Var}}\left(\widehat{\chi}\right) = \frac{y_0^2 k(n-k)}{n^3}$$

*$y_0$: 95% quantile of the empirical distribution of $Y_{min},i$, $i = 1, \ldots, n$

# Model Approach

# Asset allocation problem definition

- $\boldsymbol{X} = (\boldsymbol{X}_1, \boldsymbol{X}_2, \dots, \boldsymbol{X}_d)$ negative returns of *d* assets in our universe
- **Loss** of a linear **portfolio** *Z(w)* with allocation $\boldsymbol{w} = (w_1, w_2, \dots, w_d)$ is:

$$Z(\boldsymbol{w}) = \sum_{i=1}^{d} w_i X_i, 0 \leq w_i \leq 1$$

Optimal Asset allocation: find ***w**** which minimizes the risk of *Z(w)* (either the $VaR_\alpha$ or the expected shortfall $S_\alpha$) with confidence level $\alpha$ (design parameter)

$$w^* = \underset{w}{\operatorname{argmin}} \; RM_\alpha(Z(\boldsymbol{w})),$$

$$Constraints: \sum_{i=1}^{d} w_i = 1, w_i \geq 0$$

The univariate **EVT** approach called the **structure variable method** (SVM) with structure variable = *Z(w)* ⇨ $VaR_\alpha$ or $S_\alpha$ estimation

# $VaR_\alpha$ and $S_\alpha$ estimation using EVT

Hp. Suppose $Z_1, Z_2, \ldots$ are i.i.d. with distribution function F representing the loss (negative return) of the portfolio *Z(w)*.

We apply the POT to approximate the tail of the portfolio Z(w) distribution

$$VaR_\alpha(Z) = u + \frac{\sigma}{\xi}\left\{\left[\frac{1}{\lambda_u}(1-\alpha)\right]^{-\xi} - 1\right\} \qquad S_\alpha(Z) = \frac{VaR_\alpha(Z)}{1-\xi} + \frac{\sigma - \xi u}{1-\xi}$$

# Model estimation – step procedure

## Optimum search

Varying $\boldsymbol{w} = (w_1, w_2, \dots, w_d), \forall \boldsymbol{w}$ :

- Sample of negative returns : $\boldsymbol{X}_i = (X_{i1}, X_{i2}, \dots, X_{iT}), i=1,\dots,d \rightarrow$ PTF negative returns $Z_j(\boldsymbol{w}) = \sum_{i=1}^{d} w_i X_{ij}\, , j = 1, \dots, T$
- We **estimate the parameters** $(\lambda_u, \sigma, \xi)$ of the tail portfolio distribution (GPD) using maximum likelihood.
- **Risk measure:** $VaR_\alpha$ or $S_\alpha$ calculation using previous parametric formulas

Identifying the **w** that minimize the PTF risk measures calculated in step 3.

## Threshold choice:

Delicate matter because of the trade off between bias-variance.

According with literature, we take *u* as the **95% quantile of the empirical distribution** of Z(w).

# Optimal minimum risk portfolios

## Optimum search algorithm

Increasing the number of assets, the optimum search becomes **computationally hard to manage**

Two stage methodology
as in Bradley and Taqqu (2004):

- sampling algorithm of Bensalah (2002) to pick a starting point:

  randomly sample $n_s$ portfolio weights from uniform distributions, picking the one that leads to minimal risk.

- incremental trade algorithm to step away from the starting allocation:

  The algorithm takes **steps of size $\boldsymbol{\delta_w}$** in each market away from its current position (buying and selling) and it picks the trade which is most risk reducing

# Data and results

# Data

## Sample financial data

- Daily simple return: calculated from total return equity indices
- Source: Datastream
- Period: 02-Jan-1980 to 3-Sept-2015
- Assets: d=12, 12 international equity indices representative of 12 markets

**ITALY-TOT RETURN** IND: Code TOTMKIT(RI)

# Data

| Market | Min * | Max * | Mean * | stdev * | skew | kurt | MV |
|---|---|---|---|---|---|---|---|
| HK | -34.724 | 16.832 | 0.043 | 1.648 | -1.454 | 31.767 | 1,793,291 |
| JP | -16.000 | 12.000 | 0.031 | 1.352 | -0.010 | 6.215 | 4,416,750 |
| AU | -26.157 | 8.739 | 0.031 | 1.383 | -1.191 | 20.756 | 957,806 |
| BG | -11.146 | 10.204 | 0.034 | 1.180 | -0.090 | 6.316 | 362,608 |
| CN | -12.660 | 9.988 | 0.031 | 1.110 | -0.611 | 12.607 | 1,480,807 |
| FR | -10.142 | 11.234 | 0.035 | 1.328 | -0.085 | 5.921 | 1,967,032 |
| BD | -11.733 | 17.659 | 0.033 | 1.279 | 0.077 | 8.838 | 1,678,976 |
| IT | -10.328 | 11.914 | 0.036 | 1.529 | -0.048 | 4.891 | 633,379 |
| NL | -10.852 | 10.727 | 0.036 | 1.243 | -0.125 | 7.514 | 548,821 |
| SW | -10.497 | 9.465 | 0.040 | 1.083 | -0.200 | 6.125 | 1,494,951 |
| UK | -13.528 | 12.543 | 0.033 | 1.205 | -0.194 | 8.865 | 3,287,502 |
| US | -18.705 | 11.518 | 0.039 | 1.084 | -0.649 | 17.879 | 21,305,090 |

* Values expressed as a percentage.

Table 1. Descriptive statistics based on 9,307 observations of daily simple periodic returns determined from the price index. MV in millions of US dollars.

# Results – Extremal dependence analysis

Considered **66** different **pairs** of market and studied the dependence **analysing** both the **left and** the **right tail**.

Showing below dependences only between G5 countries (Loss tail)

- Japan: the only market that shows asymptotic independence
- US: lowest asymptotic dependence strength
- France: market with the highest dependence

| | $\overline{\chi}$ | $\sigma(\overline{\chi})$ | $H_0 : \overline{\chi} = 1$ *(pvalue)* | $\chi$ | $\sigma(\chi)$ | $\rho$ |
|---|---|---|---|---|---|---|
| **JPvsFR** | 0.4529 | 0.0674 | 0.000 | 0.278 | 0.013 | 0.249 |
| **JPvsBD** | 0.4051 | 0.0652 | 0.000 | 0.284 | 0.013 | 0.257 |
| **JPvsUK** | 0.4270 | 0.0662 | 0.000 | 0.283 | 0.013 | 0.265 |
| **JPvsUS** | 0.3639 | 0.0633 | 0.000 | 0.228 | 0.010 | 0.062 |
| **FRvsBD** | 1.0000 | 0.0972 | 0.500 | 0.568 | 0.026 | 0.762 |
| **FRvsUK** | 1.0000 | 0.0937 | 0.500 | 0.524 | 0.024 | 0.721 |
| **FRvsUS** | 0.9184 | 0.0890 | 0.180 | 0.334 | 0.015 | 0.402 |
| **BDvsUK** | 1.0000 | 0.0954 | 0.500 | 0.463 | 0.021 | 0.663 |
| **BDvsUS** | 0.9803 | 0.0918 | 0.415 | 0.340 | 0.015 | 0.424 |
| **UKvsUS** | 0.8749 | 0.0870 | 0.075 | 0.345 | 0.016 | 0.412 |

asymptotic independence

# Results - Asset allocation

**Risk Measure**: Expected Shortfall

**Assets**: equity indices of G5 countries

Optimal $S_\alpha$ allocations of the two stage incremental trade algorithm for the G5 markets as a function of high confidence level $\alpha$.

| $ES_\alpha$ | Hp normal returns | $\alpha_1$=0,95 | | $\alpha_2$=0,99 | | $\alpha_3$=0,999 | | $\alpha_4$=0,9999 | |
|---|---|---|---|---|---|---|---|---|---|
| | $W^N$ | $w^*_{\alpha 1}$ | $S_{\alpha 1}$ | $w^*_{\alpha 2}$ | $S_{\alpha 2}$ | $w^*_{\alpha 3}$ | $S_{\alpha 3}$ | $w^*_{\alpha 4}$ | $S_{\alpha 4}$ |
| US | 45.30% | 46.78% | 2.525 | 43.41% | 4.375 | 32.28% | 8.789 | 33.32% | 16.774 |
| JP | 30.27% | 33.56% | 2.989 | 36.17% | 4.700 | 57.62% | 7.866 | 59.47% | 12.144 |
| UK | 16.28% | 11.09% | 2.777 | 17.20% | 4.562 | 0.68% | 8.177 | 0.70% | 13.598 |
| FR | 1.19% | 1.02% | 3.092 | 0.37% | 4.982 | 1.12% | 8.354 | 0.14% | 12.719 |
| BD | 6.97% | 7.55% | 2.950 | 2.84% | 4.718 | 8.30% | 8.015 | 6.37% | 12.508 |
| $S_\alpha(N)$ | | 1.91986 | | 3.25481 | | 6.41318 | | 12.07465 | |
| $S_\alpha(O)$ | | 1.91733 | | 3.20894 | | 5.67920 | | 8.83772 | |
| $S_\alpha(N)/S_\alpha(O)$ | | 1.001 | | 1.014 | | 1.129 | | 1.366 | |

# Conclusions and future developments

- The **optimal allocation is quantile-based**, i.e. depends on $\alpha$, confirming the findings of Bensalah (2002) and Bradley and Taqqu (2004);
- Using the EVT dependence measures we find that **almost half pairs of the twelve equity markets** examined here are **asymptotically independent**.
- A surprising result is the **robustness of the assumption of normality** on the allocation problem at standard **confidence levels ($\boldsymbol{\alpha} = 0.95, 0.99$)**.
- When moving to more extreme quantiles ($\boldsymbol{\alpha} = \mathbf{0.999, 0.9999}$), the **difference between the two approaches can no longer be ignored**

## Future developments

- Move to a Dynamic asset allocation managing the underlying portfolio of a unit linked policy;
- Backtesting the model using conditional EVT for estimating the risk measures through methodologies that consider the possibility of non stationary series see Chavez-Demoulin, V., Embrechts, P., Sardy (2014)

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Thank you!

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