A FRACTAL PROBABILITY DISTRIBUTION FOR FINANCIAL RISK APPLICATIONS

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Synopsis

Despite around forty years of empirical evidence that capital market returns in general, and equity market returns in particular, not only have far fatter tails than the commonly assumed normal and lognormal distributions but also are not independent over successive time periods, most of the risk models of present day finance theory are based on simplistic "independent and identically distributed" models of return that bear little resemblance to what actually happens in the real financial world. The collapse of Long-Term Capital Management in the third quarter of 1998 is cited as a classic example of the acute systemic risk that can result from presently accepted methodologies. The objective of this paper is to build the foundations of an alternative approach which in particular encapsulates both the "Noah effect" (the fatter tail syndrome) and the "Joseph effect" (a long-term memory or dependence property as reflected by a Hurst exponent significantly in excess of the Brownian Motion benchmark value of 0.5). Starting from a binary process conjecture based on a simulation procedure suggested by Hurst, a probability distribution which can be described as Geometric Brownian Motion is constructed as a guideline for a possible better way forward. It is shown that this Geometric Brownian Motion model has advantages over the Fractional Brownian Motion approach suggested by Mandelbrot as a means of incorporating both the "Noah effect" and the "Joseph effect".

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1. Introduction

> "The classical theorists resemble geometers in a non-Euclidean world who, discovering that in experience straight lines apparently parallel often meet, rebuke the lines for not keeping straight – as the only remedy for the unfortunate collisions which are occurring. Yet, in truth, there is no remedy except to throw over the axiom of parallels and to work out a non-Euclidean geometry. Something similar is required today in economics."
>
> John Maynard Keynes

A pervasive simplifying assumption of present day finance theory – namely that key probability distributions are "independent and identically distributed" – leads to precisely the same situation today as Keynes described in his "General Theory" more than sixty years ago. Despite around forty years of strong empirical evidence that distributions of return in the real financial world not only have far fatter tails than the normal or lognormal distributions but also are not independent, theorists continue – often with catastrophic results – to construct "scientific" risk models using totally inappropriate probability distributions.

Nowhere has this use of fatally flawed theory been more evident in recent years than in the case of Long-Term Capital Management (LTCM), the small US bank which, when it collapsed in the third quarter of 1998, threatened to create a trillion-dollar "black hole" in the worldwide banking system. In his highly detailed account of the rise and fall of LTCM, Lowenstein (2001) quotes the conclusion drawn by Fama (1965) regarding the existence of vastly more extreme outcomes than predicted by conventional theory:

> "If the population of price changes is strictly normal, on the average for any stock … an observation more than five standard deviations from the mean should be observed about once every 7,000 years. In fact such observations seem to occur about once every three to four years."

Perhaps the most compelling evidence that successive price changes are not independent is provided by the statistical methodology devised by the hydrologist Harold Edwin Hurst. A series that is truly random has a Hurst exponent of 0.5, whereas the values of around 0.72 observed up to a horizon of about four years for all major equity markets, as documented for example in Peters (1994), indicate the existence of a strong long-term memory effect. More generally, Mandelbrot and others have looked for scaling laws that might lead to more satisfactory descriptions of security prices over the longer term. A comprehensive review of the search for such scaling laws over the past forty years has recently been produced by Walter (2001).

A cornerstone of actuarial science is the use of observed real world probabilities, such as mortality rates, whereas much of present day financial theory is, as von Hayek observed in a more general context in his Nobel Memorial Lecture in December 1974, "decidedly unscientific in the true sense of the word, in that it involves a mechanical and uncritical application of habits of thought to fields different from those in which they have been formed". The objective of this paper is accordingly to investigate whether an actuarial approach can identify a better underlying distribution for security price changes than the normal or lognormal distributions, with particular emphasis being placed on the seemingly universal value of the Hurst exponent of around 0.72. An "acid test" for any such distribution will be whether, in conformity with the Fama (1965) and other empirical data, the probability of extreme outcomes is of the order of several thousand times higher than expected on the basis of the normal distribution.

2. Bachelier, Mandelbrot and Peters

This section describes in outline the search for models of security prices, with particular emphasis on scaling laws, by discussing the main contributions of Bachelier, Mandelbrot and Peters in this area. For a rigorous survey of the search for scaling laws over the past forty years the reader is referred to Walter (2001).

The first formal model for security price changes was put forward by Bachelier (1900). His price difference process in essence sets out the mathematics of Brownian Motion before Einstein and Wiener rediscovered his results in 1905 and 1923 in the context of physical particles, and in particular generates a normal (i.e. Gaussian) distribution where variance increases proportionally with time. A crucial assumption of Bachelier's approach is that successive price changes are independent. His dissertation, which was awarded only a "mention honorable" rather than the "mention très honorable" that was essential for recognition in the academic world, remained unknown to the financial world until Osborne (1959), who made no reference to Bachelier's work, rediscovered Brownian Motion as a plausible model for security price changes.

In 1963 the famous mathematician Mandelbrot produced a paper pointing out that the tails of security price distributions are far fatter than those of normal distributions (what he called the "Noah effect" in reference to the deluge in biblical times) and recommending instead a class of independent and identically distributed "alpha-stable" Paretian distributions with infinite variance. Towards the end of the paper Mandelbrot observes that the independence assumption in his suggested model does not fully reflect reality in that "on closer inspection … large changes tend to be followed by large changes – of either sign – and small changes tend to be followed by small changes." Mandelbrot later called this the "Joseph effect" in reference to the biblical account of seven years of plentiful harvests in Egypt followed by seven years of famine. Such a sequence of events would have had an exceptionally low probability of taking place if harvest yields in successive years were independent. While considering how best to model this dependence effect, Mandelbrot came across the work of Hurst (1951, 1955) which dealt with a very strong dependence in natural events such as river flows (particularly in the case of the Nile) from one year to another and developed the Hurst exponent H as a robust statistical measure of dependence. Mandelbrot's new model of Fractional Brownian Motion, which is described in detail in Mandelbrot & van Ness (1968), is defined by an equation which incorporates the Hurst exponent H. Many financial economists, particularly Cootner (1964), were highly critical of Mandelbrot's work, mainly because – if he was correct about normal distributions being seriously inconsistent with reality – most of their earlier statistical work, particularly in tests of the Capital Asset Pricing Model and the Efficient Market Hypothesis, would be invalid. Indeed, in his seminal review work on stockmarket efficiency, Fama (1970) describes how non-normal stable distributions of precisely the type advocated by Mandelbrot are more realistic than standard distributions but then observes:

> "Economists have, however, been reluctant to accept these results, primarily because of the wealth of statistical techniques available for dealing with normal variables and the relative paucity of such techniques for non-normal stable variables."

Partly because of estimation problems with alpha-stable Paretian distributions and the mathematical complexity of Fractional Brownian Motion, and partly because of the conclusion in Lo (1991) that standard distributions might give an adequate representation of reality, Mandelbrot's two suggested new models failed to make a major impact on finance theory, and he essentially left the financial scene to pursue other interests such as fractal geometry. However, in his "Fractal Geometry of Nature", Mandelbrot (1982) commented on what he regarded as the "suicidal" statistical methodologies that were standard in finance theory:

> "Faced with a statistical test that rejects the Brownian hypothesis that price changes are Gaussian, the economist can try one modification after another until the test is fooled. A popular fix is censorship, hypocritically called 'rejection of outliers'. One distinguishes the ordinary 'small' price changes from the large changes that defeat Alexander's filters. The former are viewed as random and Gaussian, and treasures of ingenuity are devoted to them … The latter are handled separately, as 'nonstochastic'."

Shortly after the "Noah effect" manifested itself with extreme severity in the collapse of Long-Term Capital Management, Mandelbrot (1999) produced a brief article, the cover story of the February 1999 issue of "Scientific American", in which he used nautical analogies to highlight the foolhardy nature of standard risk models that assumed independent normal distributions. He also pointed out that a more realistic depiction of market fluctuations, namely Fractional Brownian Motion in multifractal trading time, already existed.

The US investment manager, Edgar Peters, a frequent lecturer and seminarian to professional investment audiences, not only believed that the teachings of mainstream finance theory were fatally flawed but also saw great potential in new methodologies that took explicit account of the Hurst exponent being very significantly in excess of the Brownian Motion benchmark value of 0.5. In his 1989 "Financial Analysts Journal" article he estimates Hurst exponent values for the S&P 500 index. Peters (1991) then sets out, with rather more emphasis on chaos theory than on rescaled range analysis, his reasons for rejecting mainstream finance theory. This latter work, which is essentially in plain English but with relevant highly technical appendices, succeeded in bringing the works of Hurst and Mandelbrot to a far wider investment audience. Subsequently Peters (1994) expands on his previous works and in particular includes much more practical detail on rescaled range analysis and Fractional Brownian Motion.

Peters' contributions to the financial literature in general and to the quest for relevant scaling laws in particular have been criticised by some for lack of rigour. Perhaps the strongest such criticism is in Walter (2001), where the general approach of Peters (1991) is described as "spontaneous rather than rigorous" and it is suggested that, by linking fractals and chaos theory in an erroneous manner, both Peters (1991) and Peters (1994) have hindered, rather than helped, the case for a fractal or scaling law approach as a far more realistic alternative to standard distributions such as the normal and lognormal distributions.

3. The Hurst Exponent

Harold Edwin Hurst (1880 – 1978) left school at 15, and, after attending evening classes to further his education, won a scholarship to Oxford, where – largely as a result of his very great strengths in practical work – he was awarded a first-class honours degree. Almost his entire working career was spent as a hydrologist in Egypt struggling with the problem of reservoir control. As Peters (1991) observes:

> "An ideal reservoir would never overflow; a policy would be put into place to discharge a certain amount of water each year. However, if the influx from the river were too low, then the reservoir level would become dangerously low. The problem was: What policy of discharges could be set, such that the reservoir never overflowed or emptied?"

There are obvious parallels with pension scheme funding, where the investment return corresponds to the influx of water from unpredictable levels of rainfall within the catchment area, while the difference between payments to beneficiaries and contributions from employer and employees corresponds to the controlled level of discharge of water from the dam. The pension funding problem is to find a reasonably stable strategy that does not lead either to excess surplus (the dam overflowing) or to financial or regulatory insolvency (the reservoir emptying). Hurst studied how the range of the reservoir level fluctuated around its average level; if successive influxes were random (i.e. statistically independent) this range – as with standard deviation in the Black-Scholes option pricing model – would increase over time in line with the square root of time. Hurst obtained a dimensionless statistical exponent by dividing the adjusted range by the standard deviation of the observations, and hence his approach is generally referred to as rescaled range (R/S) analysis. By taking logarithms, we obtain the Hurst exponent H from the following equation:

$$H \log(N) = \log(R/S) + \text{constant}$$

where N is the number of observations and R/S is the rescaled range. In practice the best way to obtain an estimate of H is to find the gradient of the log/log plot of R/S against N. In strict contrast to the "statistical mechanics" independence value of 0.5 for H, Hurst found not only that for almost all rivers the exponent for the influx was well in excess of 0.5 (0.9 for the Nile!) but also that for a vast range of other quite distinct natural phenomena, from temperatures to sunspots, the estimates of H clustered very closely around the value of 0.72, indicating the existence of a "long-term memory" causal dependence.

Hurst's crucial observation, which he stated many times in his main papers, was that "although many natural phenomena have a nearly normal distribution, this is only the case when their order of occurrence is ignored". A classic demonstration that this is generally also true for "excess returns" in equity markets can be seen in Dimson, Marsh and Staunton (2001), which documents 101 years of investment returns in a large number of countries. Figure 51, which is a histogram built up from rectangles for each year in which the difference between the returns on UK equities and treasury bills fell into various 10% bands, bears at first glance a striking resemblance to a random normal distribution except for the existence of two outliers – the extreme low of 1974 and the extreme high of 1975. The 0 to 10% range contains the maximum number of years, namely 29; there are 22 years both in the –10% to 0 range and in the 10% to 20% range; there are 7 years both in the –20% to –10% range and in the 20% to 30% range; and there are 4 years both in the –30% to –20% range and in the 30%

to 40% range. However, certain features of the order of occurrence of these differential returns are of great significance. In the 10% to 20% range, the one above where the mean lies, no fewer than 12 of the 22 values are triplets of three consecutive years – 1934, 1935, 1936; 1941, 1942, 1943; 1982, 1983, 1984; and 1995, 1996, 1997. Furthermore, in each case the outcome for the subsequent year is in a lower range. There are no "runs" in this 10% to 20% range of either two years or of four or more years. Since such a pattern of outcomes has an infinitesimally small probability of occurring by mere random chance, it points to two very general features of equity market returns – a tendency for moderately above-average returns to create a continuing upward trend for several years until a setback from an overvalued position results, and a cycle length that is often of the order of four years, roughly the length of a typical economic cycle in the U.K. or the United States. Such behaviour mirrors exactly the results set out in Peters (1991) for Hurst exponent analyses of the four largest equity markets, namely those of the United States, the U.K., Japan and Germany using Morgan Stanley Capital International index data from January 1959 to February 1990. The average Hurst exponent over the four markets was 0.715 up to a duration of on average 46 months, after which it tended to decrease back toward the value of 0.5 that is characteristic of a truly random (i.e. independent and identically distributed) time series.

4. Four pointers

In this section we describe four mathematical phenomena that the present author regarded as being highly relevant in pointing out the general area in which a search for a new and more realistic probability distribution for security price changes should be concentrated.

The first pointer is the compelling evidence in Hurst (1951) that, for an exceptionally wide range of natural phenomena (such as rainfall, sunspots, mud sediments and tree rings) Hurst obtained exponents from very long time series very close to the value of 0.72 he obtained for river discharges. In particular, Figure 4 in this paper (reproduced in Peters (1994)) provides exceptionally compelling visual evidence that this "universal" measure of long-term dependence is of the utmost significance. It seems intuitively obvious that some "invisible hand" (to use Adam Smith's immortal metaphor) is at work and that some unifying principle, conceptually similar to Newton's inverse squares law of gravity, might be the key to understanding real world patterns of dependence between successive time periods.

The second pointer is that Hurst was able, by using a biased deck of playing cards, to simulate a time series that had a Hurst exponent of 0.72, precisely as he had found in nature. He began with a deck of cards with ±1, ±3, ±5, ±7 and ±9 in approximately the same proportions as they occur in a normal distribution. After shuffling the deck, he would cut it and note the number, let us say +5. Next, he would deal out two hands to give Deck A and Deck B, and then – given that the initial cut was +5 – would transfer the five highest cards in Deck A to Deck B and remove the five lowest cards of Deck B. Deck B can now be said to be biased to a level of +5. Finally a joker is added to Deck B. This biased deck, with the joker added, is used as the time series generator. It is shuffled and cut successively until the joker appears, at which point a new biased deck is set up. After 1,000 trials, Hurst calculated H = 0.72. He also carried out 1,000 trials without incorporating any bias and in this case obtained H = 0.50, exactly as expected for any random series.

The third pointer is the manner in which the familiar bell-shaped normal (or Gaussian) distribution can be obtained as the limiting case from the much more tractable binominal distribution, which in turn can be generated with remarkable simplicity through Pascal's

Triangle, where each value is the sum of the two values immediately above to right and to left. It can be observed in passing that one of the best known fractal structures, the Sierpinski Triangle, can be constructed by colouring the area around each odd and even entry in Pascal's Triangle black and white respectively.

The fourth pointer is the well known fractal pattern, Mandelbrot's Set. The important point is the simplicity of the rule which determines whether or not a particular point in the Argand plane of complex numbers is in the set. The introduction of the concept of "remaining bounded" to a sequence of complex numbers adds the "extra dimension" needed to generate the most intricate of patterns that is invariant no matter how high the magnification of the set is raised.

5. Actuarial potential

There are a number of reasons why an actuarial approach may be successful in identifying a satisfactory general model for longer term projections of security prices even although many eminent economists and mathematicians have searched in vain for such a model over the past forty years or so.

The first, and most obvious reason, is that actuaries begin by compiling empirical data about how the real world actually works and only thereafter consider how to obtain a practical answer, to a known degree of accuracy, as to how future experience is likely to evolve. Economists, on the other hand, tend to begin by building theoretical models based on the way they think the world behaves. Similarly, pure mathematicians tend to begin with theoretical models based on some branch of mathematics in which they are highly proficient.

The second reason is that actuarial training develops the highly commendable pattern of thought that can be called "general reasoning"; if the key results of employing a particular mathematical approach cannot be explained in plain English then it is possible that this approach is unsound in principle and could lead to imprudent courses of action if applied in practice.

The third reason, and one that is particularly relevant in the context of the present paper, is the actuary's proficiency in finite differences as a mathematical tool as well as a computational device. If, like the normal distribution as the limiting case of the binominal distribution, a probability distribution for the long-term dynamics of security prices can be constructed as the limiting case of a much more tractable short series of events, then it is highly likely that it would lead to a far better understanding of the situation than would be the case with a highly esoteric approach such as asymptotic sampling theory or functional central limit theory.

6. Binary process conjecture

The considerations discussed above, and in particular those in Section 4, strongly suggest not only what the Hurst exponent should guide the search for a new "universal" fractal distribution but also that some binary process extrapolated from Hurst's biased deck of cards offers the highest likelihood of finding a mathematical formulation to which traditional actuarial tools such as finite differences can be successfully applied.

We note first of all that the initial average frequency distribution of Deck B for cards of increasing value is the line in Pascal's Triangle corresponding to N = 9, namely 1, 9, 36, 84, 126, 126, 84, 36, 9, 1. Secondly, it seems intuitively obvious that a Hurst exponent of approximately 0.72 will also be obtained for similar biased decks of cards based on lines of Pascal's Triangle for other odd values of such as 3, 5, 7, 9, 11, 13 etc. The conceptual leap adopted here to give a binary process is the conjecture that N = 1 will also lead to a Hurst exponent of approximately 0.72.

For N = 1, let us assume for Deck B the average frequency of one card with value +1 and one card with value –1. If the cut to obtain the trend is +1, there are two cards of value +1 plus the joker, a total of only three cards. Similarly, if the cut to obtain the trend is –1 there are two cards with value –1 plus the joker.

For a positive trend, there is a probability of 2/3 that the value +1 appears and a probability of 1/3 that the joker appears. The latter event changes the trend to either positive or negative with a 50:50 chance of each. It then follows that, for the first card to emerge that is not the joker, there is a probability of 5/6 that it is +1 and a probability of 1/6 that it is –1. For a negative trend initially, these probabilities are reversed.

This binary process involves an "extra dimension" in that the addition of the trend-generating joker increases the number of cards in the initial (non-random) Deck B from two to three. It would be highly aesthetical in mathematical terms, but perhaps too much to hope for, if the ratio 2/3 played some explicit role in the subsequent numerical analysis. As described in later sections, this ratio does indeed play an absolutely central role in terms of convergence properties.

7. Geometric generalisation of Pascal's Triangle

For an initial positive trend, we enumerate below the probabilities of the various possible cumulative outcomes after 2, 3, 4, 5 … drawings of cards other than the joker. The results for an initial negative trend are the obvious mirror image.

For the second drawing of a card (other than the joker), there is a two-way branching from each of the possible outcomes from the first drawing. If this was +1, the trend is positive, and the conditional probabilities of the second drawing being +1 or –1 are 5/6 and 1/6 respectively, giving 25/36 and 5/36 respectively for the overall probabilities for outcomes of +2 and 0. Similarly, for an initial outcome of –1 and a negative trend, the probabilities of cumulative outcomes of 0 and –2 and 1/36 and 5/36 respectively. In summary, we have:

Cumulative value	Trend after two drawings	Probability
2	+	25/36
2	-	0
0	+	1/36
0	-	5/36
-2	+	0
-2	-	5/36

Clearly the trend after two drawings, as well as the cumulative outcome, will affect the next outcome. The situation after the third drawing is similarly worked out as:

Cumulative value	Trend after two drawings	Probability
3	+	125/216
3	-	0
1	+	10/216
1	-	25/216
-1	+	5/216
-1	-	26/216
-3	+	0
-3	-	25/216

For a general value of $c = 5$, as against the particular value of conjectured in Section 6, and omitting the denominators of $(c + 1)^N$, these outcomes for N = 1, 2 and 3 can be portrayed very neatly as an array similar to Pascal's Triangle:

N							
0				$(1, 0)$			
1			$(0, 1)$		$(c, 0)$		
2		$(0, c)$		$(1, c)$		$(c^2, 0)$	
3	$(0, c)$		$(c, c^2 +1)$		$(2c, c^2)$		$(c^3, 0)$
	-3	-2	-1	0	1	2	3

Then, denoting the cumulative value by x, we can express the first of each pair of numbers, 6^N times the probability of a cumulative value of x from a positive trend at the last drawing, as A (c, N, x) and the second (the corresponding result from a negative trend) as B (c, N, x). We note that:

$$\begin{aligned} &\text{A}\ (c, 0, 0) &&= 1 \\ &\text{A}\ (c, \text{N}, -\text{N}) &&= 0 \text{ for N greater than 0,} \\ \text{and}\ &\text{B}\ (c, \text{N}, \text{N}) &&= 0 \text{ for all values of N.} \end{aligned}$$

The following recurrence relationships hold for all other cases:

$$\begin{aligned} &\text{A}\ (c, \text{N}+1, x+1) &&= c\,\text{A}\ (c, \text{N}, x) + \text{B}\ (c, \text{N}, x) \\ \text{and}\ &\text{B}\ (c, \text{N}+1, x-1) &&= \text{A}\ (c, \text{N}, x) + c\,\text{B}\ (c, \text{N}, x) \end{aligned}$$

These five rules then generate the complete array. In general, when moving to the next line, a value in the line above is multiplied by c when the direction (to right or left) corresponds to the trend being positive or negative respectively, and by 1 otherwise. Since in Pascal's Triangle the multiplier is 1 in both cases, the array defined above can be accurately described as a geometric generalisation of Pascal's Triangle. Clearly Pascal's Triangle is the special case for $c = 1$, with A $(1, N, x)$ and B $(1, N, x)$ both equal to half the binomial coefficient that would occupy the same place in the array.

By extending the above array to, say, N = 10, it is possible to recognise various beautiful symmetries involving binomial coefficients that would allow highly complex general expressions for A (c, N, x) and B (c, N, x) to be formulated. However, the above algorithmic approach is far more satisfactory for practical work.

As explained in Section 6, the basic conjecture is that c will be equal to 5. However, in evaluating the general properties of the fractal probability distribution defined by the limiting case for large N it is convenient to work some of the time with a general value of c.

8. Mean, weighted sum of squares, and variance

For very small values of N it is easy to express the mean, or expected value, E (N) as a function of c. A little algebraic manipulation gives:

$$
\begin{aligned}
E(0) &= 0,\\
E(1) &= (c-1)/(c+1),\\
E(2) &= E(1) + (c-1)^2/(c+1)^2,\\
E(3) &= E(2) + (c-1)^3/(c+1)^3,\\
\text{and}\quad E(4) &= E(3) + (c-1)^4/(c+1)^4.
\end{aligned}
$$

Putting $d = (c-1)/(c+1)$, the successive first differences are d, d^2, d^3 and d^4. We now have:

$$
\begin{aligned}
E(N) &= d + d^2 + d^3 + \ldots. + d^N\\
&= (c-1)(1-d)^N/2,
\end{aligned}
$$

which tends towards $(c-1)/2$ as N tends towards infinity. The limiting value is 2 for the special case $c = 5$. Not only is this result an elementary application of finite differences, but convergence to the limiting value for the special case of $c = 5$ is geometric with constant ratio 2/3. A formal proof of the result by induction is trivial.

For the weighted sum of squares S(N), the algebra is more complex, and it is sensible to begin with numerical values. For $c = 5$ we obtain:

N	S (N)	First difference	Second difference
0	0.00000		
1	1.00000	1.00000	
2	3.33333	2.33333	1.33333
3	6.55556	3.22222	0.88889
4	10.37036	3.81481	0.79259
5	14.58024	4.20988	0.39507

Successive second differences decrease by a constant ratio of 2/3, and S (N), the sum of the first N –1 first differences, is an arithmetico – geometric series. Summing this series gives:

$$S(N) = 5N - 12 + 2^{N+2}/3^{N-1}.$$

For a general value of c we obtain:

$$S(N) = cN - (c^2 - 1)/2 + (c-1)^2\, d^{N-1}/2.$$

The variance, which is S (N) less the square of E (N), is approximately equal to *c* times N for large values of N. This suggests that as N becomes large the resulting probability distribution remains invariant apart from a scaling constant and hence is fractal in nature.

9. Evaluating the Hurst exponent

The binary process conjecture in Section 6 must be rejected forthwith if it cannot be shown that the Hurst exponent of the probability distribution represented by the geometric generalisation of Pascal's Triangle tends towards a limit of approximately 0.72 as N tends towards infinity. Since a rigorous evaluation procedure for the Hurst exponent in the case of a deterministic as opposed to empirical time series has not, to the present author's knowledge, been documented in the finance theory literature to date, this section begins with some practical examples which set out, in plain English and elementary mathematics, the underlying principles and the necessary practical detail.

Consider first of all, for N = 6 and an initial positive trend, those paths which lead to a cumulative outcome of 0. There are 20 such paths, this being the corresponding binominal coefficient in Pascal's Triangle. Consider in particular the path that can be described in an obvious notation as (+ - - + - +). Since each step in the path is a movement of either 1 or –1, the set of successive cumulative outcomes it (1, 0, -1, 0, -1, 0), from which it follows that maximum is 1, the minimum is –1, and the range is 2. The standard deviation is clearly 1.

In terms of the probability of this path arising, the first step, being positive, is in the direction of the existing trend and hence, for *c* = 5, has probability 5/6. The second step, being negative, is against the previous trend and hence has probability 1/6. Similarly, the remaining steps have probabilities of 5/6, 1/6, 1/6 and 1/6 respectively, giving an overall probability of 5 to the power of 2 divided by 6 to the power of 6.

The relevant power of 5 in this compound probability can in the general case be derived as the value of N (in this case 6), less the number of distinct positive or negative sub-paths (in this case 5), plus 1 (as in this case) only if the initial step is positive. In probability terms, the path (+ - - + - +) can be said to have a power of 2.

The ranges and powers of all 20 possible paths are enumerated below:

Path	Range	Power	Path	Range	Power
+ + + - - -	3	5	+ + - + - -	2	3
+ + - - - +	3	4	+ - + - - +	2	2
+ - - - + +	3	4	- + - - + +	2	2
- - - + + +	3	4	+ - + + - -	2	3
+ + - - + -	2	3	- + + - - +	2	2
+ - - + - +	2	2	- + + + - -	3	3
- - + - + +	2	2	+ - + - + -	1	1
+ - - + + -	2	3	- + - + - +	1	0
- - + + - +	2	2	- + - + + -	2	1
- - + + + -	3	3	- + + - + -	2	1

The R/S rescaled range value for this set of paths is then the probability-weighted value of the various ranges. For a general value of c, the numerator of this value (ignoring the probability divisor of 6 to the power of 6, which cancels out) is:

$$3c^5 + 9c^4 + 14c^3 + 12c^2 + 5c + 1$$

and the denominator is:

$$c^5 + 3c^4 + 6c^3 + 6c^2 + 3c + 1$$

This gives values of 2.8874 for $c = 5$ and 2.2000 for $c = 1$. Since the logarithms (to the base of 10) of these values are 0.46035 and 0.34242 respectively, dividing by 0.77815 (the logarithm of 6) gives for these paths Hurst exponents of 0.592 and 0.439 respectively.

Consider now, for N = 6 and an initial positive trend, the 15 paths which lead to a cumulative value of 2. Since rescaled range analysis begins by removing any positive or negative trend, the mean value of 1/3 for each step is subtracted to give an adjusted value of 2/3 for each positive step and an adjusted value of –4/3 for each negative step. Also, the standard deviation for each path is one third of the square root of 8, and ranges derived from the adjusted values at each step are divided by this standard deviation to obtain the rescaled range for each path. Using r to represent the maximum rescaled range, namely the square root of 8, the rescaled ranges and powers for the 15 paths are as below:

Path	Rescaled range	Power	Path	Rescaled range	Power
+ + + + - -	r	5	- + - + + +	$0.75r$	2
+ + + - - +	r	4	+ + - + + -	$0.5r$	3
+ + - - + +	r	4	+ - + + - +	$0.5r$	2
+ - - + + +	r	4	- + + - + +	$0.5r$	2
- - + + + +	r	4	+ - + + + -	$0.75r$	3
+ + + - + -	$0.75r$	3	- + + + - +	$0.75r$	2
+ + - + - +	$0.75r$	2	- + + + + -	r	3
+ - + - + +	$0.75r$	2			

This gives R/S rescaled ranges of 2.7495 for $c = 5$ and 2.2627 for $c = 1$, and corresponding Hurst exponents of 0.564 and 0.456 respectively.

Applying these evaluation procedures to all possible paths for values of N from 3 to 6 gives the following Hurst exponents for $c = 5$ and $c – 1$.

	Hurst exponent	
N	$c = 5$	$c = 1$
3	-0.764	0.053
4	-0.195	0.288
5	0.083	0.362
6	0.244	0.432

While the values for $c = 1$ show signs of tending towards the expected statistical mechanics value of 0.5 as N increases, the values for $c = 5$ are, contrary to what might have been expected, lower than the values for $c = 1$ for each of these values of N. However, in the course of the detailed numerical work it was noticed that, for $c = 1$, the rescaled range appeared to vary very little with the end value of the path. For these values to be virtually constant for a given value of N would be a very surprising result, somewhat reminiscent of Einstein's "special relativity" result that the speed of light is constant to all observers regardless of the velocity relative to any given frame of reference. To examine the situation further, what can be called the Hurst sub-components were calculated for each end value, precisely as was shown above by way of illustration for N = 6 and end values of 0 and 2. For $c = 1$, the pattern up to N = 10 is shown below with the decimal point omitted throughout, and the end value x along the bottom.:

N	Hurst sub-exponent $c = 1$																				
2									-		000		-								
3								-		296		296		-							
4							-		396		368		396		-						
5						-		431		387		387		431		-					
6					-		449		456		439		456		449		-				
7				-		460		477		473		473		477		460		-			
8			-		468		482		488		470		488		482		468		-		
9		-		473		492		489		485		485		489		491		473		-	
10	-		477		493		490		492		486		492		490		493		477		-
	-10		–8		–6		–4		–2		0		2		4		6		8		10

There is no contribution to the rescaled range for end values of $\pm N$ since the adjusted range is zero in these cases. For $\pm(N\text{-}2)$ the rescaled range is the square root of N-1, so that these Hurst sub-exponents increase towards a limit of 0.5 as N increases. Also, for a given value of N, the other Hurst sub-exponents are also very close to these values for $\pm(N\text{-}2)$. We also note that, for end values of 0, 0.5 minus the Hurst sub-exponent decreases very nearly in a geometric progression with constant ratio 0.5 from $N = 6$ to $N = 8$ and from $N= 8$ to $N = 10$. Since the probability of the end value being N or $-N$ is the inverse of 2^N in both cases, we conclude that the Hurst exponent tends towards 0.5 from below as N tends towards infinity.

Consider now, using "general reasoning", what might be expected for $c = 5$. For an end value x other than $\pm N$, the weighted average rescaled range is very heavily weighted in probability terms towards the maximum value, namely the square root of $r(N\text{-}r)$ where $r = (N\text{-}x)/2$, since it is obvious that there is a very strong correlation between adjusted range and probability power. Accordingly, these Hurst sub-components will be higher than in the case of $c = 1$. First, and much more importantly, the sum of the probabilities of the path corresponding to N and $-N$ is $(5/6)^{N-1}$, which is vastly greater than the corresponding probability of $(1/2)^{N-1}$ for $c = 1$. For $N = 6$, for example, these values are 0.40188 and 0.03125 respectively. Second, for small values of N the maximum rescaled range of $N/2$ (for even values of N) is not significantly greater than the square root of N. This second factor can be expected to lead to slower upward convergence towards a limit as N increases.

For c = 5, the corresponding table of Hurst sub-exponents is as below:

N	Hurst sub-exponent $c = 5$																				
2									-		000		-								
3								-		296		296		-							
4							-		396		488		396		-						
5						-		431		544		544		431		-					
6					-		449		563		592		564		449		-				
7				-		460		574		614		615		575		460		-			
8			-		468		577		617		632		618		579		468		-		
9		-		473		579		627		647		648		628		582		473		-	
10	-		477		580		624		651		659		652		625		582		477		-
	-10		–8		–6		–4		–2		0		2		4		6		8		10

As for c = 1, the value for ±(N-2) therefore the square root of N-1. For other end values the Hurst sub-exponent increases rapidly from each extreme to a maximum end value 0. Also, for given end value, it increases with N. In particular, the difference between the values for N = 6 and N = 8 is very nearly 2/3 of the difference between the values for N = 8 and N = 10. This strongly suggests that geometric convergence is taking place with a constant ratio of 2/3, precisely as was the case for the mean and the sum if squares.

For an end value of 0, the Hurst sub-exponents for N = 12, 14 and 16 are 0.676, 0.688 and 0.697 respectively, which are – as conjectured – converging geometrically with a constant ratio of 2/3 to a value very close to 0.72. Since the values of around 0.72 other than towards the extreme of end value will dominate as N becomes larger, we infer that the overall Hurst exponent, which is derived from the weighted average rescaled range over all possible end values, will increase – albeit very slowly compared to the convergence for c = 1 – to a limit of approximately 0.72, thereby confirming the conjecture in Section 6 that a geometric generalisation of Pascal's Triangle will define a probability distribution with a Hurst exponent equal to the "universal" empirically derived value of around 0.72.

In view of the computational effort involved (in particular, the case N = 16 involves 65,536 possible paths) some minor approximations were used in estimating the above values. A priority for further development is to construct a computer program for the evaluation of the Hurst exponent so that the inferred convergence to around 0.72 can be tested rigorously.

The above evaluation of the Hurst exponent is deterministic rather than, as is usually the case in theoretical work, based on Monte Carlo simulation. It is instructive to note that a very small deviation from the theoretically expected number of large rescaled ranges can cause serious distortions. For example, for N = 6 and an end value of 0, if the number of paths with a probability power of 5 is one more or one less than the deterministic number of 3, with the number of paths with a range of 2 and a probability power of 3 changing to compensate, the sampled Hurst sub-exponents would be 0.678 and 0.489 respectively as against the deterministic value of 0.592.

10. Geometric Brownian Motion

The discrete probability distribution corresponding to the geometric generalisation of Pascal's Triangle can accurately be described as Geometric Brownian Motion. Brief comments on four aspects of the general features and properties are set out below.

First, as N tends towards infinity the distribution tends towards a symmetric continuous distribution with a mean value of zero. Although this has still to be investigated rigorously, it seems likely that this limiting distribution will be similar in certain respects to Fractional Brownian Motion.

Second, it might in some applications be appropriate to use the discrete version to incorporate any positive or negative bias that is believed to exist initially; in the limiting case the effect of the initial bias collapses to zero. For example, at the height of the worldwide speculative bubble in TMT (technology media and telecom) stocks during 2000 it might have been thought appropriate to assume an initial negative trend. Similarly, after the stockmarket falls in the two weeks after the terrorist atrocities of 11th September 2001 it might have been thought that sufficient economic and other uncertainties has been disconnected at the lower price levels for it to have been appropriate to assume an initial positive trend.

Third, the mathematical formulation of Geometric Brownian Motion is very considerably simpler and more transparent than that of Fractional Brownian Motion, whereby the highly complex defining equation seriously hinders the more widespread practical use of the latter process. A similar situation exists in the field of option pricing, where binominal lattice models are far more tractable than complex formulae of the Black-Scholes type.

Fourthl, Geometric Brownian Motion has a readily unstood generating process, namely a very small biased deck of playing cards. A relevant analogy in the physical sciences is that, although Kepler's three laws of Planetary Motion proved to be a correct description of reality, the mathematical statements (particularly the Third Law that the squares of the periodic times of planets around the sun are proportional to the cubes of their mean distances from the sun) were so esoteric that most astronomers were reluctant to accept and apply them. The real breakthrough in astronomy came when Newton formulated his "universal inverse squares law of gravity" and showed that all three of Kepler's Laws could be derived from this simple result, which could be understood by everyone.

11. Select and ultimate periods

The empirical evidence for capital markets in developed countries suggests that the value of the Hurst exponent is generally of the order of 0.72 for the first four years or so, which approximates to the length of a typical economic cycle, and then decreases towards the theoretical statistical mechanics value of 0.5. This is reminiscent of "select" and "ultimate" experience in mortality studies where, for "selected" lives that have passed an underwriting process to be accepted for life assurance at normal rates, there is a strong bias for several years towards lower than "average" or "ultimate" rates of mortality.

In the case of equity markets, it seems that various biases and instances of "irrational behaviour" can cause prices to diverge markedly from "fair" or "equilibrium" values over the medium term of several years. In the longer term, however, economic realities act as constraints and tend to force the prices back towards central values. This suggests that in practical work it may be desirable to use a value of $c = 5$ only up to a horizon of, say, 3, 4 or 5 years, after which a downward trend towards 0.5 can be incorporated. This presents no computational difficulties whatsoever.

Since in practical applications it is envisaged that a downside approach to risk will be adopted, such as outlined in Clarkson (2000, 2001), particular attention has to be paid to the "downside tail" of the distribution when the select period is being chosen.

12. Financial risk application

After the Hurst exponent the next most important acid test of a new class of probability distribution for capital market risk applications is the existence of exceptionally fatter tails than the normal or lognormal distributions. For 25 periods with an initial positive trend, the "simplistic" standard deviation is 5 (based on the conventional extrapolation of the single period standard deviation of 1), the probability of an outcome of 21, 23 or 25 (i.e. above 4 "standard deviations") is 0.041 and the probability of an outcome of –21, -23 or –25 (i.e. more than 4 standard deviations below the mean) is 0.026. On the basis of the binominal distribution (as a good approximation to normality) the corresponding probabilities are both approximately 0.00001, i.e. only one in a hundred thousand.

Using Geometric Brownian Motion, the resulting measures of financial risk will be vastly higher than on conventional VaR (Value at Risk) approaches. However, whereas central banks and financial regulators might be in favour of a significant move towards this much more prudent approach, many individual banks and security houses could be strongly opposed on account of the much higher capital requirement implications.

13. Behavioural finance

Over the past ten years of so considerable empirical research has been carried out to investigate what appeared to be violations of the standard "rational expectations" cornerstone of economic science. This new branch, behavioural finance, has identified several recurring patterns of non-optimal behaviour such as "over-confidence" (as exemplified by the well known "irrational exuberance" comments from Alan Greenspan, Chairman of the US Federal Reserve), "over-reaction bias", and "myopic loss aversion". All three of these traits involve, to a greater or lesser extent, the extrapolation in the investor's mind of the recent past into the future and are accordingly biases which cause a trend – once initiated – to continue for far longer than would be likely if most investors and other economic agents were essentially "rational" in their actions. The Geometric Brownian Motion model is of course fully consistent with the empirical findings in the behavioural finance field.

14. Commentary on Lo (1991)

Lo (1991) suggests that rescaled range analysis as pioneered by Hurst (1951) and later refined by Mandelbrot can be seriously distorted by short-range dependence, making deductions as to the existence of long-term dependence unsound. Lo proposes a modified approach which replaces the usual standard deviation denominator by one that also includes weighted autocovariance terms. The null hypothesis is highly complex and comprises four conditions, only one of which is standard. The new values obtained for the modified rescaled range statistic fall back close to the 0.5 statistical mechanics value.

The present author's view is this highly complex modified approach is unnecessary, in that, as described in Section 9, Hurst exponents are not only of little or no significance for short-term series but can also be seriously distorted by unavoidable sampling errors. Furthermore, Mandelbrot's comments on how some economists try various modifications until "conventional" teachings appear justified come to mind on reading some of Lo's concluding remarks:

> "The absence of strong dependence in stock returns should not be surprising from an economic standpoint, given the frequency with which financial asset prices clear. Surely financial security prices must be immune to persistent informational asymmetries, especially over longer term time spans."

15. Actuarial judgement

A crucial feature of the Geometric Brownian Motion model is that, for short term projections, the results may differ significantly depending on whether it is considered that the initial trend (i.e. the bias within the system) is in a positive or negative direction. This runs completely contrary to the view held in some quarters that financial economics allows an actuary to dispense with judgement and estimate all relevant factors for future projections on the basis of a statistical description of the past.

The situation is, in the present author's view, akin to Einstein's deeply held reservations about the purely statistical descriptions of quantum theory. He was convinced that underlying casual mechanisms were at work and, in personal correspondence cited in Popper (1980), suggested that we should not have to be satisfied for ever with "so loose and flimsy a description of nature". As mentioned in Section I0, the Geometric Brownian Motion model has an inbuilt causal mechanism which generates a Hurst exponent well in excess of 0.5, whereas Fractional Brownian Motion, for example, does not.

16. Conclusion

A binary process conjecture based on a simulation approach suggested by Hurst (1951) leads first of all to a geometric generalisation of Pascal's Triangle and then to a probability distribution which can be described as Geometric Brownian Motion. This new model, which – before taking account of select and ultimate periods as in actuarial mortality studies – is characterised by a Hurst exponent very close to the seemingly universal value of around 0.72, could be an important first step forward in formulating a better model to encapsulate the observed "Noah" and "Joseph" effects in capital market returns. Whilst it seems likely that this new actuarial model of Geometric Brownian Motion, with appropriate adjustment for select and ultimate periods, might be preferable on grounds of both empirical accuracy and mathematical tractability to existing models such as Fractional Brownian Motion, there is clearly a vast amount of further development work to be carried out before this could be shown beyond any reasonable doubt to be the case.

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