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One-year estimation uncertainty in some claim development models

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Abstract

The paper studies the one-year estimation uncertainty associated with using credibility-based loss reserving methods, when claim development can be described by the models of Bühlmann-Straub or Hesselager-Witting.

1 Introduction

The Solvency II regime requires specification of the *one-year uncertainty* of estimates of outstanding claims, or to be more precise, the uncertainty associated with one year's development of *estimated ultimate claim cost*.

For a given cohort of claims, the *estimated ultimate claim cost* at any valuation date is the sum of

1. claim payments already made since the cohort came into existence, plus
2. an estimate of outstanding claim payments needed to settle all claims of the cohort.

A cohort of claims normally consists of the claims attached to an underwriting year or the claims incurred during an accident year, but other periods of origin may also be envisaged (e.g., reporting year).

The development of estimated ultimate claim cost between two valuation dates is the sum of

1. claim payments made between the two valuation dates, plus
2. the change in the estimate of outstanding claim payments.

If the sum is positive, the estimated ultimate claim cost for the cohort of claims has increased between the valuation dates; if it is negative, the estimated ultimate claim cost has decreased. Insurers refer to the negative of the development of estimated claim cost, as the run-off result.

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Some mathematical results exist about the uncertainty associated with the estimate of outstanding claim payments needed to settle the cohort, or the *ultimate uncertainty*. The most influential paper by Mack (1993) studies the ultimate uncertainty associated with using the chain ladder method, albeit within a restrictive model. Norberg (1986) studies ultimate uncertainty associated with several estimation methods in a random parameter model. More recent papers exist, too.

Moro & Lo (2014) have recently challenged the actuarial community to develop new models that will allow calculation of the one-year uncertainty when using different estimation methods, to comply with the Solvendy II regime. This author believes that no new models are required; it is quite feasible to assess the variability of the one-year claim development when using different methods, analytically or by simulation, within the framework of existing models. However, the author also believes that the quest for one-year uncertainty is misguided and can lead to perverse conclusions.

This paper shows how one can calculate the variance of the one-year claim development, when using a credibility-based loss reserving method at the beginning and at the end of the development year, under the assumption that the stochastic mechanism of claim development can be described by the models of Bühlmann-Straub or Hesselager-Witting. Credibility-based loss reserving methods are commonly used and include, as limiting cases, the standard chain ladder and Bornhuetter-Ferguson methods. The models of Bühlmann-Straub and Hesselager-Witting seem to provide a reasonable description of the uncertainties one faces when charged with estimation outstanding claims.

2 The models

The Bühlmann-Straub model was originally presented in a paper on loss ratios (Bühlmann&Straub, 1970). The model combines the notion of a known risk exposure with the notion of an à priori unknown claim rate. Bühlmann and Straub model the unobserved claim rate as a random variable and derive an optimal credibility estimator to estimate the claim rate in the light of emerging claim experience.

Hesselager & Witting (1988) later extended the model to comprise not only an à priori unknown claim rate, but also an à priori unknown claim development pattern. One of the important results of the paper is that the credibility assigned to the claim experience in the optimal credibility estimator, decreases when the claim development pattern is random. The Hesselager-Witting model includes the Bühlmann-Straub model as a limiting case, in which the claim development pattern is known and fixed.

This paper is not concerned with the quest for an optimal credibility estimator of the ultimate claim cost, but with calculating the mean squared error of prediction (MSEP) of arbitrary credibility estimators of outstanding claims, under the assumptions of those models.

Ample literature exists on the Bühlmann-Straub model, for which reason it will be presented quite briefly here.

2.1 Motivation

This small preamble is only intended to motivate the structure of the models that follow.

Let us start with one claim cohort that we denote by j. The claim payments emanating from that cohort, let us denote by $\{X_{je} : e = 0, 1, \cdots\}$, where e is the development period. Let the quantity π_e denote the expected proportion of ultimate claim cost that will be paid in development period e, and assume that $\sum_{e=0}^{\infty} \pi_e = 1$. Let the quantity p_j denote an observed measure of the risk exposure that is generating claim payments in cohort j.

We start with a compound Poisson distribution. Imagine that claim payments X_{je} have a compound Poisson distribution with frequency parameter $p_j \pi_e f$ and severity distribution $F(\cdot)$. Then, using the well-known properties of the compound Poisson distribution, we find that

$$\mathrm{E}\left(X_{je}\right) = p_j \pi_e f \int y dF\left(y\right) =: p_j \pi_e b \tag{1}$$

and

$$\mathrm{Var}\left(X_{je}\right) = p_j \pi_e f \int y^2 dF\left(y\right) =: p_j \pi_e v. \tag{2}$$

In these two equations, the risk exposure p_j should be a known quantity. One can think of p_j as the number of similar risks that generate claims in cohort j. Bühlmann and Straub take p_j to be the earned premium pertaining to claim cohort j. Risk exposure can be measured in a variety of ways, as long as one measures consistently over time. The payment proportion π_e will normally have to be estimated. Also the parameters b and v will be unknown quantities, as they depend on the claim frequency and claim severity.

The compound Poisson process being a standard model of collective risk theory, this seems to be a reasonable description of the payment distribution. The only exception that one could take to the assumptions is that the severity distribution $F(\cdot)$ ought to be allowed to vary with the payment delay e. But let's not complicate things more than necessary.

2.2 The Bühlmann-Straub model

The assumptions of Bühlmann-Straub's model are:

Pertaining to claim cohort j is a risk parameter that we denote by Θ_j. Given Θ_j, the claim payments $\{X_{je} : e = 0, 1, \cdots\}$ are stochastically independent with conditional mean

$$\mathrm{E}\left(X_{je}|\Theta_j\right) = p_j \pi_e b\left(\Theta_j\right) \tag{3}$$

and variance

$$\mathrm{Var}\left(X_{je}|\Theta_j\right) = p_j \pi_e v\left(\Theta_j\right). \tag{4}$$

The unobserved risk parameter Θ_j is taken to be the outcome of a random variable. We denote the mean and variance of the function $b\left(\Theta_j\right)$ by

$$\beta = \mathrm{E}\left(b\left(\Theta_j\right)\right) \tag{5}$$

and

$$\lambda = \mathrm{Var}\left(b\left(\Theta_j\right)\right). \tag{6}$$

Let us further denote the mean of the function $v\left(\Theta_j\right)$ by

$$\varphi = \mathrm{E}\left(v\left(\Theta_j\right)\right). \tag{7}$$

Using these definitions it is easy to express the unconditional first and second order moments of the payments:

$$\mathrm{E}\left(X_{je}\right) = p_j \pi_e \beta \tag{8}$$

and

$$\mathrm{Cov}\left(X_{je}, X_{jd}\right) = \delta_{e,d} \cdot p_j \pi_e \varphi + p_j^2 \pi_e \pi_d \lambda. \tag{9}$$

We assume that the delay probabilities $\{\pi_e : e = 0, 1, \cdots\}$ and the distribution moments $(\beta, \lambda, \varphi)$ are known. If they have been estimated aforehand, we treat the estimates as if they were the true values.

Assume that the payments $\{X_{je} : e = 0, 1, \cdots, D(j)\}$ have been observed for cohort j. Without any ado, let us postulate that a predictor of future payments X_{je} for $e > D(j)$, should be of the form

$$\overline{X}_{je} = p_j \pi_e \bar{b}_j, \tag{10}$$

where $\bar{b}_j$ is an estimator of $b\left(\Theta_j\right)$. This form unifies many commonly used estimation methods for outstanding claims. Schmidt and Zocher (2008) call (10) the *Bornhuetter-Ferguson principle.*

We restrict the estimator $\bar{b}_j$ to be a linear combination of a chain-ladder estimate and the à priori mean:

$$\bar{b}_j = z_j \widehat{b}_j + \left(1 - z_j\right) \beta \tag{11}$$

where the chain-ladder estimate is

$$\widehat{b}_j = \frac{X_{j,\leq D(j)}}{p_j \pi_{\leq D(j)}}. \tag{12}$$

An inequality in a subscript signifies summation of terms for the values of the subscript that satisfy the inequality.

It is easy to verify that for an arbitrary choice of z_j, the mean squared error of the estimator $\bar{b}_j$ is

$$Q(z_j) = \mathrm{E}\left(\bar{b}_j - b(\Theta_j)\right)^2 = z_j^2 \frac{\varphi}{p_j \pi_{\leq D(j)}} + (1 - z_j)^2 \lambda. \tag{13}$$

The mean squared error of the predictor (10) for $e > D(j)$ is

$$\mathrm{E}\left(\overline{X}_{je} - X_{je}\right)^2 = (p_j \pi_e)^2 Q(z_j) + p_j \pi_e \varphi. \tag{14}$$

An estimator of the total outstanding claim payments is

$$\overline{X}_{j,>D(j)} = p_j \bar{b}_j \pi_{>D(j)}, \tag{15}$$

with mean squared error

$$\mathrm{E}\left(\overline{X}_{j,>D(j)} - X_{j,>D(j)}\right)^2 = \left(p_j \pi_{>D(j)}\right)^2 Q(z_j) + p_j \pi_{>D(j)} \varphi. \tag{16}$$

Minimising (13), is easy to see that the optimal choice of z_j is

$$\zeta_j = \frac{p_j \pi_{\leq D(j)} \lambda}{p_j \pi_{\leq D(j)} \lambda + \varphi}. \tag{17}$$

This is the famous Bühlmann-Straub credibility factor.

2.3 The Hesselager-Witting model

The assumptions of Hesselager-Witting's model are:

Pertaining to claim cohort j is a risk parameter that we denote by Θ_j and a payment pattern that we denote by $\mathbf{\Pi}_j = \{\Pi_{je} : e = 0, 1, \cdots\}$ .Given Θ_j and $\mathbf{\Pi}_j$, the claim payments $\{X_{je} : e = 0, 1, \cdots\}$ are stochastically independent with conditional mean

$$\mathrm{E}(X_{je} | \Theta_j, \mathbf{\Pi}_j) = p_j \Pi_{je} b(\Theta_j) \tag{18}$$

and variance

$$\mathrm{Var}(X_{je} | \Theta_j, \mathbf{\Pi}_j) = p_j \Pi_{je} v(\Theta_j). \tag{19}$$

The unobserved risk parameter Θ_j is taken to be the outcome of a random variable. As in the Bühlmann-Straub model, we define $\beta = \mathrm{E}(b(\Theta_j))$, $\lambda = \mathrm{Var}(b(\Theta_j))$ and $\varphi = \mathrm{E}(v(\Theta_j))$.

Let D denote the maximum payment delay. The payment pattern $\mathbf{\Pi}_j = \{\Pi_{j0}, \cdots, \Pi_{jD}\}$ is taken to be the outcome of a random vector that is stochastically independent of Θ_j, and with a Dirichlet distribution. The Dirichlet distribution with parameters $\alpha_0, \cdots, \alpha_D \geq 0$ is a generalisation of the beta distribution to D dimensions, with a density function in D-space, of

$$f(x_0, \cdots, x_D) = \frac{\Gamma(\alpha)}{\Gamma(\alpha_0) \cdots \Gamma(\alpha_D)} x_0^{\alpha_0 - 1} \cdots x_D^{\alpha_D - 1} \cdot I(x_0 + \cdots + x_D = 1). \tag{20}$$

Using similar calculations as in the beta distribution, one can verify that the first and second order moments in a Dirichlet distribution are

$$\mathrm{E}\left(\Pi_{je}\right) = \pi_e \tag{21}$$

and

$$\mathrm{Cov}\left(\Pi_{je}, \Pi_{jd}\right) = \frac{\delta_{e,d} \cdot \pi_e - \pi_e \pi_d}{\alpha + 1}, \tag{22}$$

where we have defined $\alpha = \sum_{e=0}^{D} \alpha_e$ and

$$\pi_e = \alpha_e / \alpha. \tag{23}$$

It requires a fair amount of algebraic manipulation to express the unconditional first and second order moments of the payments, and those operations will not be repeated here. The result is as follows:

$$\mathrm{E}\left(X_{je}\right) = p_j \pi_e \beta \tag{24}$$

and

$$\mathrm{Cov}\left(X_{je}, X_{jd}\right) = \delta_{e,d} \cdot p_j \pi_e \varphi_j(\alpha) + p_j^2 \pi_e \pi_d \lambda(\alpha), \tag{25}$$

where we have defined

$$\varphi_j(\alpha) = \varphi + p_j \left(\frac{\lambda + \beta^2}{\alpha + 1} \right) \tag{26}$$

and

$$\lambda(\alpha) = \frac{\lambda\alpha - \beta^2}{\alpha + 1}. \tag{27}$$

The important observation is that the first and second order moment structure of the payments in the Hesselager-Witting model (24) – (25) is isomorphic to the first and second order moment structure in the Bühlmann-Straub model (8)–(9) , if one makes the transformation $\varphi \to \varphi_j(\alpha)$ and $\lambda \to \lambda(\alpha)$. That means that the results about the Bühlmann-Straub model (10) – (17) , which only involve first and second order moments, also apply in the Hesselager-Witting model, mutatis mutandis. In particular, the optimal choice of credibility factor in a credibility estimator of the form (11) – (12) becomes

$$\zeta_j(\alpha) = \frac{p_j \pi_{\leq D(j)} \lambda(\alpha)}{p_j \pi_{\leq D(j)} \lambda(\alpha) + \varphi(\alpha)} \tag{28}$$

in the Hesselager-Witting model. It is easy to verify that $\lambda(\alpha) \leq \lambda$ and $\varphi_j(\alpha) \geq \varphi$, so that

$$\zeta_j(\alpha) \leq \zeta_j. \tag{29}$$

This formalises the untuitively obvious result, that one ought to assign less credibility to the emerging payments if there less certainty about the payment pattern. Indeed one can show that the optimal credibility factor (28) can turn negative.

The Bühlmann-Straub model is a limiting case of Hesselager-Witting's (larger) model that one arrives at by letting $\alpha_e/\alpha \to \pi_e$ while $\alpha \to \infty$.

3 One-year uncertainty of estimates

In this paper we are not concerned with the optimal estimators in either model. In what follows, our focus will be on calculating the uncertainty associated with one year's development of estimated claim cost, when

- using a credibility *method* of the form $(10)-(12)$, and
- assuming that the claim development *mechanism* behaves in accordance with a Hesselager-Witting *model*.

Note that the form $(10)-(12)$ includes the chain ladder method $(z_j = 1)$, the Bornhuetter-Ferguson method $(z_j = 0)$ and Benktander's method $\left(z_j = \pi_{\leq D(j)}\right)$.

The reason for selecting the Hesselager-Witting model is that it provides a reasonable description of the uncertainties one faces when trying to estimate outstanding claim payments.

Before we start, let us introduce some new notation.

We shall consider the development of only one cohort of claims and omit the subscript j that was used previously to denote the cohort. We shall study the passing from development period e to development period $e+1$, while using a certain credibility formula. The credibility formula is written as

$$\bar{b}_e = z_e \widehat{b}_e + (1 - z_e)\beta \tag{30}$$

where

$$\widehat{b}_e = \frac{X_{\leq e}}{p\pi_{\leq e}}, \tag{31}$$

and similarly at development stage $e+1$.

Remark 1 *Note that $\bar{b}_e$, z_e and $\widehat{b}_e$ in this section are indexed with the development stage e, whereas in the previous section they were indexed with the cohort number j, that we omit in this section. The relabelling has the purpose of saving some notation in the lengthy equations that follow.*

The development of estimated claim cost in period $e+1$ is the sum of the incremental claim payments in period $e+1$, and the change in the estimated outstanding claim payments:

$$R_{e+1} = X_{e+1} + p\left(\bar{b}_{e+1}\pi_{>e+1} - \bar{b}_e\pi_{>e}\right). \tag{32}$$

Let us derive an expression for the mean squared claim development cost.

Proposition 2 *In the Hesselager-Witting model, the mean squared claim development cost* (38) *can be expressed in the following form:*

$$E\left(R_{e+1}^2\right) = Var\left(R_{e+1}\right) = p^2 L \cdot \lambda(\alpha) + pF \cdot \varphi(\alpha), \tag{33}$$

where

$$L = \left(\omega_{e+1} - \omega_e + \pi_{e+1}\right)^2, \tag{34}$$

$$F = \pi_{e+1} + \frac{\omega_{e+1}}{\pi_{\leq e+1}}\left(\omega_{e+1} - 2\omega_e + 2\pi_{e+1}\right) + \frac{\omega_e^2}{\pi_{\leq e}}, \tag{35}$$

and we have defined the abbreviations

$$\omega_e = z_e \pi_{>e}, \ \omega_{e+1} = z_{e+1}\pi_{>e+1}. \tag{36}$$

Similar equations apply in the Bühlmann-Straub model, with the argument (α) *omitted from* (33).

Proof. We start by writing:

$$R_{e+1} = X_{e+1} + p\left(\left(z_{e+1}\widehat{b}_{e+1} + \left(1 - z_{e+1}\right)\beta\right)\pi_{>e+1} - \left(z_e\widehat{b}_e + \left(1 - z_e\right)\beta\right)\pi_{>e}\right) \tag{37}$$

It is easy to verify that $\mathrm{E}(R_{e+1}) = 0$. As a consequence, the mean squared claim development cost is

$$\mathrm{E}\left(R_{e+1}^2\right) = \mathrm{Var}\left(R_{e+1}\right). \tag{38}$$

Ignoring the non-stochastic terms in (37), we find the variance:

$$\begin{aligned} \mathrm{Var}\left(R_{e+1}\right) &= \mathrm{Var}\left(X_{e+1} + p\omega_{e+1}\widehat{b}_{e+1} - p\omega_e\widehat{b}_e\right) \\ &= \mathrm{Var}\left(X_{e+1}\right) + \left(p\omega_{e+1}\right)^2 \mathrm{Var}\left(\widehat{b}_{e+1}\right) + \left(p\omega_e\right)^2 \mathrm{Var}\left(\widehat{b}_e\right) \\ &\quad +2p\omega_{e+1}\mathrm{Cov}\left(X_{e+1}, \widehat{b}_{e+1}\right) \\ &\quad -2p\omega_e\mathrm{Cov}\left(X_{e+1}, \widehat{b}_e\right) \\ &\quad -2p^2\omega_e\omega_{e+1}\mathrm{Cov}\left(\widehat{b}_e, \widehat{b}_{e+1}\right). \end{aligned} \tag{39}$$

We now calculate the variances and covariances in the above expression (see the appendix for some more detail). The first three variance terms are simple:

$$\mathrm{Var}\left(X_{e+1}\right) = \left(p\pi_{e+1}\right)^2 \lambda(\alpha) + p\pi_{e+1}\varphi(\alpha), \tag{40}$$

$$\mathrm{Var}\left(\widehat{b}_{e+1}\right) = \lambda(\alpha) + \frac{\varphi(\alpha)}{p\pi_{\leq e+1}}, \tag{41}$$

$$\mathrm{Var}\left(\widehat{b}_e\right) = \lambda(\alpha) + \frac{\varphi(\alpha)}{p\pi_{\leq e}}, \tag{42}$$

Next, the covariances:

$$\operatorname{Cov}\left(X_{e+1}, \widehat{b}_e\right) = p\pi_{e+1}\lambda(\alpha), \tag{43}$$

$$\operatorname{Cov}\left(X_{e+1}, \widehat{b}_{e+1}\right) = p\pi_{e+1}\lambda(\alpha) + \left(\pi_{e+1}/\pi_{\leq e+1}\right)\varphi(\alpha) \tag{44}$$

$$\operatorname{Cov}\left(\widehat{b}_e, \widehat{b}_{e+1}\right) = \lambda(\alpha) + \frac{\varphi(\alpha)}{p\pi_{\leq e+1}} \tag{45}$$

Collecting all the terms $(40) - (45)$ in (39) and re-grouping them, we find

$$\operatorname{E}\left(R_{e+1}^2\right) = \operatorname{Var}\left(R_{e+1}\right) = p^2 L \cdot \lambda(\alpha) + pF \cdot \varphi(\alpha) \tag{46}$$

where L and F are defined in (34)-(35). ■

4 Minimising the one-year uncertainty

Since the Solvency II regulation defines the one-year uncertainty of estimates of outstanding claims as a *risk* to be measured and controlled, and having derived a formula for it, what seems more natural than trying to minimise that risk? Superficially, minimising the one-year uncertainty appears to be just as sensible as minimising the ultimate uncertainty (16) by the Bühlmann-Straub credibility factor (17) or its counterpart (28) for the Hesselager-Witting model.

The author is of the opinion that focus on the one-year uncertainty is misdirected, and will explain that view later. With this disclaimer in mind, let us notwithstanding consider what results one achieves when minimising the one-year uncertainty with a credibility formula.

Two forms of minimisation will be considered:

1. Univariate minimisation of (33) as a function of z_{e+1}, when z_e is fixed.
2. Bivariate minimisation of (33) as a function of both z_e and z_{e+1}.

The first scenario mimics the situation where initial estimates of claim cost have been calculated by a credibility formula of the form $(30)-(31)$ with a known value of z_e, and the actuary is pondering how to select z_{e+1} when updating the claim cost estimates with one more year of claim statistics, with the view to minimising the "risk".

The second scenario mimics the situation where actuary à priori wants to determine credibility factors to use at the end of development years e and $e+1$, with the objective of minimising the one-year uncertainty in development year $e+1$.

4.1 Univariate minimisation

Assume that the ultimate claim cost of the cohort has been estimated at the end of development year e with a credibility formula of the form $(30)-(31)$. Using a similar formula at the end of development year $e+1$, with an arbitrary but fixed value z_{e+1}, one incurs one-year uncertainty given by (33). We differentiate (33) by ω_{e+1} to find the minimising value.

$$\frac{\partial \mathrm{Var}\left(R_{e+1}\right)}{\partial \omega_{e+1}} = p^2 \lambda(\alpha) \frac{\partial L}{\partial \omega_{e+1}} + p\varphi(\alpha) \frac{\partial F}{\partial \omega_{e+1}} \tag{47}$$

Then we find that the derivatives of L and F are identical up to a multiplicative factor:

$$\frac{\partial L}{\partial \omega_{e+1}} = 2\left(\omega_{e+1} - \omega_e + \pi_{e+1}\right) \tag{48}$$

and

$$\frac{\partial F}{\partial \omega_{e+1}} = \frac{2}{\pi_{\leq e+1}}\left(\omega_{e+1} - \omega_e + \pi_{e+1}\right) \tag{49}$$

The derivative (47) is zero and non-decreasing if, and only if,

$$\omega_{e+1} = \omega_e - \pi_{e+1}. \tag{50}$$

This translates to the "optimal" credibility factor

$$z_{e+1} = z_e - (1-z_e)\frac{\pi_{e+1}}{\pi_{>e+1}} = 1 - (1-z_e)\frac{\pi_{>e}}{\pi_{>e+1}}, \tag{51}$$

which can also be expressed in the following way

$$(1-z_{e+1}) = (1-z_e)\frac{\pi_{>e}}{\pi_{>e+1}}. \tag{52}$$

Note that $z_{e+1} \leq z_e$ if $0 \leq z_e \leq 1$, and that z_{e+1} does not depend on $\lambda(\alpha)$ and $\varphi(\alpha)$. This behaviour is at odds with the evolution of the optimal credibility factors (17) and (28) for estimating the ultimate claim cost.

Using the credibility estimator with z_{e+1} as credibility factor, we obtain the following estimate of outstanding claims at the end of development period $e+1$:

$$\begin{aligned}
\overline{X}_{>e+1} &= p\bar{b}_{e+1}\pi_{>e+1} && (53)\\
&= p\left(z_{e+1}\widehat{b}_{e+1} + (1-z_{e+1})\beta\right)\pi_{>e+1}\\
&= p\left(\left(z_e - (1-z_e)\frac{\pi_{e+1}}{\pi_{>e+1}}\right)\widehat{b}_{e+1} + (1-z_e)\frac{\pi_{>e}}{\pi_{>e+1}}\beta\right)\pi_{>e+1}\\
&= p\left(z_e\widehat{b}_{e+1}\pi_{>e+1} + (1-z_e)\left(\beta\pi_{>e} - \widehat{b}_{e+1}\pi_{e+1}\right)\right). && (54)
\end{aligned}$$

The one-year claim cost development in development period $e+1$ becomes

$$
\begin{aligned}
R_{e+1} &= X_{e+1} + p\bar{b}_{e+1}\pi_{>e+1} - p\bar{b}_e\pi_{>e} \quad (55)\\
&= X_{e+1} + p\left(z_e\widehat{b}_{e+1}\pi_{>e+1} + (1-z_e)\left(\beta\pi_{>e} - \widehat{b}_{e+1}\pi_{e+1}\right)\right) - p\left(z_e\widehat{b}_e + (1-z_e)\beta\right)\pi_{>e}\\
&= X_{e+1} + p\left(z_e\left(\widehat{b}_{e+1}\pi_{>e+1} - \widehat{b}_e\pi_{>e}\right) - (1-z_e)\widehat{b}_{e+1}\pi_{e+1}\right).
\end{aligned}
$$

Note that the prior mean disappears from the equation for the one-year claim cost development.

Let us next consider a few special cases.

4.1.1 Starting with the Bornhuetter-Ferguson method

Bornhuetter-Ferguson's method is characterised by $z_e = 0$. The one-year uncertainty is minimised by selecting

$$z_{e+1} = 1 - \frac{\pi_{>e}}{\pi_{>e+1}} = -\frac{\pi_{e+1}}{\pi_{>e+1}}, \quad (56)$$

a negative credibility factor. The equations (53) – (55) reduce to $\overline{X}_{>e+1} = p\left(\beta\pi_{>e} - \widehat{b}_{e+1}\pi_{e+1}\right)$ and $R_{e+1} = X_{e+1} - p\widehat{b}_{e+1}\pi_{e+1}$.

4.1.2 Starting with the chain ladder method

The chain ladder method is characterised by $z_e = 1$. The one-year uncertainty is minimised by selecting $z_{e+1} = 1$. Expressed in words this means "*if you have used the chain-ladder once, you must continue using the chain-ladder method, if minimising one-year uncertainty is your governing criterion*". The equations (53) – (55) reduce to $\overline{X}_{>e+1} = p\widehat{b}_{e+1}\pi_{>e+1}$ and $R_{e+1} = X_{e+1} + p\left(\widehat{b}_{e+1}\pi_{>e+1} - \widehat{b}_e\pi_{>e}\right)$ that we know from the chain ladder method.

4.1.3 Starting with the Benktander method

Benktander's method is characterised by $z_e = \pi_{\leq e} = 1 - \pi_{>e}$. The one-year uncertainty is minimised by selecting $z_{e+1} = 1 - \frac{\pi^2_{>e}}{\pi_{>e+1}}$. The equations (53)–(55) reduce to

$$
\begin{aligned}
\overline{X}_{>e+1} &= p\left(\left(1-\pi_{>e}\right)\widehat{b}_{e+1}\pi_{>e+1} + \pi_{>e}\left(\beta\pi_{>e} - \widehat{b}_{e+1}\pi_{e+1}\right)\right) \quad (57)\\
&= p\left(\widehat{b}_{e+1}\left(\pi_{>e+1} - \pi^2_{>e}\right) + \beta\pi^2_{>e}\right)\\
&= p\left(\widehat{b}_{e+1}\left(1 - \pi^2_{>e}\right) + \beta\pi^2_{>e} - \widehat{b}_{e+1}\pi_{\leq e+1}\right)
\end{aligned}
$$

and

$$R_{e+1} = X_{e+1} + p\left(z_e\left(\widehat{b}_{e+1}\pi_{>e+1} - \widehat{b}_e\pi_{>e}\right) - \widehat{b}_{e+1}\pi_{e+1}\pi_{>e}\right) \quad (58)$$

4.2 Bivariate minimisation

Let us briefly consider the bivariate minimisation of (33) as a function of both z_e and z_{e+1}. This mimics the situation where actuary à priori wants to determine credibility factors to use at the end of development years e and $e+1$, with the objective of minimising the one-year uncertainty in development year $e+1$.

From (47) – (50) we already know that $\frac{\partial \text{Var}(R_{e+1})}{\partial \omega_{e+1}} = 0$ if, and only if $\omega_{e+1} = \omega_e - \pi_{e+1}$. Next, consider

$$\frac{\partial \text{Var}\,(R_{e+1})}{\partial \omega_e} = p^2 \lambda(\alpha) \frac{\partial L}{\partial \omega_e} + p\varphi(\alpha) \frac{\partial F}{\partial \omega_e}, \tag{59}$$

with

$$\frac{\partial L}{\partial \omega_e} = -2\,(\omega_{e+1} - \omega_e + \pi_{e+1}) \tag{60}$$

and

$$\frac{\partial F}{\partial \omega_e} = -2 \left(\frac{\omega_{e+1}}{\pi_{\leq e+1}} - \frac{\omega_e}{\pi_{\leq e}} \right). \tag{61}$$

Inserting $\omega_{e+1} = \omega_e - \pi_{e+1}$ makes the first term (60) vanish and reduces the second term (61) to

$$\frac{\partial F}{\partial \omega_e} = 2\pi_{e+1} \left(\frac{\omega_e - \pi_{\leq e}}{\pi_{\leq e} \pi_{\leq e+1}} \right). \tag{62}$$

Thus $\frac{\partial \text{Var}(R_{e+1})}{\partial \omega_e} = \frac{\partial \text{Var}(R_{e+1})}{\partial \omega_{e+1}} = 0$ if, and only if $\omega_e = \pi_{\leq e}$ and $\omega_{e+1} = \omega_e - \pi_{e+1}$, which translates to

$$\begin{aligned} z_e &= \frac{\pi_{\leq e}}{\pi_{>e}}, \\ z_{e+1} &= \frac{\pi_{\leq e} - \pi_{e+1}}{\pi_{>e+1}}. \end{aligned} \tag{63}$$

Not a great deal can be said about these formulas, except the following:

- The minimising pair (63) does not depend on the structural parameters $\lambda(\alpha)$ and $\varphi(\alpha)$.
- Early in the development while $\pi_{>e} \approx 1$, the credibility factor z_e is near Benktander's.
- The "optimal" z_{e+1} at the end of development period $(e, e+1]$ is not equal to the optimum starting value for subsequent development period $(e+1, e+2]$.Thus it is not possible to form a coherent sequence of "optimal" credibility factors for development periods $e = 0, 1, \cdots$, with the criterion of minimising the one-year uncertainty.

5 Conclusion

Beauty is the first test: there is no permanent place in the world for ugly mathematics. (G.H. Hardy)

By a small *tour de force*, we have derived formulas for the one-year uncertainty of credibility-based estimates of outstanding claims, within the models of Bühlmann-Straub and Hesselager-Witting. We have also derived the "optimal" credibility factors that could, if required, minimise the one-year uncertainty.

Unfortunately, the formulas (33) – (36) for the one-year uncertainty are unsightly and unenlightening, and the "optimal" credibility factors with respect to one-year uncertainty have an evolution that is at odds with the evolution of the optimal credibility factors for estimating the ultimate claim cost. To the author this confirms that mathematics has an uncanny ability to inform its disciple that *"if you ask a stupid question, you get an ugly answer".*

In the view of this author, the quest for the one-year uncertainty of estimates of outstanding claims is misguided. The Solvency II regulation's focus on the one-year uncertainty is probably due to two reasons: the first being that its architects have been hidebound to a period accounting view, as opposed to a balance sheet view. The other reason is that, unfortunately, many actuaries are generating a surfeit of one-year uncertainty by using methods that magnify the impact of what may be random fluctuations.

Hesselager (1995) has shown that the standard discretisation involved in determining the claim cohort (accident year) and development period (calendar year minus accident year) leads to spurious randomness in the development patterns, that could be avoided with continuous time modelling. Discretisation makes modelling more complex than it needs to be, as anyone who has tried to change the discretisation (for example from yearly to quarterly) can attest to. Now add to that discretisation a requirement to pay special attention to *one year* of future claim development.

In the claim development of a line of insurance, one year is a totally arbitrary time period. In a short-tailed line of insurance, one year's development reveals the ultimate claim cost; any prior estimates will be wide of the mark to a larger or lesser extent, and one can see it. Thus, a short-tailed line crystallises uncertainty. In a long-tailed line of insurance, one may hardly know any more after one year has passed; the ultimate cost estimate will be more or less unchanged. Does that mean that a long-tailed line of insurance is less risky than a short-tailed line? That is exactly what the one-year focus of Solvency II seems to imply. Drawing the example to extremes, one could say that it is easy to minimise one-year uncertainty in a long-tailed line of insurance: simply don't revise the estimates.

The fact that something can be calculated does not imply that it should be calculated.

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Appendix. First and second order moments

To assist the unprepared reader and for the author's own peace of mind, the first and second order moments in the Bühlmann-Straub and Hesselager-Witting models are briefly calculated here.

Bühlmann-Straub model

Equation (8) :

$$\mathrm{E}\left(X_{je}\right) = \mathrm{E}\left(\mathrm{E}\left(X_{je}|\Theta_j\right)\right) = \mathrm{E}\left(p_j\pi_e b\left(\Theta_j\right)\right) = p_j\pi_e\beta.$$

Equation (9) :

$$\begin{aligned}
\mathrm{Cov}\left(X_{je}, X_{jd}\right) &= \mathrm{E}\left(\mathrm{Cov}\left(X_{je}, X_{jd}|\Theta_j\right)\right) + \mathrm{Cov}\left(\mathrm{E}\left(X_{je}|\Theta_j\right), \mathrm{E}\left(X_{jd}|\Theta_j\right)\right) \\
&\quad [\text{use conditional independence of } X_{je} \text{ and } X_{jd}, \text{ given } \Theta_j] \\
&= \mathrm{E}\left(\delta_{e,d}\cdot \mathrm{Var}\left(X_{je}|\Theta_j\right)\right) + \mathrm{Cov}\left(p_j\pi_e b\left(\Theta_j\right), p_j\pi_d b\left(\Theta_j\right)\right) \\
&= \mathrm{E}\left(\delta_{e,d}\cdot p_j\pi_e v\left(\Theta_j\right)\right) + \mathrm{Cov}\left(p_j\pi_e b\left(\Theta_j\right), p_j\pi_d b\left(\Theta_j\right)\right) \\
&= \delta_{e,d}\cdot p_j\pi_e\varphi + p_j^2\pi_e\pi_d\lambda.
\end{aligned}$$

Preparing for equation (13) :

$$\begin{aligned}
\mathrm{E}\left(\widehat{b}_j|\Theta_j\right) &= \mathrm{E}\left(\frac{X_{j,\le D(j)}}{p_j\pi_{\le D(j)}}|\Theta_j\right) \\
&= \frac{p_j\pi_{\le D(j)}b\left(\Theta_j\right)}{p_j\pi_{\le D(j)}} \\
&= b\left(\Theta_j\right).
\end{aligned}$$

$$\begin{aligned}
\mathrm{Var}\left(\widehat{b}_j|\Theta_j\right) &= \mathrm{Var}\left(\frac{X_{j,\le D(j)}}{p_j\pi_{\le D(j)}}|\Theta_j\right) \\
&= \frac{p_j\pi_{\le D(j)}v\left(\Theta_j\right)}{\left(p_j\pi_{\le D(j)}\right)^2} \\
&= \frac{v\left(\Theta_j\right)}{p_j\pi_{\le D(j)}}.
\end{aligned}$$

Equation (13) :

$$
\begin{aligned}
Q\left(z_j\right) &= \mathrm{E}\left(\bar{b}_j - b\left(\Theta_j\right)\right)^2 \\
&= \mathrm{E}\left(z_j \widehat{b}_j + \left(1-z_j\right)\beta - b\left(\Theta_j\right)\right)^2 \\
&= \mathrm{E}\left(z_j\left(\widehat{b}_j - b\left(\Theta_j\right)\right) + \left(1-z_j\right)\left(\beta - b\left(\Theta_j\right)\right)\right)^2 \\
&= z_j^2 \mathrm{E}\left(\widehat{b}_j - b\left(\Theta_j\right)\right)^2 + \left(1-z_j\right)^2 \mathrm{E}\left(\beta - b\left(\Theta_j\right)\right)^2 \\
&\quad +2z_j\left(1-z_j\right)\mathrm{E}\left(\left(\widehat{b}_j - b\left(\Theta_j\right)\right)\left(\beta - b\left(\Theta_j\right)\right)\right) \\
&= z_j^2 \mathrm{E}\left(\mathrm{E}\left(\left(\widehat{b}_j - b\left(\Theta_j\right)\right)^2 |\Theta_j\right)\right) + \left(1-z_j\right)^2 \mathrm{Var}\left(b\left(\Theta_j\right)\right) \\
&\quad +2z_j\left(1-z_j\right)\mathrm{E}\left(\mathrm{E}\left(\left(\widehat{b}_j - b\left(\Theta_j\right)\right)\left(\beta - b\left(\Theta_j\right)\right)|\Theta_j\right)\right) \\
&= z_j^2 \mathrm{E}\left(\mathrm{Var}\left(\widehat{b}_j|\Theta_j\right)\right) + \left(1-z_j\right)^2\lambda + 2z_j\left(1-z_j\right)\cdot 0 \\
&= z_j^2 \frac{\varphi}{p_j \pi_{\leq D(j)}} + \left(1-z_j\right)^2 \lambda.
\end{aligned}
$$

Equation (14) **and similarly** (16) :

$$
\begin{aligned}
\mathrm{E}\left(\overline{X}_{je} - X_{je}\right)^2 &= \mathrm{E}\left(p_j \pi_e \bar{b}_j - X_{je}\right)^2 \\
&= \mathrm{E}\left(p_j \pi_e\left(\bar{b}_j - b\left(\Theta_j\right)\right) - \left(X_{je} - p_j\pi_e b\left(\Theta_j\right)\right)\right)^2 \\
&= \left(p_j\pi_e\right)^2 \mathrm{E}\left(\bar{b}_j - b\left(\Theta_j\right)\right)^2 + \mathrm{E}\left(X_{je} - p_j\pi_e b\left(\Theta_j\right)\right)^2 \\
&\quad -2p_j\pi_e \mathrm{E}\left(\left(\bar{b}_j - b\left(\Theta_j\right)\right)\left(X_{je} - p_j\pi_e b\left(\Theta_j\right)\right)\right) \\
&= \left(p_j\pi_e\right)^2 \mathrm{E}\left(\bar{b}_j - b\left(\Theta_j\right)\right)^2 + \mathrm{E}\left(\mathrm{E}\left(\left(X_{je} - p_j\pi_e b\left(\Theta_j\right)\right)^2|\Theta_j\right)\right) \\
&\quad -2p_j\pi_e \mathrm{E}\left(\mathrm{E}\left(\left(\bar{b}_j - b\left(\Theta_j\right)\right)\left(X_{je} - p_j\pi_e b\left(\Theta_j\right)\right)|\Theta_j\right)\right) \\
&\quad [\text{use conditional independence of } X_{je} \text{ and } \bar{b}_j \text{, given } \Theta_j] \\
&= \left(p_j\pi_e\right)^2 Q\left(z_j\right) + p_j\pi_e\varphi.
\end{aligned}
$$

Hesselager-Witting model

Equation (24) :

$$\mathrm{E}\left(X_{je}\right)=\mathrm{E}\left(\mathrm{E}\left(X_{je}|\Theta_j,\mathbf{\Pi}_j\right)\right)=\mathrm{E}\left(p_j\Pi_{je}b\left(\Theta_j\right)\right)=p_j\pi_e\beta.$$

Preparing for equation (25) :

$$\begin{aligned}\mathrm{Cov}\left(b\left(\Theta_j\right)\Pi_{je},b\left(\Theta_j\right)\Pi_{jd}\right) &= \mathrm{E}\left(\mathrm{Cov}\left(b\left(\Theta_j\right)\Pi_{je},b\left(\Theta_j\right)\Pi_{jd}|\Theta_j\right)\right)\\ &\quad+\mathrm{Cov}\left(\mathrm{E}\left(b\left(\Theta_j\right)\Pi_{je}|\Theta_j\right),\mathrm{E}\left(b\left(\Theta_j\right)\Pi_{jd}|\Theta_j\right)\right)\\ &= \mathrm{E}\left(b^2\left(\Theta_j\right)\mathrm{Cov}\left(\Pi_{je},\Pi_{jd}\right)\right)\\ &\quad+\mathrm{Cov}\left(b\left(\Theta_j\right)\pi_e,b\left(\Theta_j\right)\pi_d\right)\\ &= \left(\lambda+\beta^2\right)\frac{\delta_{e,d}\cdot\pi_e-\pi_e\pi_d}{\alpha+1}+\lambda\pi_e\pi_d\\ &= \delta_{e,d}\cdot\pi_e\left(\frac{\lambda+\beta^2}{\alpha+1}\right)+\left(\frac{\lambda\alpha-\beta^2}{\alpha+1}\right)\pi_e\pi_d\end{aligned}$$

Equation (25) :

$$\begin{aligned}\mathrm{Cov}\left(X_{je},X_{jd}\right) &= \mathrm{E}\left(\mathrm{Cov}\left(X_{je},X_{jd}|\Theta_j,\mathbf{\Pi}_j\right)\right)+\mathrm{Cov}\left(\mathrm{E}\left(X_{je}|\Theta_j,\mathbf{\Pi}_j\right),\mathrm{E}\left(X_{jd}|\Theta_j,\mathbf{\Pi}_j\right)\right)\\ &= \mathrm{E}\left(\delta_{e,d}\cdot p_jv\left(\Theta_j\right)\Pi_{je}\right)+\mathrm{Cov}\left(p_jb\left(\Theta_j\right)\Pi_{je},p_jb\left(\Theta_j\right)\Pi_{jd}\right)\\ &= \delta_{e,d}\cdot p_j\varphi\pi_e+p_j^2\left(\delta_{e,d}\cdot\pi_e\left(\frac{\lambda+\beta^2}{\alpha+1}\right)+\left(\frac{\lambda\alpha-\beta^2}{\alpha+1}\right)\pi_e\pi_d\right)\\ &= \delta_{e,d}\cdot p_j\pi_e\left(\varphi+p_j\left(\frac{\lambda+\beta^2}{\alpha+1}\right)\right)+p_j^2\pi_e\pi_d\left(\frac{\lambda\alpha-\beta^2}{\alpha+1}\right)\\ &= \delta_{e,d}\cdot p_j\pi_e\varphi_j(\alpha)+p_j^2\pi_e\pi_d\lambda(\alpha)\end{aligned}$$

Equation (42) **and similarly** (41) :

$$\begin{aligned}\mathrm{Var}\left(\widehat{b}_e\right) &= \frac{1}{\left(p\pi_{\le e}\right)^2}\mathrm{Cov}\left(X_{\le e},X_{\le e}\right)\\ &= \frac{1}{\left(p\pi_{\le e}\right)^2}\sum_{e'\le e}\sum_{d'\le e}\left(\delta_{e',d'}\cdot p\pi_{e'}\varphi(\alpha)+p^2\pi_{e'}\pi_{d'}\lambda(\alpha)\right)\\ &= \lambda(\alpha)+\frac{\varphi(\alpha)}{p\pi_{\le e}},\end{aligned}$$

Equation (43) :

$$\begin{aligned}\mathrm{Cov}\left(X_{e+1},\widehat{b}_e\right) &= \frac{1}{p\pi_{\le e}}\mathrm{Cov}\left(X_{e+1},X_{\le e}\right)\\ &= \frac{1}{p\pi_{\le e}}\sum_{d'\le e}p^2\pi_{e+1}\pi_{d'}\lambda(\alpha)\\ &= p\pi_{e+1}\lambda(\alpha),\end{aligned}$$

Equation (44) :

$$\begin{aligned}
\mathrm{Cov}\left(X_{e+1},\widehat{b}_{e+1}\right) &= \frac{1}{p\pi_{\le e+1}}\mathrm{Cov}\left(X_{e+1},X_{\le e+1}\right)\\
&= \frac{1}{p\pi_{\le e+1}}\sum_{d'\le e+1}\left(\delta_{e+1,d'}\cdot p\pi_{e+1}\varphi(\alpha)+p^2\pi_{e+1}\pi_{d'}\lambda(\alpha)\right)\\
&= p\pi_{e+1}\lambda(\alpha)+\left(\pi_{e+1}/\pi_{\le e+1}\right)\varphi(\alpha)
\end{aligned}$$

Equation (45) :

$$\begin{aligned}
\mathrm{Cov}\left(\widehat{b}_e,\widehat{b}_{e+1}\right) &= \frac{1}{p\pi_{\le e}\pi_{\le e+1}}\mathrm{Cov}\left(X_{\le e},X_{\le e+1}\right)\\
&= \frac{1}{p\pi_{\le e}\pi_{\le e+1}}\sum_{e'\le e}\sum_{d'\le e+1}\left(\delta_{e',d'}\cdot p\pi_{e'}\varphi(\alpha)+p^2\pi_{e'}\pi_{d'}\lambda(\alpha)\right)\\
&= \frac{1}{p\pi_{\le e}\pi_{\le e+1}}\left(p\pi_{\le e}\varphi(\alpha)+p^2\pi_{\le e}\pi_{\le e+1}\lambda(\alpha)\right)\\
&= \lambda(\alpha)+\frac{\varphi(\alpha)}{p\pi_{\le e+1}}
\end{aligned}$$