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Optimizing transition rules of bonus-malus system under different criteria

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Preface

Bonus-malus systems are used as a tools of posterior premiums differentiation in risk assessment process in automobile insurance. While the tools of systems analysis and premium calculation criteria are well-described in the literature, relatively little space is devoted to the optimization of transition rules between classes of a bonus-malus system. The problem seems to be particularly interesting when designing a bonus-malus system. The possibility of building a system that meets the specified optimality criterion in advance seems to be desirable.

In our research we optimize transition rules of bonus-malus systems with respect to different criteria, to achieve possibly best system against different measures. This issue constitutes a nonlinear nonconvex discrete optimization problem. To solve this problem, we engage improved greedy optimization algorithm, similar to the one proposed by Marlock [1984]. We use premium scales given by the minimization of mean square error proposed by Norberg [1976]. Optimization is carried for nine portfolios that differ by the risk structure, that is mean claim ratio and claim variance.

We try to determine if optimization of transition rules can improve and objectify the process of building a bonus-malus system, and if the aim to obtain a bonus-malus system of good statistical properties goes hand in hand with the desired market utility performance of the system. We also try to check if it is possible to eliminate to main disadvantages of bonus-malus system which are being widely criticised, that policyholders tend to cluster in 'better classes' and that bonus-malus systems have low premium elasticity.

Considering above reflections, two main research questions are:

- Can we improve and objectify process of building of bonus-malus system by optimizing transition rules?
- Can we eliminate disadvantages of bonus-malus system by optimizing transition rules?

Risk structure

To model risk process, we use two random variables:

K– number of claims, having Poisson distribution, K ~ *Poisson*(λ),

Λ – claim rate, having Inverse Gaussian distribution[1], Λ ~ Inverse Gaussian *IG*(μ, θ), with probability density function $u(\lambda)$ and distribution function $U(\lambda)$.

Function is $u(\lambda)$ called **risk structure** function.

Conditional probability of k claims in unitary period (one year) is

$$P_k(\lambda) = P(K = k|\Lambda = k) = \frac{e^{-\lambda}\lambda^k}{k!}$$

and unconditional probability of k claims in unitary period (one year) is

$$P_k = P(K = k) = \int_0^\infty \frac{e^{-\lambda}\lambda^k}{k!} u(\lambda) d\lambda = \int_0^\infty \frac{e^{-\lambda}\lambda^k}{k!} dU(\lambda).$$

With above assumptions we have:

$$EK = \mu,$$
$$VarK = \mu + \mu\theta,$$
$$E\Lambda = \mu,$$
$$Var\Lambda = \mu\theta.$$

We also introduce following propositions:

- we assume that the amount of a claim and the number of claims are independent,
- we assume expected claim amount = 1 (claim rate λ is a measure of risk of a single insured),
- we assume heterogeneous portfolio (insured differ by claim rate λ) with overdispersion ($EK < VarK$),
- we assume there is no bonus hunger.

Using Inverse Gaussian distribution allows well represent real portfolios and easy describe their main characteristics using only distribution parameters μ and θ, as $E\Lambda = \mu$ and $Var\Lambda = \mu\theta$.

Bonus-malus system (BMS)

After Lemaire [1985] we assume that bonus-malus system consists of:

- Finite number of classes $i \epsilon S, S = \{1, 2, \ldots, s\}$ such as insured belongs to one and only one class in unitary period and the class in the next period depends only on the class and the number of claims reported in the current period – rudiments that describe how insures move between classes are called **transition rules**,
- Premiums b_i specified for each class,

[1] Willmot [1987].

- Specified starting class for those who insure for the first time (unnecessary condition for stationary state analysis).

Furthermore, we assume that:

- The best class is class number 1 (best - with the lowest premium and the most favourable transition rules),
- The worst class is class number *s*.

Typically, transition rules can be represented by a transition table or transition matrix $\boldsymbol{T} = [t_{ik}]$, which shows to which class insured passes from class *i* after *k* claims.

Example of transition table

	k =	*0*	*1*	*2*	*3+*
class	*1*	1	2	3	5
	2	1	3	5	5
	3	2	5	6	6
	4	3	6	6	7
	5	4	6	7	7
	6	5	7	7	8
	7	6	7	8	8
	8	7	8	8	9
	9	8	9	9	10
	10	9	10	10	10

Example of transition matrix

$$\boldsymbol{T} = [t_{ik}] = \begin{bmatrix} 1 & 2 & 3 & 5 \\ 1 & 3 & 5 & 5 \\ 2 & 5 & 6 & 6 \\ 3 & 6 & 6 & 7 \\ 4 & 6 & 7 & 7 \\ 5 & 7 & 7 & 8 \\ 6 & 7 & 8 & 8 \\ 7 & 8 & 8 & 9 \\ 8 & 9 & 9 & 10 \\ 9 & 10 & 10 & 10 \end{bmatrix}$$

Model of a bonus-malus system

As bonus-malus system possess Markov property (class in the next period depends only on the class and the number of claims in the previous period) it is usually modelled by suitable Markov chain (Lemaire [1985], [1995]).

Transformation matrix is a matrix $\boldsymbol{T}_k = [t_{ij}(k)]$, where:

$$t_{ij}(k) = \begin{cases} 1 & dla \;\; t_{ik} = j \\ 0 & dla \;\; t_{ik} \neq j \end{cases}$$

Probability of transition from class i to class j (depending on claim rate λ)

$$p_{ij}(\lambda) = \sum_{k=0}^{\infty} p_k(\lambda) t_{ij}(k)$$

The transition probability matrix of Markov chain

$$\boldsymbol{P}(\lambda) = [p_{ij}(\lambda)] = \sum_{k=0}^{\infty} p_k(\lambda) \boldsymbol{T}_k$$

For regular transition probability matrix, after sufficient time the chain tends to stationary state (Kemeny [1976]) with stationary distribution:

$$\boldsymbol{e}(\lambda) = [e_1(\lambda), \dots, e_s(\lambda)]$$

that fulfils conditions given by equation system

$$\begin{cases} \boldsymbol{e}(\lambda)\boldsymbol{P}(\lambda) = \boldsymbol{e}(\lambda) \\ \boldsymbol{e}(\lambda)\mathbf{1} = 1 \end{cases}$$

Unconditional stationary distribution is given by

$$\boldsymbol{e} = [e_1, \dots, e_s] = [\int_0^\infty e_1(\lambda)u(\lambda)d\lambda, \dots, \int_0^\infty e_s(\lambda)u(\lambda)d\lambda].$$

Permissible systems

To ensure real live application and operatives driven by common sense we limit ourselves to systems which fulfil below conditions:

- elements in rows of transition matrix $\boldsymbol{T}$ are non-decreasing (weak monotonicity in rows) - in each class penalty[2] for more claims is no less than for fewer claims,
- elements in columns of transition matrix $\boldsymbol{T}$ are non-decreasing (weak monotonicity in columns) – penalty for the same number of claims in the worse class cannot be less than in the better class (with the exception of the worst class)[3],
- systems are irreducible (are modelled by an irreducible Markov chain) - none of elements of stationary distribution equal zero,
- systems are ergodic (are modelled by an ergodic Markov chain) - stationary distribution does not depend on starting class.

Systems which fulfil above conditions are called **permissible systems**.

Premiums

We use Norberg criterion of premiums calculation (Norberg [1976]), based on minimising square error of risk assessment

$$Q(\boldsymbol{b}) = \int_0^\infty \sum_{j=1}^{s} \left(b_j - \lambda\right)^2 e_j(\lambda)u(\lambda)d\lambda \rightarrow min$$

[2] At this point penalty is understood in terms of transition to a worse class, we do not address premiums yet.
[3] This is enhanced definition of 'systems justice in terms of transition rules' (Podgórska et al. [2006]).

It gives so called *Q*-optimal premiums (where b_j is a premium to be paid in class j)

$$b_j = \frac{\int_0^\infty \lambda e_j(\lambda)u(\lambda)d\lambda}{\int_0^\infty e_j(\lambda)u(\lambda)d\lambda} = \frac{\int_0^\infty \lambda e_j(\lambda)u(\lambda)d\lambda}{e_j}$$

Premiums can be represented as a vector $\boldsymbol{b} = \left[b_j\right]$. For *Q*-optimal premiums, system is financially balanced, that is stationary premium equals expected claim rate for the portfolio (and equals μ for *IG* risk structure function).

$$b_e = \sum_{j=1}^{s} e_j b_j = E\Lambda = EK = \mu$$

Characteristics of a bonus-malus system

To monitor properties of bonus-malus system we use following measures (characteristics):

- Stationary premium (Loimaranta [1972])

  $$b_e = \sum_{j=1}^{s} e_j b_j$$

- Volatility coefficient of the stationary premium (Lemaire [1985], [1995])

  $$V_{b_e} = \frac{\sqrt{\sum_{j=1}^{s}\left(b_j - b_e\right)^2 e_j}}{b_e}$$

- RSAL – Relative stationary average level (Lemaire [1985], [1995])

  $$RSAL = \frac{b_e - b_1}{b_s - b_1}$$

- Elasticity of the stationary premium (Loimaranta [1972])

  $$\eta(\lambda) = \frac{\partial b_e}{b_e} \Big/ \frac{\partial \lambda}{\lambda} = \frac{\partial b_e}{\partial \lambda}\frac{\lambda}{b_e}$$

- Global elasticity of the stationary premium (De Pril [1977])

  $$\eta = \int_0^\infty \eta(\lambda)u(\lambda)d\lambda$$

Optimization criteria

We want to find systems which are ‘the best’ on different areas. It means that we need to optimize transition rules with respect to different criteria. We consider three areas: goodness of risk assessment, elasticity and (stationary) premium volatility.

Goodness of risk assessment is measured by the mean square error of assessment, which is the principle of premium calculation using Norberg method (Norberg [1976]). Thus our **first optimization criterion** is

- to minimize mean square error of assessment

$$Q(\boldsymbol{b}) = \int_0^\infty \sum_{j=1}^{s} \left(b_j - \lambda\right)^2 e_j(\lambda) u(\lambda) d\lambda \rightarrow min \qquad (1)$$

As values of $Q(\boldsymbol{b})$ depend on the portfolio (risk structure function), this measure can't be directly used to compare accuracy of risk assessment for different distribution of risk parameter Λ. To depict quality of assessment for different portfolios additionally we use normalized measure of accuracy of risk assessment QN for Q-optimal premium scales[4] (Bernardelli, Topolewski [2015]), which is calculated as

$$QN = \frac{\sum_{j=1}^{s} e_j b_j^2 - (E\Lambda)^2}{E\Lambda^2 - (E\Lambda)^2}$$

This measure takes values between 0 and 1. Values closer to one show better accuracy (lower error of risk assessment) while values closer to zero reflect worse accuracy.

Most favourite value for elasticity is 1. It means that change of stationary premium is proportional to change in risk measured by claim ratio λ. Generally, values of elasticity lower and higher than one are recognized as not appropriate, meaning respectively under-reaction and over-reaction of stationary premium for change of claim ratio. In reality systems usually show low values of elasticity $\eta(\lambda)$ for large range of λ, and as a concequence low values of global elasticity η. Hence there is a temptation to design transition rules that force high values (higher than one) of point elasticity $\eta(\lambda)$ to achieve higher value of global elasticity. To avoid this unfavourable situation in our approach best global elasticity is achieved by minimizing distance of elasticity $\eta(\lambda)$ to 1, weighted by the risk structure function $u(\lambda)$. Thus our **second optimization criterion** is

- to minimize mean absolute error of elasticity (mean absolute distance from 1)

$$MAE_\eta = \int_0^\infty |1 - \eta(\lambda)| u(\lambda) d\lambda \rightarrow min \qquad (2)$$

The volatility of stationary premium is measured by volatility coefficient V_{b_e}. Low values of this coefficient indicate low risk differentiation by the system, while high values are associated with transferring huge portion of risk to the insured instead of insurer. Although there is no universally recognized value of this coefficient, some authors indicate that values higher than 1 can be hard to accept by customers (Lemaire, Zi [1994])). In order to achieve

[4] This measoure is adequate for Q-optimal premium scales only.

reasonable trade-off between risk differentiation and financial stability of insured we set the most wanted value for volatility coefficient for 1. Thus our **third optimization criterion** is

- to minimize absolute error of premium volatility (absolute distance from 1)

$$MAE_V = |1 - V_{b_e}| \to min \qquad (3)$$

Optimization of transition rules with respect to above functions is nonlinear and non-convex discrete optimization problem.

The algorithm

We use greedy algorithm similar to one used by Marlock [1984] but with some alteration:

- we consider stationary state (stationary distribution),
- we impose weak monotonicity conditions, both in rows and in columns in the table of bonus-malus system (permissible systems),
- we limit ourselves to irreducible and ergodic systems,
- we use different directions of optimization (rows, columns, diagonals).

Subsequently for each element t_{ik} of transition matrix $\boldsymbol{T}$ we change its value (taking into account the conditions for irreducibility, ergodicity and monotonicity of the system), for each value we calculate value of optimizing criterion (goal function) and we choose t_{ik} which minimizes our optimization criterion. After optimization of all elements of the matrix $\boldsymbol{T}$ procedure is repeated and we compare the results with the previous iteration. If in two subsequent iterative steps algorithm will show the same solution, we stop the procedure. What is important, we apply above algorithm in three ways, changing values of $[t_{ik}]$ elements in rows, columns and by diagonals starting from different points (different $\boldsymbol{T}$ matrices). Solutions may differ – this is a greedy algorithm and may not always give globally optimal solution for each way. For each portfolio we optimize each criterion separately.

Research

Using above algorithm, we optimise transition rules of bonus-malus systems:

- for systems with 10 classes that count up to 3 claims (more than 3 is treated as 3).
- for risk structure function given by Inverse Gaussian distribution - $IG(\mu, \theta)$,
- for different mean and variance of claim rate (different parameters of inverse Gaussian distribution), to screen portfolios with low and high claim rate and claim variance,
- for stationary state.

Portfolios are shown below:

Portfolio 1	Portfolio 2	Portfolio 3
μ = 0.05 θ = 0.01	μ = 0.05 θ = 0.05	μ = 0.05 θ = 0.15

Portfolio 4	Portfolio 5	Portfolio 6
μ = 0.15 θ = 0.01	μ = 0.15 θ = 0.05	μ = 0.15 θ = 0.15

Portfolio 7	Portfolio 8	Portfolio 9
μ = 0.30 θ = 0.01	μ = 0.30 θ = 0.05	μ = 0.30 θ = 0.15

Results

Results are shown in tables. Each table contains optimal transition rules for all three criteria, together with stationary distributions[5], percentage premium scales, system characteristics and values of optimization criteria. Percentage premium scales were calculated based on Q-optimal scales assuming that starting class is class 7, which is subjective (we believe that this way of premium presentation is closer to what is observed in real bonus-malus systems).

Table 1. Portfolio 1, *IG*(0.05; 0.01).

$Q(\boldsymbol{b}) \to min$			$MAE_\eta \to min$			$MAE_V \to min$		
T	***e***	***b %***	***T***	***e***	***b %***	***T***	***e***	***b %***
1 2 4 6	0.9296	7.1	1 8 10 10	0.7861	18.1	1 10 10 10	0.7641	17.5
1 4 6 7	0.0315	25.5	1 10 10 10	0.0178	58.1	1 10 10 10	0.0169	57.2
2 6 7 7	0.0058	49.0	2 10 10 10	0.0192	63.2	2 10 10 10	0.0181	62.3
3 6 7 8	0.0073	53.5	3 10 10 10	0.0208	69.3	3 10 10 10	0.0196	68.5
4 7 8 8	0.0042	73.5	4 10 10 10	0.0227	76.9	4 10 10 10	0.0214	76.3
5 7 8 9	0.0060	80.0	5 10 10 10	0.0251	86.8	5 10 10 10	0.0236	86.4
6 8 9 9	0.0051	100.0	6 10 10 10	0.0281	100.0	6 10 10 10	0.0265	100.0
7 9 9 10	0.0041	125.3	7 10 10 10	0.0320	118.9	7 10 10 10	0.0302	119.6
8 9 10 10	0.0043	157.3	8 10 10 10	0.0204	226.6	8 10 10 10	0.0356	150.1
9 10 10 10	0.0022	237.1	9 10 10 10	0.0278	295.4	9 10 10 10	0.0440	205.3
	Q =	0.0046		*Q* =	0.0076		*Q* =	0.0089
	MAEη =	0.85		*MAEη* =	0.6449		*MAEη* =	0.6529
	V_{be} =	1.7751		V_{be} =	1.3956		V_{be} =	1.1938
	QN =	0.6302		*QN* =	0.3895		*QN* =	0.2851
	η =	0.2132		*η* =	0.3551		*η* =	0.3471
	RSAL =	0.0173		*RSAL* =	0.0836		*RSAL* =	0.1257

[5] Probabilities shown in vectors ***e*** may seem not sum up to one due to rounding.

Table 2. Portfolio 2, *IG*(0.05; 0.05).

$Q(\boldsymbol{b}) \to min$			$MAE_{\eta} \to min$			$MAE_{V} \to min$		
T	***e***	***b %***	***T***	***e***	***b %***	***T***	***e***	***b %***
1 3 8 10	0.8612	30.6	1 5 10 10	0.7854	31.7	1 3 8 10	0.8612	30.6
1 8 10 10	0.0366	53.8	1 10 10 10	0.0312	55.2	1 8 10 10	0.0366	53.8
2 8 10 10	0.0394	56.0	2 10 10 10	0.0334	57.4	2 8 10 10	0.0394	56.0
3 10 10 10	0.0072	86.4	3 10 10 10	0.0359	59.8	3 10 10 10	0.0072	86.4
4 10 10 10	0.0081	90.4	4 10 10 10	0.0386	62.5	4 10 10 10	0.0081	90.4
5 10 10 10	0.0092	95.0	5 10 10 10	0.0116	95.0	5 10 10 10	0.0092	95.0
6 10 10 10	0.0105	100.0	6 10 10 10	0.0130	100.0	6 10 10 10	0.0105	100.0
7 10 10 10	0.0120	105.6	7 10 10 10	0.0147	105.7	7 10 10 10	0.0120	105.6
8 10 10 10	0.0071	138.6	8 10 10 10	0.0168	112.1	8 10 10 10	0.0071	138.6
9 10 10 10	0.0086	147.0	9 10 10 10	0.0193	119.6	9 10 10 10	0.0086	147.0
	Q =	0.0018		*Q* =	0.0018		*Q* =	0.0018
	MAEη =	0.79		*MAEη* =	0.79		*MAEη* =	0.79
	V_{be} =	0.5409		V_{be} =	0.5409		V_{be} =	0.5409
	QN =	0.2925		*QN* =	0.2925		*QN* =	0.2925
	η =	0.21		*η* =	0.21		*η* =	0.21
	RSAL =	0.0018		*RSAL* =	0.0018		*RSAL* =	0.0018

Table 3. Portfolio 3, *IG*(0.05 ;0.15).

$Q(\boldsymbol{b}) \to min$			$MAE_{\eta} \to min$			$MAE_{V} \to min$		
T	***e***	***b %***	***T***	***e***	***b %***	***T***	***e***	***b %***
1 5 10 10	0.7762	57.0	1 6 10 10	0.7455	57.5	1 5 10 10	0.7762	57.0
1 10 10 10	0.0362	74.2	1 10 10 10	0.0344	74.8	1 10 10 10	0.0362	74.2
2 10 10 10	0.0384	75.5	2 10 10 10	0.0365	76.0	2 10 10 10	0.0384	75.5
3 10 10 10	0.0408	76.7	3 10 10 10	0.0387	77.3	3 10 10 10	0.0408	76.7
4 10 10 10	0.0433	78.1	4 10 10 10	0.0412	78.7	4 10 10 10	0.0433	78.1
5 10 10 10	0.0110	98.0	5 10 10 10	0.0438	80.1	5 10 10 10	0.0110	98.0
6 10 10 10	0.0119	100.0	6 10 10 10	0.0132	100.0	6 10 10 10	0.0119	100.0
7 10 10 10	0.0129	102.1	7 10 10 10	0.0143	102.0	7 10 10 10	0.0129	102.1
8 10 10 10	0.0140	104.2	8 10 10 10	0.0155	104.2	8 10 10 10	0.0140	104.2
9 10 10 10	0.0152	106.5	9 10 10 10	0.0168	106.4	9 10 10 10	0.0152	106.5
	Q =	0.0007		*Q* =	0.0007		*Q* =	0.0007
	MAEη =	0.8877		*MAEη* =	0.8873		*MAEη* =	0.8877
	V_{be} =	0.1993		V_{be} =	0.1977		V_{be} =	0.1993
	QN =	0.1192		*QN* =	0.1172		*QN* =	0.1192
	η =	0.1123		*η* =	0.1127		*η* =	0.1123
	RSAL =	0.1214		*RSAL* =	0.1358		*RSAL* =	0.1214

Table 4. Portfolio 4, *IG*(0.15; 0.01).

$Q(\boldsymbol{b}) \to min$			$MAE_{\eta} \to min$			$MAE_{V} \to min$		
T	***e***	***b*** *%*	***T***	***e***	***b*** *%*	***T***	***e***	***b*** *%*
1 1 2 3	0.9447	2.1	1 9 10 10	0.7056	13.8	1 10 10 10	0.6957	13.5
1 3 4 4	0.0093	19.6	1 10 10 10	0.0173	47.7	1 10 10 10	0.0169	47.2
2 4 4 4	0.0094	28.6	2 10 10 10	0.0189	52.9	2 10 10 10	0.0184	52.4
3 4 4 5	0.0160	38.3	3 10 10 10	0.0208	59.5	3 10 10 10	0.0202	59.1
4 5 5 6	0.0090	64.7	4 10 10 10	0.0232	68.4	4 10 10 10	0.0226	68.0
5 5 5 7	0.0036	86.3	5 10 10 10	0.0263	80.9	5 10 10 10	0.0256	80.7
5 5 6 8	0.0016	100.0	6 10 10 10	0.0307	100.	6 10 10 10	0.0299	100.0
6 6 6 9	0.0013	124.9	7 10 10 10	0.0374	133.1	7 10 10 10	0.0365	133.5
6 6 6 10	0.0009	135.4	8 10 10 10	0.0495	207.2	8 10 10 10	0.0485	208.2
7 8 8 10	0.0044	221.2	9 10 10 10	0.0703	750.8	9 10 10 10	0.0858	618.7
	Q =	0.0678		*Q* =	0.233		*Q* =	0.2516
	MAEη =	0.9058		*MAEη* =	0.4891		*MAEη* =	0.4895
	V_{be} =	3.462		V_{be} =	2.1554		V_{be} =	1.9538
	QN =	0.799		*QN* =	0.3097		*QN* =	0.2545
	η =	0.1976		*η* =	0.5109		*η* =	0.5105
	RSAL =	0.0148		*RSAL* =	0.1002		*RSAL* =	0.1214

Table 5. Portfolio 5, *IG*(0.15; 0.05).

$Q(\boldsymbol{b}) \to min$			$MAE_{\eta} \to min$			$MAE_{V} \to min$		
T	***e***	***b*** *%*	***T***	***e***	***b*** *%*	***T***	***e***	***b*** *%*
1 2 3 5	0.8204	10.8	1 5 9 10	0.6047	20.7	1 9 10 10	0.5373	23.4
1 3 5 5	0.0718	24.8	1 9 10 10	0.0341	41.4	1 10 10 10	0.0298	50.1
2 5 6 6	0.0217	41.4	2 9 10 10	0.0381	44.7	2 10 10 10	0.0335	55.1
3 6 6 7	0.0111	57.5	3 9 10 10	0.0430	48.7	3 10 10 10	0.0381	61.5
4 6 7 7	0.0173	63.8	4 10 10 10	0.0490	53.8	4 10 10 10	0.0441	69.9
5 7 7 8	0.0191	79.4	5 10 10 10	0.0244	87.3	5 10 10 10	0.0523	81.8
6 7 8 8	0.0198	100.0	6 10 10 10	0.0311	100.0	6 10 10 10	0.0639	100.0
7 8 8 9	0.0104	137.4	7 10 10 10	0.0411	118.7	7 10 10 10	0.0825	131.6
8 9 9 10	0.0038	196.7	8 10 10 10	0.0579	150.1	8 10 10 10	0.0254	44.3
9 10 10 10	0.0047	284.9	9 10 10 10	0.0767	247.8	8 10 10 10	0.0931	247.5
	Q =	0.0175		*Q* =	0.0387		*Q* =	0.0449
	MAEη =	0.7394		*MAEη* =	0.5125		*MAEη* =	0.5678
	V_{be} =	1.4913		V_{be} =	1.1322		V_{be} =	1.0014
	QN =	0.7413		*QN* =	0.4273		*QN* =	0.3342
	η =	0.3438		*η* =	0.4875		*η* =	0.4322
	RSAL =	0.0346		*RSAL* =	0.165		*RSAL* =	0.1947

Table 6. Portfolio 6, *IG*(0.15; 0.15).

$Q(\mathbf{b}) \to min$			$MAE_\eta \to min$			$MAE_V \to min$		
T	***e***	***b %***	***T***	***e***	***b %***	***T***	***e***	***b %***
1 2 3 5	0.7898	25.9	1 3 7 9	0.6105	36.2	1 2 3 5	0.7898	25.9
1 3 5 7	0.0925	41.2	1 7 9 10	0.0570	54.7	1 3 5 7	0.0925	41.2
2 5 7 8	0.0255	58.0	2 7 9 10	0.0653	57.7	2 5 7 8	0.0255	58.0
3 7 8 8	0.0099	74.4	3 9 10 10	0.0220	80.2	3 7 8 8	0.0099	74.4
4 7 8 9	0.0136	79.0	4 10 10 10	0.0268	85.6	4 7 8 9	0.0136	79.0
5 8 9 9	0.0104	93.8	5 10 10 10	0.0332	92.1	5 8 9 9	0.0104	93.8
6 8 9 10	0.0156	100.0	6 10 10 10	0.0419	100.	6 8 9 10	0.0156	100.0
7 9 10 10	0.0154	116.5	7 10 10 10	0.0339	129.1	7 9 10 10	0.0154	116.5
8 10 10 10	0.0132	139.2	8 10 10 10	0.0471	144.2	8 10 10 10	0.0132	139.2
9 10 10 10	0.0140	169.7	9 10 10 10	0.0623	171.5	9 10 10 10	0.0140	169.7
	Q =	0.01		Q =	0.0124		Q =	0.01
	$MAE\eta$ =	0.7455		$MAE\eta$ =	0.5738		$MAE\eta$ =	0.7455
	V_{be} =	0.7466		V_{be} =	0.6716		V_{be} =	0.7466
	QN =	0.5575		QN =	0.451		QN =	0.5575
	η =	0.3343		η =	0.4262		η =	0.3343
	$RSAL$ =	0.071		$RSAL$ =	0.1917		$RSAL$ =	0.071

Table 7. Portfolio 7, *IG*(0.3; 0.01).

$Q(\mathbf{b}) \to min$			$MAE_\eta \to min$			$MAE_V \to min$		
T	***e***	***b %***	***T***	***e***	***b %***	***T***	***e***	***b %***
1 1 1 2	0.9580	2.1	1 10 10 10	0.6755	13.1	1 10 10 10	0.6755	13.1
1 2 2 3	0.0177	42.5	1 10 10 10	0.0165	46.0	1 10 10 10	0.0165	46.0
2 2 2 4	0.0057	62.1	2 10 10 10	0.0180	51.2	2 10 10 10	0.0180	51.2
2 2 2 5	0.0027	75.2	3 10 10 10	0.0199	57.9	3 10 10 10	0.0199	57.9
2 2 2 6	0.0016	85.3	4 10 10 10	0.0223	67.0	4 10 10 10	0.0223	67.0
2 2 2 7	0.0011	93.4	5 10 10 10	0.0254	79.9	5 10 10 10	0.0254	79.9
2 2 2 8	0.0008	100.0	6 10 10 10	0.0298	100.0	6 10 10 10	0.0298	100.0
2 2 2 9	0.0006	105.6	7 10 10 10	0.0367	135.9	7 10 10 10	0.0367	135.9
2 2 2 10	0.0005	110.3	8 10 10 10	0.0499	221.7	8 10 10 10	0.0499	221.7
2 3 3 10	0.0113	265.8	9 10 10 10	0.1059	1251.9	9 10 10 10	0.1059	1251.9
	Q =	0.9695		Q =	2.2403		Q =	2.2403
	$MAE\eta$ =	0.9496		$MAE\eta$ =	0.4050		$MAE\eta$ =	0.405
	V_{be} =	4.385		V_{be} =	2.2599		V_{be} =	2.2599
	QN =	0.6409		QN =	0.1702		QN =	0.1702
	η =	0.1581		η =	0.595		η =	0.595
	$RSAL$ =	0.0177		$RSAL$ =	0.1241		$RSAL$ =	0.1241

Table 8. Portfolio 8, *IG*(0.3; 0.05).

$Q(\boldsymbol{b}) \to min$			$MAE_\eta \to min$			$MAE_V \to min$		
T	***e***	***b*** *%*	***T***	***e***	***b*** *%*	***T***	***e***	***b*** *%*
1 1 2 3	0.8867	4.6	1 6 10 10	0.5091	18.6	1 10 10 10	0.4509	24.1
1 3 3 4	0.0241	21.7	1 10 10 10	0.0291	38.3	1 10 10 10	0.0247	51.3
2 3 4 4	0.0274	30.7	2 10 10 10	0.0327	41.8	2 10 10 10	0.0278	56.3
3 4 4 5	0.0216	44.8	3 10 10 10	0.0371	46.1	3 10 10 10	0.0315	62.6
4 5 5 6	0.0172	67.4	4 10 10 10	0.0428	51.8	4 10 10 10	0.0364	70.9
5 5 5 7	0.0071	85.7	5 10 10 10	0.0503	59.7	5 10 10 10	0.0429	82.6
5 5 6 8	0.0034	100.0	6 10 10 10	0.0333	100.0	6 10 10 10	0.0522	100.0
6 6 6 9	0.0028	123.2	7 10 10 10	0.0460	125.5	7 10 10 10	0.0666	129.6
6 6 7 10	0.0018	133.4	8 10 10 10	0.0707	178.3	8 10 10 10	0.0933	192.9
7 8 8 10	0.0079	206.	9 10 10 10	0.1489	420.2	9 10 10 10	0.1737	492.7
	Q =	0.1057		*Q* =	0.3741		*Q* =	0.4057
	MAEη =	0.8122		*MAEη* =	0.4075		*MAEη* =	0.4902
	V_{be} =	2.1967		V_{be} =	1.3575		V_{be} =	1.2213
	QN =	0.8043		*QN* =	0.3071		*QN* =	0.2486
	η =	0.307		*η* =	0.596		*η* =	0.5098
	RSAL =	0.0304		*RSAL* =	0.2105		*RSAL* =	0.2457

Table 9. Portfolio 9, *IG*(0.3; 0.15).

$Q(\boldsymbol{b}) \to min$			$MAE_\eta \to min$			$MAE_V \to min$		
T	***e***	***b*** *%*	***T***	***e***	***b*** *%*	***T***	***e***	***b*** *%*
1 1 2 4	0.8176	15.0	1 4 9 10	0.4190	28.1	1 1 2 2	0.728	27.0
1 4 5 5	0.0208	36.8	1 9 9 10	0.0433	45.1	1 10 10 10	0.0107	56.4
2 5 5 6	0.0135	47.8	2 9 9 10	0.0508	48.7	2 10 10 10	0.0035	71.8
3 5 6 6	0.0233	52.7	3 9 9 10	0.0604	53.1	3 10 10 10	0.0051	76.8
4 6 6 7	0.0315	62.8	4 9 9 10	0.0330	77.2	4 10 10 10	0.0075	82.8
5 6 7 7	0.0396	76.2	5 9 9 10	0.0436	86.8	5 10 10 10	0.0115	90.2
6 7 7 8	0.0259	100.0	6 9 9 10	0.0599	100.0	6 10 10 10	0.0183	100.0
7 8 8 9	0.0110	136.0	7 9 10 10	0.0869	119.5	7 10 10 10	0.0309	114.3
8 8 9 10	0.0045	169.3	8 9 10 10	0.1374	152.4	8 10 10 10	0.0572	138.7
9 10 10 10	0.0123	238.7	9 10 10 10	0.0657	379.1	9 10 10 10	0.1271	199.0
	Q =	0.0406		*Q* =	0.0872		*Q* =	0.0902
	MAEη =	0.7295		*MAEη* =	0.4387		*MAEη* =	0.7337
	V_{be} =	1.2444		V_{be} =	1.0154		V_{be} =	0.9992
	QN =	0.7743		*QN* =	0.5155		*QN* =	0.4992
	η =	0.4109		*η* =	0.5613		*η* =	0.5049
	RSAL =	0.0567		*RSAL* =	0.1706		*RSAL* =	0.1986

Conclusions

Analysing the normalized measure of the accuracy of the assessment QN, it is clear that the various portfolios (different risk structure function parameters) allow for different accuracy of assessment. Generally, for low claim rate (majority of real portfolios) accuracy is low and system needs tough transition rules to achieve optimal accuracy of assessment. Considering low values of $RSAL$ and observing stationary distributions $\boldsymbol{e}$ we can conclude that optimal accuracy of assessment forces insured to cluster in better (chipper) classes – frequently criticized feature of bonus-malus system. Summarising:

- Optimal accuracy of assessment is associated with tough transition rules, especially for low claim rate (typical portfolios).
- Better accuracy of assessment is achieved for portfolios with high claim rate.
- Better accuracy of assessment goes together with clustering of insured in better classes.

We find that for relatively low claim ratio it is impossible to achieve satisfactory level of global elasticity with this particular premium calculation principle, even for the toughest transition rules due to not sufficient premium span. Notice, that high global elasticity is achieved by imposition of tough transition rules. The highest value for global elasticity were achieved for portfolios 7 and 8, then for portfolio 9, all this portfolios have high claim rate 0.3. The lowest global elasticities come for portfolios with low claim rate 0.05. Again frequently criticized feature of having low premium elasticity seems to be not possible to overcome for typical portfolios with rather low claim rate. Summarising:

- Higher global elasticity is associated with tough transition rules.
- Satisfactory values of global elasticity were achieved only for portfolios with high claim rate.
- For low claim rate global elasticity is on low level – for majority of real portfolios it is impossible to have high global elasticity.

To have optimal values of volatility coefficient we have to choose systems with the toughest transition rules, especially for portfolios with low claim variance (namely for any claim in any class insured goes to the worst class). Situation is very similar for portfolios with medium claim variance. Profound difference is observed only for portfolio 6, where optimal system is clearly different and resembles ones that can be observed in reality. Optimization of volatility coefficient significantly lowers accuracy of assessment for most portfolios accept 2, 3 (for which values of QN and V_{b_e} are anyway unsatisfactory) and portfolio 6, the only one portfolio scoring reasonably well for all three optimisation criteria. Generally, it is easier to achieve volatility coefficient closer to one for portfolios with moderate claim rate to claim variance ratio than with extreme one (low or high ratio). For higher claim rate to claim variance ratio optimal volatility coefficient goes lower. Summarising:

- Optimal volatility coefficient is associated with tough transition rules.
- Satisfactory values of volatility coefficient were achieved for portfolios with moderate claim rate to claim variance ratio.

For most of analysed portfolios optimization with respect to different criteria gives different bonus-malus systems. There was no portfolio that gave the same optimal systems for all criteria. We had pairwise same optimal systems for particular portfolios and particular criteria:

portfolio 2 - criterion 1 and 3,
portfolio 3 - criterion 1 and 3 (system for criterion 2 differs just by one entry of transition matrix),
portfolio 6 - criterion 1 and 3,
portfolio 7 - criterion 2 and 3.

It is worth noting, that portfolio 3 that had almost the same systems for all of criteria is characterised by the worst scores on all characteristics. This is portfolio with low claim rate and high claim variance.

Considering above findings, we can conclude that using *Q*-optimal premiums it is hard to eliminate frequently criticised feathers of bonus-malus systems, that is insured clustering in 'better classes' and low premium elasticity, as systems optimal with respect to global elasticity tend to cluster customers in better classes. For most of the typical portfolios it is also impossible to achieve high values of global elasticity and high accuracy of risk assessment, as optimal systems for portfolios with lower claim ratio in our analysis had rather low elasticity and low normalized accuracy of assessment. Optimisation with respect to different criteria gives usually different systems, and even if in some cases we have same or similar systems, their market implementation is rather impossible due to very tough transition rules (marketing reasons) or poor characteristics.

Limitations

We use one premium calculation principle only.
We use one size of bonus-malus system (10 by 3) only.

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