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Fishing for scenarios

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Fishing for scenarios

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Abstract: The new solvency frameworks commonly include the requirement to estimate scenarios to test if the risk distributions include all potential negative deviations. Even though scenarios can be estimated in terms of loss amounts, it can be difficult to find out the return period associated with such scenarios. In particular, when scenarios are prescribed and impact several lines of business following one risk factor which is implicitly embedded in the risk distribution but for which there is no explicit model, the return period is not directly readable in the risk distribution. As a result, this paper aims at providing a methodology to look for a proxy for estimating the return period of such scenario using some light assumptions.

Keywords: Cornish-Fisher expansion; Skewness; Variance; Gaussian copula; Correlations.

1. INTRODUCTION

1.1 Background

With the development of the new solvency frameworks (e.g. Solvency 2 in Europe, Comframe from the International Association of Insurance Supervisors (see IAIS 2015), Swiss Solvency Test in Switzerland (see FINMA 2014)), there are more and more requests to evaluate adverse scenarios and to test if these scenarios are embedded in the risk distributions of internal models (see IAA 2015). Whenever the scenarios are effectively embedded in the risk distribution, the return period (ie the average frequency of occurrence of the scenario estimated in number of years) of the scenario is often requested for disclosure purpose. However, it often happens that a scenario seems to be embedded in the distribution but the return period cannot be read directly from the risk distribution. Good examples of such situations include the estimation of the impact on non-life reserves following:

- an increase of inflation,
- a decrease of the mortality and its impact on the future payable annuities (e.g. on French and UK Motor business),
- regulatory changes regarding the discount rate of lump sum payments related to future medical care (e.g. nursing care for UK medical malpractice).

In the case of an inflation scenario, the impact on reserves is estimated on applying an an inflation yield curve to the future payment cash-flows of non-life reserves. Such scenario is relatively easy to estimate. In order to get the return period of such scenario, the usual solution consists in looking at the reserve risk distribution coming from the internal model. However, the reserve risk distribution is usually calculated on aggregating the reserve distributions of each line of business. A line of business reserve distribution is commonly modelled as a lognormal distribution with mean equal to the best estimate and volatility estimated using a Mack model on a paid or incurred triangle (see Mack 1993). Such volatility may contain an inflation component but certainly is influenced by many other factors.

As a result, when estimating the return period of an inflation scenario on the basis of the overall reserves risk distribution, there is a bias in the estimation. Such bias is also true for the other scenarios given as examples above. This paper intends to give the reader a way to estimate the bias so that the return period provided for a given scenario is more accurate. In this sense, the reader will go to fish for his scenario within the risk distribution.

1.2 Outline

This paper is divided into the following sections:

- Section 2 provides the general methodology used in this paper.
- Section 3 provides basic properties of Cornish-Fisher expansion and of Gaussian copulas.
- Section 4 establishes the mathematical framework in which the scenario fishing can be done.
- Section 5 provides a numerical example.

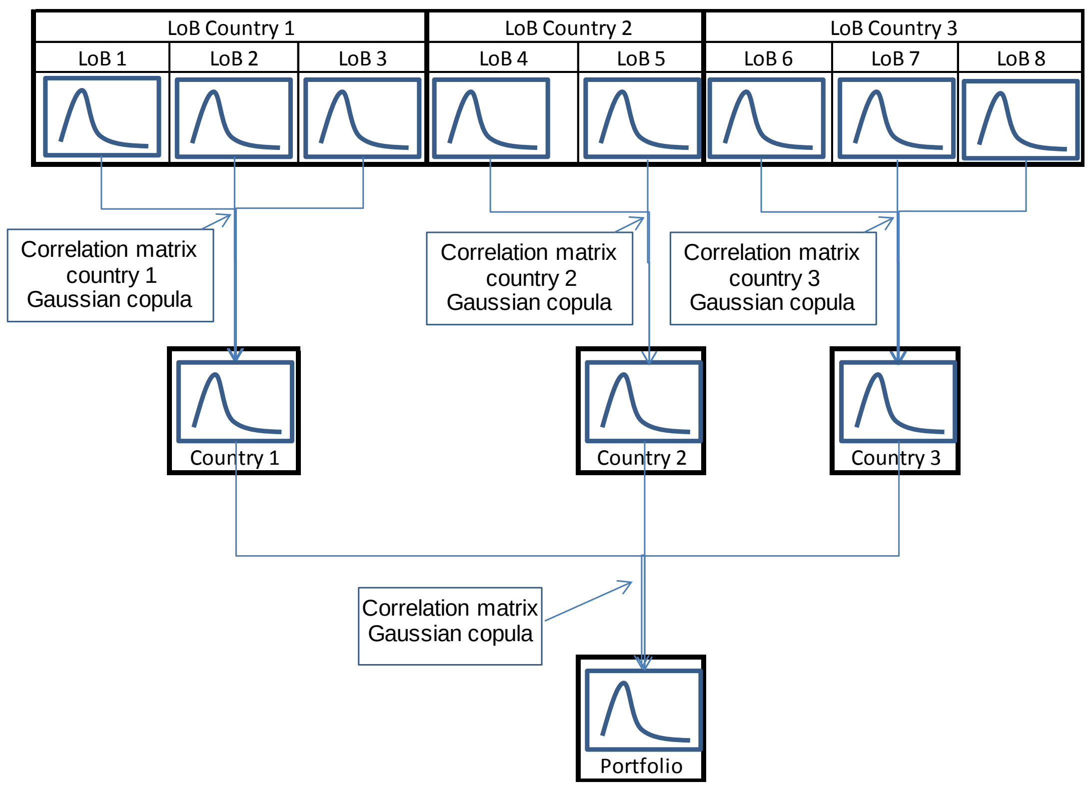
Remark: An excel sheet developed to estimate the presented formula is available on the URL:
https://drive.google.com/file/d/0B6piPKdUSkYIbEs5QmdFQ2xBdGM/view?usp=sharing

2. GENERAL METHODOLOGY

The general methodology is based on the following requirements:

- The first requirement is that there exists an overall reserve risk distribution coming from the internal model;
- The second requirement is that the loss amount related to the adverse scenario is known and has been estimated using external assumptions and tools.

As for the first requirement, the internal model will be assumed to follow the aggregation tree described below:



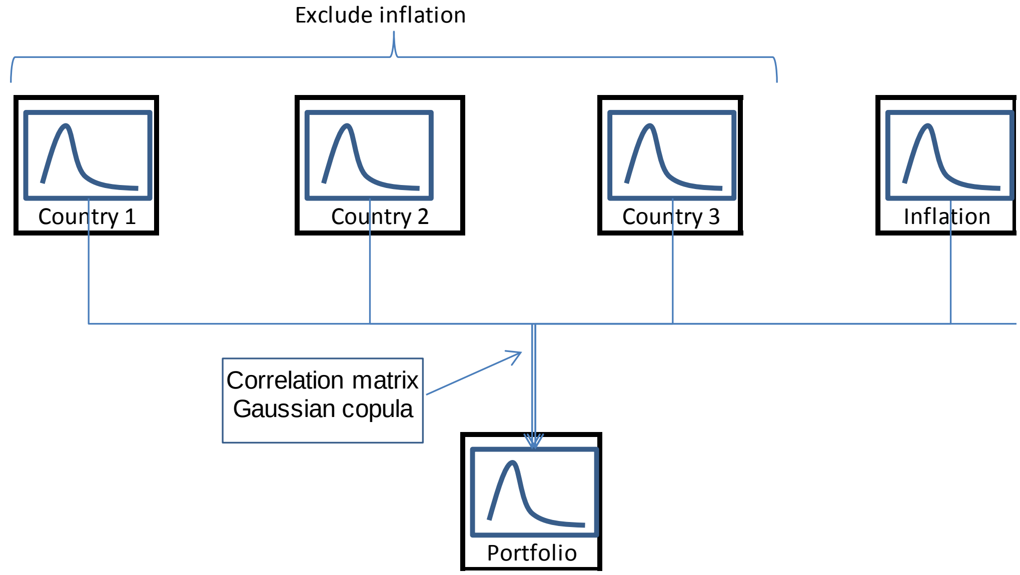
The above aggregation tree is based on a 2 step aggregation, first aggregating the loss distributions at country level and then aggregating the country reserve risk distributions. For the aggregation steps, a Gaussian copula (see next sections) is used for which the correlation matrix is assumed to be known. As a result of the aggregation, the loss amount of the proposed scenario can be found at different levels in the above aggregation tree.

For example, in the case of inflation, the loss amount (e.g. 250) and its return period can be found in the risk distribution of the overall portfolio as it can be seen below:

Scenario	Loss
1	45
2	47
…	
750	250
…	
1000	1200

The return period read from the overall risk distribution would correspond to $\frac{1}{1-75\%}=4$, i.e. a scenario which can occur every 4 years.

However, the return period which comes from the overall risk distribution will not correspond to the real return period of the inflation scenario as it incorporates more contributing factors than just inflation. The idea is therefore to try and separate the contributing factor of the scenario so as to have a proxy of the return period for this scenario, all other factors being neutral. Therefore, the aggregation tree will be modified as follows (in the case of inflation):



In order to separate inflation, a choice of the Lines of Business most exposed to inflation will have to be done. Such lines could include the long tail lines such as medical malpractice while the other lines would be kept within their respective countries.

Following the proposed separation, the distributions for each country and for inflation will be characterized by:

- their best estimates,
- their coefficients of variations,
- their skewness.

Knowing these 3 parameters, the Cornish-Fisher expansions (see next sections) will be used to have a proxy of the full distributions. The next sections will therefore focus on describing the properties of the Cornish-Fisher expansion and of the Gaussian copula.

3. SOME BASIC PROPERTIES

2.1 Cornish-Fisher expansion

In 1938, the Cornish-Fisher expansion (see Cornish et al 1938) introduced an approximation for the quantiles of a random variable based on its first few cumulants. For this expansion, let X be a random variable with density function f(x) with mean 0 and variance 1. Let β_1 be the skewness of this distribution. Let Z be a normally distributed random variable and let z_α be the α^{th} quantile of this distribution. Then the α^{th} quantile ω_α of the distribution X can be approximated by:

$$\omega_\alpha = z_\alpha + \frac{1}{6}\left(z_\alpha^2 - 1\right)\beta_1 \qquad (1)$$

In order to apply this formula, it is therefore necessary to have the first three moments of a distribution. These moments can be easily retrieved from the risk distributions at country level.

2.2 Gaussian copula

An n-dimensional copula (See Sklar (1959)) is a multivariate distribution function, C, with uniform distributed margins in [0,1] and the following properties:

1. $C : [0,1]^n \to [0,1]$
2. C is grounded and n-increasing
3. C has margins C_i which satisfy $C_i(u) = C(1,...,1,u,1,...,1) = u$ for all $u \in [0,1]$.

It is obvious from the above definition, that, if F_1, …, F_n are univariate distribution functions, $C(F_1(x_1),…, F_n(x_n))$ is a multivariate distribution function with margins F_1, …, F_n. Copula functions are a useful tool to construct and simulate multivariate distributions.

Sklar's theorem: Let F be an n-dimensional distribution function with continuous margins F_1, …, F_n. Then it has the following unique copula representation:

$$F(x_1,…, x_n) = C(F_1(x_1),…, F_n(x_n)) \qquad (2)$$

For the proof of the theorem, see Sklar (1996).

Corollary: Let F be an n-dimensional distribution function with continuous margins F_1, …, F_n.and copula C satisfying (1). Then, for any u=(u_1,…, u_n) in $[0,1]^n$:

$$C(u_1,...,u_n) = F\left(F_1^{-1}(u_1),...,F_n^{-1}(u_n)\right)$$

where F_i^{-1} is the generalized inverse of F_i.

Gaussian copula

If X has the stochastic representation:

$$X \stackrel{d}{=} \mu + AZ$$

where $Z_1,...,Z_n \sim \mathrm{N}(0,1)$ are independent.

Then X has an n-variate Gaussian distribution with mean vector μ and covariance matrix $\Sigma = AA^T$.

The copula of the n-variate normal distribution with linear correlation matrix Σ is:

$$C_\Sigma(u) = \Phi_\Sigma^n\left(\Phi^{-1}(u_1),...,\Phi^{-1}(u_n)\right)$$

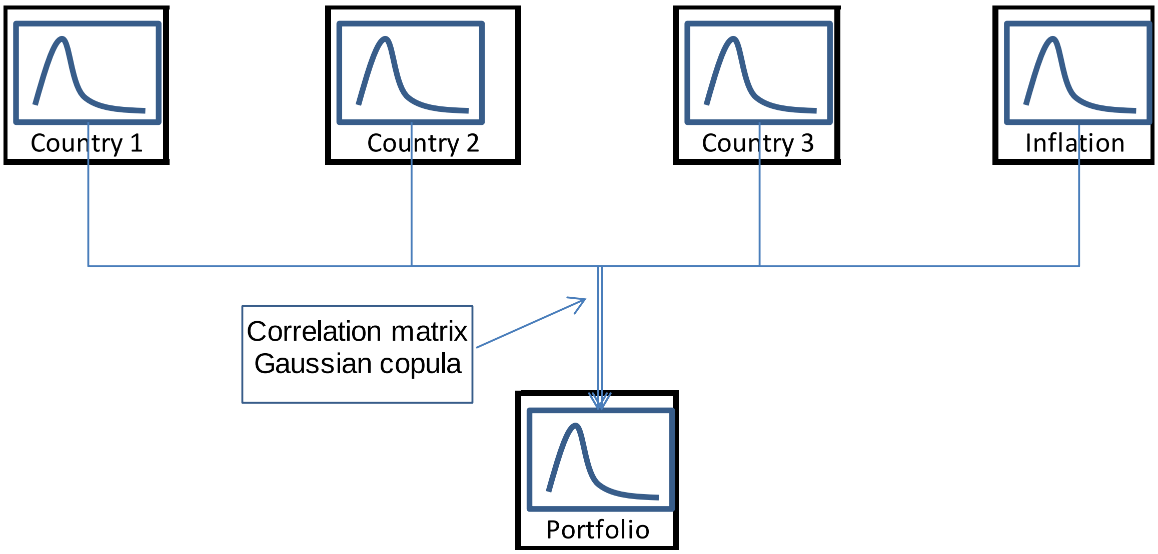
where Φ_Σ^n denotes the joint distribution function of the n-variate standard normal distribution function with linear correlation matrix Σ, and Φ^{-1} denotes the inverse of the distribution function of the univariate standard normal distribution.

To simulate random variates from the Gaussian copula, C_Σ, we can use the following algorith:

- find the Cholesky decomposition, A, of Σ so that $\Sigma = A^t A$
- simulate n independent random variates $z = \begin{pmatrix} z_1 \\ ... \\ z_n \end{pmatrix}$ from the standard normal distribution
- set the vector y = Az
- determine the components $u_i = \Phi(x_i), i = 1,...,n$
- the resulting vector is: $u = \begin{pmatrix} u_1 \\ ... \\ u_n \end{pmatrix} \sim C_\Sigma$.

4. MATHEMATICAL FRAMEWORK

The mathematical framework is going to be based on the following aggregation tree:



For the above tree, we know the following parameters:

- The correlation matrix used for the Gaussian copula is known;
- For each country and for the inflation, we know:
  - The mean denoted as μ_i. It corresponds to the reserves best estimate (excluding the reserves allocated to the inflation distribution for each country)
  - The standard deviation denoted as σ_i. This standard deviation can be obtained directly from the distribution resulting from the internal model. As for inflation, a standardized standard deviation would have to be estimated.
  - The skewness denoted as ω_i . The skewness can also be obtained directly from the distribution resulting from the internal model.

In addition, we know the loss amount resulting from the inflation scenario. It will be denoted as L_{scen}. Looking at the overall portfolio loss distribution, the return period related to this scenario is known and the quantile α on the distribution related to this scenario is also known.

On the basis of the above assumptions, we can say that:

- The mean of the overall portfolio distribution denoted M is equal to:

$$\mathrm{M} = \mu_1 + \mu_2 + \mu_3 + \mu_{\text{inflation}}$$

- The standard deviation of the overall portfolio distribution denoted Σ is equal to:

$$\Sigma = \sqrt{{}^{t}\begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_{\text{inflation}} \end{pmatrix} \begin{pmatrix} 1 & \rho_{12} & \rho_{13} & \rho_{14} \\ \rho_{21} & 1 & \rho_{23} & \rho_{24} \\ \rho_{31} & \rho_{32} & 1 & \rho_{34} \\ \rho_{41} & \rho_{42} & \rho_{43} & 1 \end{pmatrix} \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_{\text{inflation}} \end{pmatrix}}$$

- The skewness of the overall portfolio distribution denoted Ω is given from the results of the internal model distribution.

With these assumptions and using the Cornish-Fisher expansion, we can estimate that:

$$L_{scen} = \mathrm{M} + \left(\Phi^{-1}(\alpha) + \frac{1}{6}\left(\Phi^{-1}(\alpha)^2 - 1\right)\Omega \right)\Sigma$$

where Φ denotes the standard normal distribution function.

As mentioned earlier, we are looking for a proxy of the return period for this scenario, all other factors being neutral. As a result, in order to estimate the proxy:

- we assume that the factor neutralization corresponds to the VaR 50% (Value at risk) on the country risk distribution,
- and we are looking for the quantile **β** on the inflation risk distribution.

Using the Cornish-Fisher expansion again, we find:

$$\begin{aligned}
L_{scen} &= \mu_1 + \left(\Phi^{-1}(50\%) + \frac{1}{6}\left(\Phi^{-1}(50\%)^2 - 1\right)\omega_1\right)\sigma_1 \\
&\quad + \mu_2 + \left(\Phi^{-1}(50\%) + \frac{1}{6}\left(\Phi^{-1}(50\%)^2 - 1\right)\omega_2\right)\sigma_2 \\
&\quad + \mu_3 + \left(\Phi^{-1}(50\%) + \frac{1}{6}\left(\Phi^{-1}(50\%)^2 - 1\right)\omega_3\right)\sigma_3 \\
&\quad + \mu_{\text{inflation}} + \left(\Phi^{-1}(\beta) + \frac{1}{6}\left(\Phi^{-1}(\beta)^2 - 1\right)\omega_{\text{inflation}}\right)\sigma_{\text{inflation}} \\
\Rightarrow L_{scen} &= \mu_1 - \frac{1}{6}\omega_1\sigma_1 + \mu_2 - \frac{1}{6}\omega_2\sigma_2 + \mu_3 - \frac{1}{6}\omega_3\sigma_3 \\
&\quad + \mu_{\text{inflation}} + \left(\Phi^{-1}(\beta) + \frac{1}{6}\left(\Phi^{-1}(\beta)^2 - 1\right)\omega_{\text{inflation}}\right)\sigma_{\text{inflation}} \\
&= \mathrm{M} + \left(\Phi^{-1}(\alpha) + \frac{1}{6}\left(\Phi^{-1}(\alpha)^2 - 1\right)\Omega\right)\Sigma
\end{aligned}$$

As we have: $\mathrm{M} = \mu_1 + \mu_2 + \mu_3 + \mu_{\text{inflation}}$, the above equation simplifies into a standard quadratic equation with unknown $\Phi^{-1}(\beta)$:

$$\begin{aligned}
&\left(\Phi^{-1}(\alpha) + \frac{1}{6}\Omega\Phi^{-1}(\alpha)^2\right)\frac{\Sigma}{\sigma_{\text{inflation}}} + \frac{1}{6\sigma_{\text{inflation}}}\left(\omega_1\sigma_1 + \omega_2\sigma_2 + \omega_3\sigma_3 + \omega_{\text{inflation}}\sigma_{\text{inflation}} - \Omega\Sigma\right) \\
&= \Phi^{-1}(\beta) + \frac{1}{6}\omega_{\text{inflation}}\Phi^{-1}(\beta)^2
\end{aligned}$$

The above equation can be solved easily. If we set:

$$\begin{aligned}
a &= \frac{1}{6}\omega_{\text{inflation}} \\
b &= 1 \\
c &= -\left(\left(\Phi^{-1}(\alpha) + \frac{1}{6}\Omega\Phi^{-1}(\alpha)^2\right)\frac{\Sigma}{\sigma_{\text{inflation}}} + \frac{1}{6\sigma_{\text{inflation}}}\left(\omega_1\sigma_1 + \omega_2\sigma_2 + \omega_3\sigma_3 + \omega_{\text{inflation}}\sigma_{\text{inflation}} - \Omega\Sigma\right)\right)
\end{aligned}$$

then, we have:

$$\beta = \Phi\left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}\right) \qquad (3)$$

5. NUMERICAL EXAMPLE

In this section, we will illustrate the formula (3) through a small numerical example. We choose the following lognormal distributions for inflation, country 1, 2 and 3:

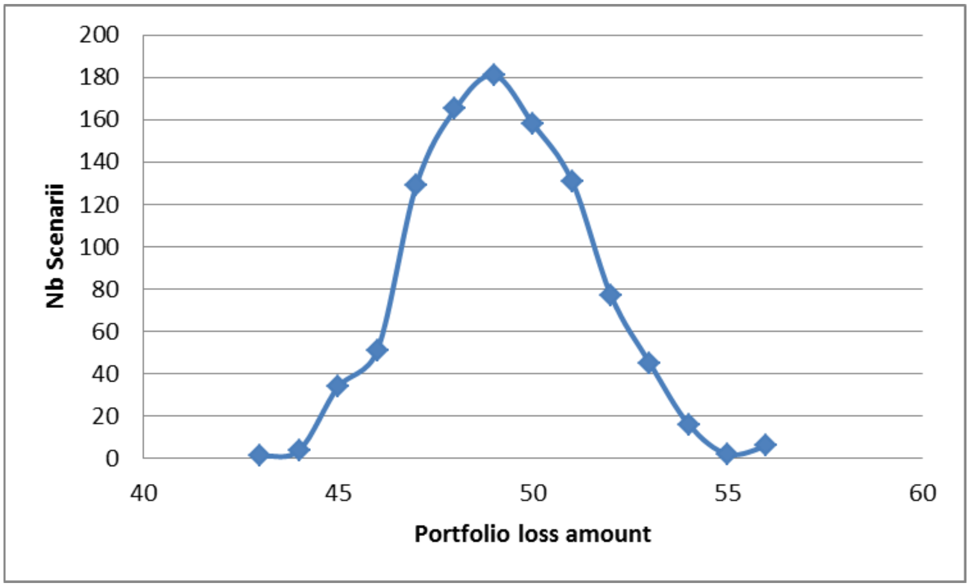
	Lognormal					
	m	s	CoV	Skewness ω	Mean μ	Std deviation σ
Inflation	2.301	0.050	0.05	0.15	10.000	0.500
Country 1	3.000	0.060	0.06	0.18	20.122	1.207
Country 2	2.500	0.040	0.04	0.12	12.192	0.488
Country 3	2.000	0.070	0.07	0.21	7.407	0.518

In this example, we propose to look for an inflation scenario corresponding an amount of 49.91 provided by an external model.

As described in the sections above, the 4 distributions are aggregated on using a Gaussian Copula which correlation matrix is provided below:

	Inflation	Country 1	Country 2	Country 3
Inflation	100%	50%	50%	50%
Country 1	50%	100%	50%	50%
Country 2	50%	50%	100%	50%
Country 3	50%	50%	50%	100%

When aggregating the 4 above distribution with the Gaussian copula, we get the aggregated distribution below:



Using this distribution, we find that the proposed scenario is found at the 55% quantile.

Finally, on using equation (3), we find that this scenario would correspond to a 70.7% quantile on the inflation distribution when it is the only contributor to the scenario.

6. CONCLUSIONS

This article proposes a proxy to get the return period for a scenario which amount is known but for which the contributing factor is implicitly embedded in the overall risk distribution. In the solvency frameworks currently being developed such proxy should prove to be useful. However, when using the proxy, the reader should bear in mind the following limitations:

- It is necessary to assume some parameters related to the contributing factors such as CoV and skewness. Such assumptions are crucial for the application of the proxy formula. However, such assumptions are usually very difficult to justify. As a result, sensitivities to the proposed assumptions should be performed.

- In order to go one step up in the aggregation tree, it is also necessary to make an assumption on the split of the reserves related to the contributing factors. This choice is usually arbitrary and, for this assumption too, sensitivities should be performed.

When keeping these limitations in mind, the proxy can help to answer to the usually difficult question of the return period of a scenario. Hopefully, such proxy can be extended to more complex scenario cases.

Acknowledgment

The views expressed in the paper are entirely my own and should not be taken as representing those of my firm.

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