# Economic valuation of defined benefit liabilities

Teemu Pennanen

King's College London

1. Hilli, Koivu and Pennanen, Cash-flow based valuation of pension liabilities, European Actuarial Journal, 2011.
2. Koivu, Pennanen and Ranne, Modeling assets and liabilities of a Finnish pension company: a VEqC approach, Scandinavian Actuarial Journal, 2005.
3. Hilli, Koivu and Pennanen, Optimal construction of a fund of funds, European Actuarial Journal, 2011.
4. Koivu and Pennanen, Galerkin methods in dynamic stochastic programming, Optimization, 2010.
5. Aro and Pennanen, User friendly approach to stochastic mortality modeling, European Actuarial Journal, 2011.
6. Pennanen, Optimal investment and contingent claim valuation in illiquid markets, Finance and Stochastics, 2014.

# The problem



- Consider an insurance portfolio with aggregate claims $c_t$ payable at time $t = 1, \dots, T$.
- Our aim is to value the current liabilities so any future additions to the insurance portfolio are ignored.
- We assume that the liabilities amortize in finite time and that the last claim will be paid at time $T$.
- After paying the claims $c_t$ at time $t$, the insurer invests the remaining wealth in financial markets over the next period $[t, t+1]$.
- What is the least amount of initial capital needed to cover the claims?

# “Best estimate”



When the claims $c = (c_t)_{t=1}^T$ and investment returns $R = (R_t)_{t=1}^T$ are deterministic, the problem can be written as

$$\begin{aligned} &\text{minimize} && V_0 \quad \text{over} \quad (V_t)_{t=0}^T \\ &\text{subject to} && V_t = R_t V_{t-1} - c_t \quad t = 1, \dots, T, \\ & && V_T \geq 0. \end{aligned}$$

In this deterministic case, the minimum value is given by the “best estimate”

$$V_0 = \sum_{t=1}^T \frac{c_t}{\prod_{s=1}^t R_s}.$$

What happens in when $c$ and $R$ are random?

# “Best estimate”



What happens in when $c$ and $R$ are random?

- It is common practice to take a single prediction (projection) of the claims $c$ and to take the discount factors from fixed-income markets.
- However, yield curves are meant for valuation of fixed-income instruments, not uncertain insurance claims.
- For example, the “best estimate” of a European call-option is much higher than its market (or Black-Scholes) value.
- Adding a “risk margin” makes things worse.
- The best estimate is procyclical: during adverse market conditions, yield curves tend to decrease which results in higher values of the “best estimate” and increased capital requirements.

# Risk sensitive valuation



- From now on, both the claims $c_t$ and the investment returns $R_t$ will be random variables on a probability space $(\Omega, \mathcal{F}, P)$.
- The probability measure $P$ models the views of the insurer (or a supervisor) concerning the future development of the underlying risk factors.
- We are still dealing with only one asset.

# Risk sensitive valuation



When the claims $c_t$ and the investment returns $R_t$ are random variables, the valuation problem can be written as

$$\begin{aligned} &\text{minimize} && V_0 \quad \text{over} \quad V \in \mathcal{N} \\ &\text{subject to} && V_t = R_t V_{t-1} - c_t \quad t = 1, \dots, T, \quad P\text{-a.s.} \\ &&& V_T \in \mathcal{A}, \end{aligned}$$

where

- $\mathcal{N}$ is the set of adapted processes (the value of $V_t$ depends only on information observed by time $t$),
- $\mathcal{A}$ is a set of random terminal positions that are acceptable.

# Risk sensitive valuation



- superhedging: $\mathcal{A} = \{V \,|\, V \geq 0 \ P\text{-a.s.}\}$.
- quantile hedging: $\mathcal{A} = \{V \,|\, P(V \geq 0) \leq \delta\}$.
- zero utility principle: $\mathcal{A} = \{V \,|\, Eu(V) \geq u(0)\}$, where $u$ is a utility function.
- acceptable hedging: $\mathcal{A} = \{V \,|\, \mathcal{R}(V) \leq 0\}$, where $\mathcal{R}$ is a risk measure. This covers e.g.
  - all the above examples
  - Conditional Value at Risk.

In general, analytical solutions to the pricing problem are not available (even in this one-asset model) but for many choices of $\mathcal{A}$ it can be solved numerically using integration quadratures (e.g. Monte Carlo) and a simple line search.

# Case study

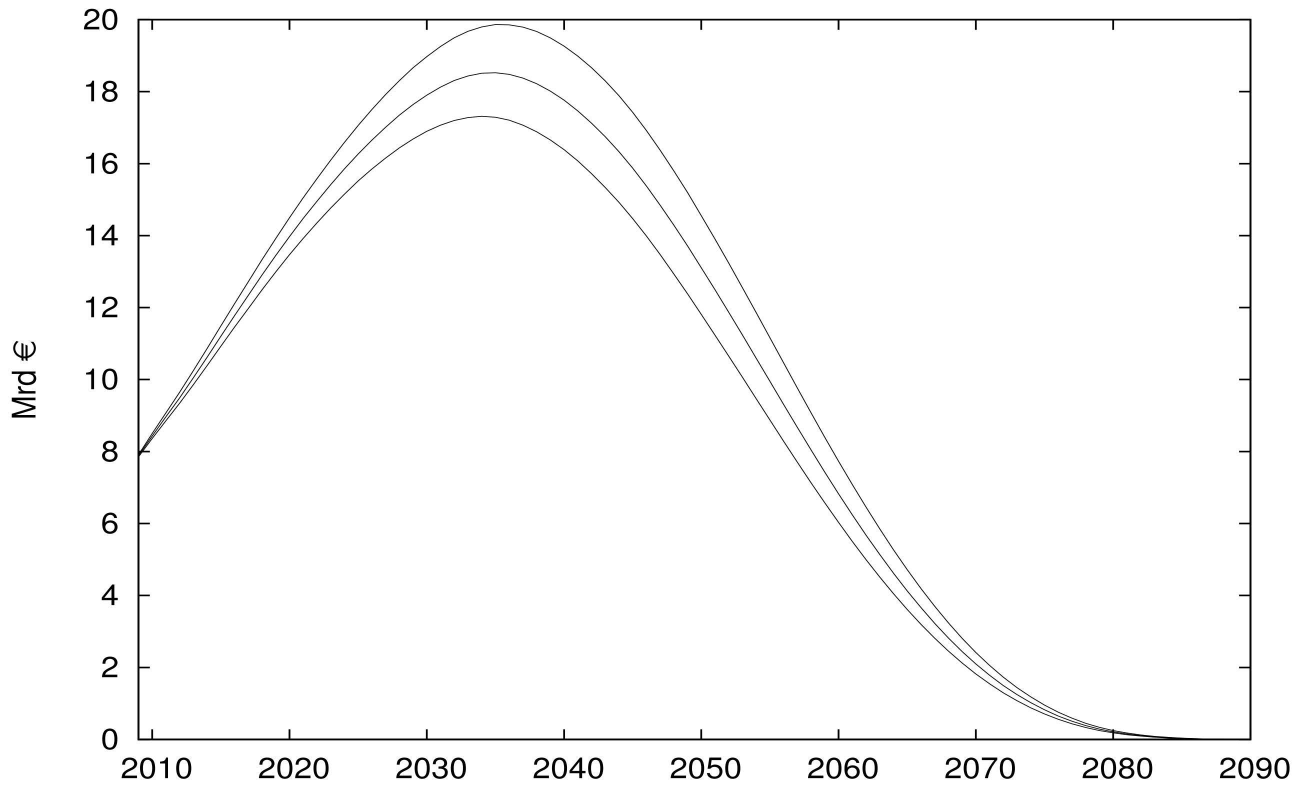


- We will study the pension insurance portfolio of the Finnish private sector occupational pension system.
- The yearly claims $c_t$ consist of aggregate old age, disability and unemployment pension benefits earned by the end of 2008 and become payable during year $t$.
- The claims depend on mortality and the price- and wage-inflation, etc.

# Case study



Figure 1: Yearly claims



# Case study



- We model the investment returns by

$$\ln R_t = \mu + \sigma \varepsilon_t,$$

where $\varepsilon_t$ are iid standard normal and the parameters $\mu$ and $\sigma$ are chosen so that the annualized logarithmic returns have a mean and standard deviation of $6\%$.

- We will use the acceptance sets

$$\mathcal{A} = \{V \mid \mathcal{R}(V) \leq 0\},$$

where $\mathcal{R}$ is either the Value at Risk or the Conditional Value at Risk with varying confidence levels.

# Case study



| | Confidence level | | | | |
|---|---|---|---|---|---|
| | 95% | 90% | 85% | 80% | 66% |
| $V@R$ | 289 | 271 | 259 | 250 | 232 |
| $CV@R$ | 305 | 288 | 276 | 268 | 252 |

Table 1: Liability values (bn€) with varying risk tolerances

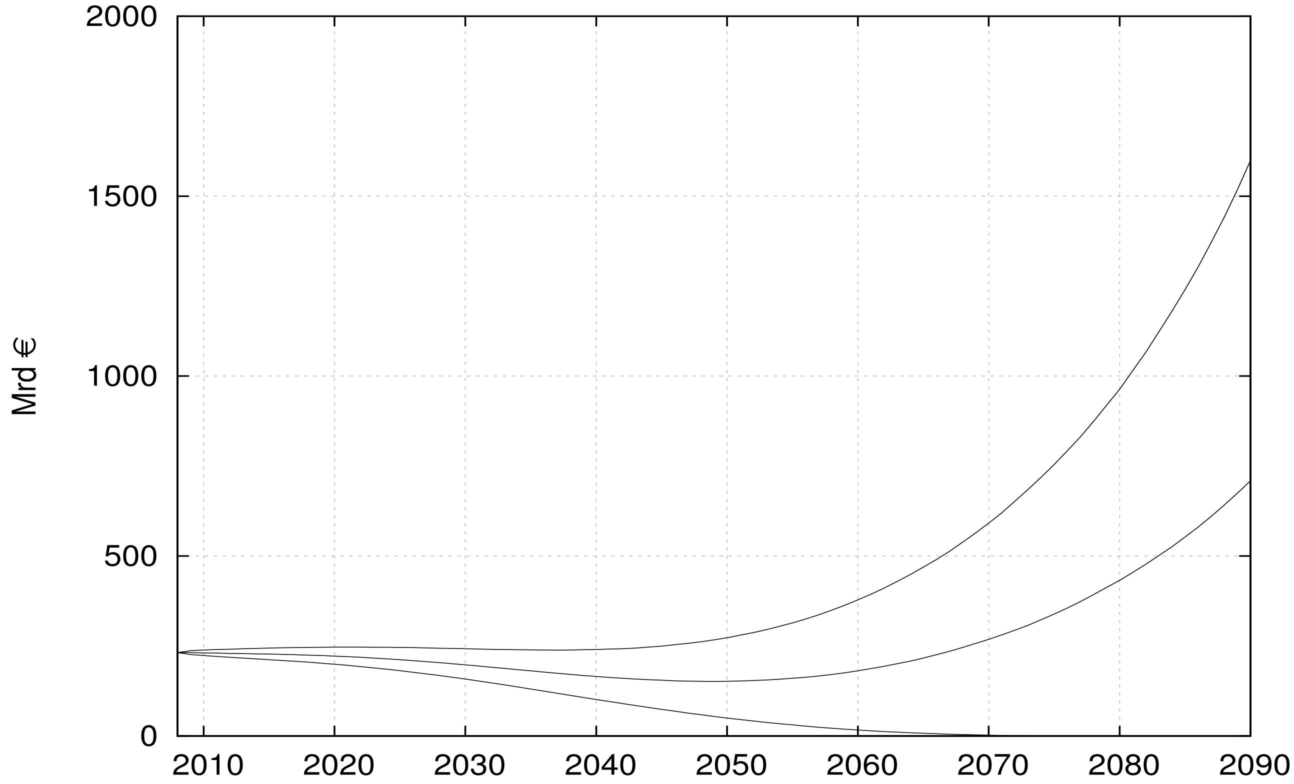
| | Confidence level | | | | |
|---|---|---|---|---|---|
| | 95% | 90% | 85% | 80% | 66% |
| $V@R$ | 24.9 | 26.6 | 27.9 | 28.9 | 31.1 |
| $CV@R$ | 23.6 | 25.1 | 26.1 | 26.7 | 28.7 |

Table 2: Corresponding funding ratios (%)

# Case study



Figure 2: The development of 34%, 50%- and 66%-quantiles of net wealth $V_t$ when $V_0$ is set according to $V@R_{66\%}$.



# Market consistent valuation

- When there are several asset classes, it may be possible to reduce the initial capital by adapting the investment strategy to the liabilities.
- For example, the Black-Scholes formula gives the initial capital required for a strategy that replicates the claim in a complete market model.
- In reality, riskless hedging is often prohibitively expensive so we will look for hedging strategies that result in acceptable terminal positions.

# Market consistent valuation



- Assume a finite set $J$ of investment classes (bonds, equities, real estate, ...).
- Denote by $R_{t,j}$ the total return on class $j \in J$ over period $[t-1, t]$.
- Let $h_{t,j}$ be the amount of wealth invested in class $j \in J$ at the beginning of period $t$.
- The portfolio $h_t = (h_{t,j})_{j \in J}$ depends only on the information observed by time $t$.

The budget constraint becomes

$$\sum_{j \in J} h_{t,j} + c_t \leq \sum_{j \in J} R_{t,j} h_{t-1,j} \quad P\text{-a.s. } t = 1, \dots, T.$$

# Market consistent valuation



The valuation problem can be written as

$$\begin{aligned}
&\text{minimize} && \sum_{j\in J} h_{0,j} \quad \text{over} \quad h \in \mathcal{N}_D \\
&\text{subject to} && \sum_{j\in J} h_{t,j} + c_t \le \sum_{j\in J} R_{t,j} h_{t-1,j} \quad t = 1,\dots,T, \\
& && \sum_{j\in J} h_{T,j} \in \mathcal{A},
\end{aligned}$$

where $\mathcal{N}_D$ denotes the set of feasible investment strategies adapted to the available information. Analytical solutions are not available, in general.

# Market consistent valuation

The above valuation framework

- extends actuarial premium principles by incorporating the possibility of dynamic trading.
- extends superreplication principles of financial mathematics by incorporating more reasonable risk tolerances.
- can be used internally and/or for regulatory purposes, depending on who chooses the probability measure $P$ and the acceptance set $\mathcal{A}$.

# Market consistent valuation



When $\mathcal{A} = \{V \mid \mathcal{R}(V) \leq 0\}$ for a convex risk measure $\mathcal{R}$, we can solve the valuation problem by "solving" few instances of

$$\begin{aligned} \text{minimize} \quad & \mathcal{R}\left(\sum_{j\in J} h_{T,j}\right) \quad \text{over} \quad h \in \mathcal{N}_D \\ \text{subject to} \quad & \sum_{j\in J} h_{0,j} \leq V_0 \\ & \sum_{j\in J} h_{t,j} + c_t \leq \sum_{j\in J} R_{t,j} h_{t-1,j} \quad t = 1, \ldots, T. \end{aligned}$$

We will apply the Galerkin method.

# Market consistent valuation



- The Galerkin method seeks optimal solutions among convex combinations of a finite set $\{h^i\}_{i\in I} \subset \mathcal{N}$ of feasible solutions (basis strategies).
- Such a problem can be written as

$$\text{minimize} \quad \mathcal{R}\left(\sum_{j\in J}\sum_{i\in I}\alpha^i h^i_{T,j}\right) \quad \text{over} \quad \alpha\in\Delta,$$

  where $\Delta = \{\alpha \in \mathbb{R}^I_+ \mid \sum_{i\in I}\alpha^i = 1\}$.
- Any convex combination of feasible solutions $h^i \in \mathcal{N}$ will be adapted and feasible.
- For certain choices of $\mathcal{R}$, the above can be solved approximately by integration quadratures and convex optimization techniques.

# Case study



- The traded assets consist of five equity indices and two bond indices.
- Yearly bond returns are modeled by

$$R_t = \exp(Y_t \Delta t - D \Delta Y_t),$$

  where $Y$ is the yield to maturity and $D$ the duration.
- Market risk factors are modeled together with the liability risk factors (mortality, price- and wage-inflation) by a stochastic difference equation of the form

$$\Delta x_t = A x_{t-1} + b + \varepsilon_t,$$

  where $x$ is the vector of (transformed) risk factors.

# Case study



- We evaluated the liabilities with 529 different investment strategies as well as with the Galerkin method.
- The strategies are based on the buy and hold, fixed proportion and constant proportion portfolio insurance rules with varying parameters.
- All strategies were modified to accommodate for claim payments and the portfolio constraints.

# Case study



| | Confidence level | | | | |
|---|---|---|---|---|---|
| | 95% | 90% | 85% | 80% | 66% |
| Best basis | 296 | 284 | 273 | 261 | 239 |
| Optimized | 288 | 271 | 254 | 236 | 202 |

Table 3: Liability values with varying risk tolerances

| | Confidence level | | | | |
|---|---|---|---|---|---|
| | 95% | 90% | 85% | 80% | 66% |
| Best basis | 24.3 | 25.4 | 26.4 | 27.6 | 30.1 |
| Optimized | 25.0 | 26.6 | 28.3 | 30.5 | 35.6 |

Table 4: Corresponding funding ratios

# Summary

The economic valuations depend on

1. views: the probabilistic description of the future development of the claims and the market.
2. risk preferences: level of risk at which the assets should cover the liabilities,
3. hedging strategy: the strategy according to which the given capital is invested in financial markets.

- All the above factors are subjective.
- Asset management is an integral part of valuation (and one of the most important tasks of an insurer).
- Dependencies between assets and liabilities determine how well investment returns can be adapted to the liabilities.