Investigation of Dependency between Short Rate and Transition Rate on Pension Buy-outs

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The Outline of the Study

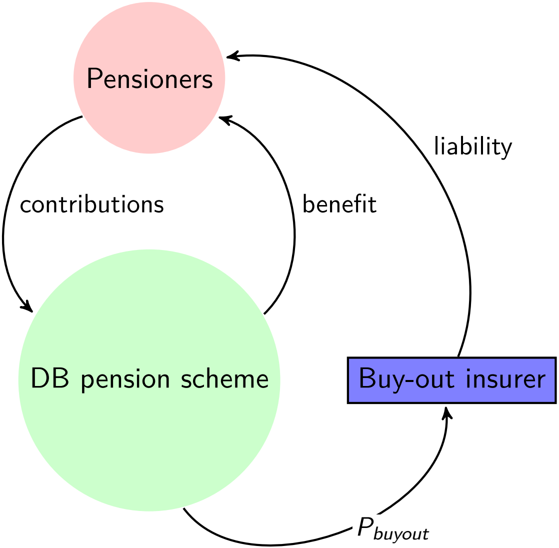
Aim of the Study

The main purpose of the study

Introduce a general valuation expression for pension buy-outs

1. as an extension of Arık et al. (2017),
2. under the dependence assumption between short rate and transition rates,
3. in a **continuous Markovian setting**.

What is a Pension Buy-out?



The Model Framework

$$P_{\text{buyout}}(t) = \frac{\sum_{t_i>t}^{t_M} E^{\mathbb{Q}}\left[e^{-\int_t^{t_i} r(s)ds} \max\{PA(t_i^+) - (PA(t_i) - N(t_i).C), 0\} | \mathcal{I}_t\right]}{L(t)}$$

- $r(\cdot)$: the stochastic short rate for $t \geq 0$.
- $L(\cdot)$ is the liability of the pension scheme as defined in Definition 1.
- $a(.)$ is the fair price of an immediate life annuity deal.
- $B(.)$ is the fair price of a zero coupon bond.
- $PA(t_i^+)$ is the value of the pension portfolio at the beginning of time t_{i+1}.

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Liability Model

Definition

(Liability Process) *The pension liability process* $L(t)$ *at time* t, $t \in [0, T]$, *is determined as*

$$L(t) = N(t) \times a(t, x),$$

where

- $N(t)$: *the number of survivors in the model at time* t *(determined according to the force of mortality rate dynamics).*
- *For the valuation of* $a(t, x)$, *see the next slides.*

An Illness-Death Model for a DB Pension Scheme

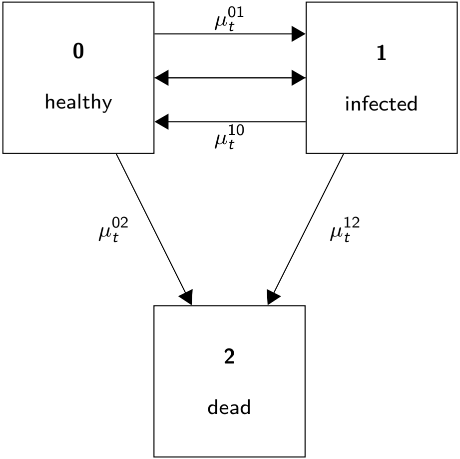

Figure 1: The illness-death model for a hypothetical DB pension scheme

Correlated Transition and Short Rates

1. Suppose $\eta(t) = \mu_t^{01}$, $\mu(t) = \mu_t^{\cdot 2}$ and $\mathbb{X}$ is a 3-dimensional affine process:

$$(\eta(t), \mu(t), r(t))' = c(t) + \Gamma(t)\mathbb{X}(t), \tag{1}$$

where $c : \mathbb{R}_+ \to \mathbb{R}^3$ and $\Gamma : \mathbb{R}_+ \to \mathbb{R}^{3\times 3}$.

2. Hence

$$\begin{pmatrix} d\eta(t) \\ d\mu(t) \\ dr(t) \end{pmatrix} = \begin{pmatrix} c_1(t) \\ c_2(t) \\ c_3(t) \end{pmatrix} dt + \begin{pmatrix} 0 & 0 & \Gamma_{13} \\ \Gamma_{21} & 0 & \Gamma_{23} \\ 0 & \Gamma_{32} & \Gamma_{33} \end{pmatrix} \begin{pmatrix} dX_1(t) \\ dX_2(t) \\ dX_3(t) \end{pmatrix}$$

3. The relevant SDEs are as

$$\begin{aligned} d\eta(t) &= c_1(t)dt + \Gamma_{13}dX_3(t), \\ d\mu(t) &= c_2(t)dt + \Gamma_{21}dX_1(t) + \Gamma_{23}dX_3(t), \\ dr(t) &= c_3(t)dt + \Gamma_{32}dX_2(t) + \Gamma_{33}dX_3(t). \end{aligned}$$

How to Derive Possible Transition Probabilities in Figure 1?

- Chapman-Kolmogorov equations are held

$$ {}_{t+h}p_x^{ij} = \sum_{k \in \mathcal{S}} {}_h p_{x+t}^{kj} \times {}_t\, p_x^{ik}, $$

where $i, j \in \mathcal{S}$ and $\mathcal{S} = \{0, 1, 2\}$.

- Hence, Kolmogorov forward differential equations are

$$
\begin{aligned}
\frac{d}{dt}({}_t p_x^{00}) &= {}_t p_x^{01} \mu_t^{10} - {}_t\, p_x^{00} \left[\mu_t^{01} + \mu_t^{02}\right] \\
\frac{d}{dt}({}_t p_x^{01}) &= {}_t p_x^{00} \mu_t^{01} - {}_t\, p_x^{01} \left[\mu_t^{10} + \mu_t^{12}\right] \\
\frac{d}{dt}({}_t p_x^{02}) &= {}_t p_x^{00} \mu_t^{02} + {}_t\, p_x^{01} \mu_t^{12} \\
\frac{d}{dt}({}_t p_x^{10}) &= {}_t p_x^{11} \mu_t^{10} - {}_t\, p_x^{10} \left[\mu_t^{01} + \mu_t^{02}\right] \\
\frac{d}{dt}({}_t p_x^{11}) &= {}_t p_x^{10} \mu_t^{01} - {}_t\, p_x^{11} \left[\mu_t^{10} + \mu_t^{12}\right] \\
\frac{d}{dt}({}_t p_x^{12}) &= {}_t p_x^{10} \mu_t^{02} + {}_t\, p_x^{11} \mu_t^{12}. \qquad (2)
\end{aligned}
$$

The Model Framework (Continued)

- The value of pension portfolio right after possible adjustments at the end of time t_i where $t_i = t + i$ for $i = 1, 2, \ldots$
- The value of pension portfolio at time t_i

$$P_{\text{buyout}}(t) = \frac{\sum_{t_i > t}^{t_M} E^{\mathbb{Q}}\left[e^{-\int_t^{t_i} r(s)ds} \max\{ PA(t_i^+) - (PA(t_i) - N(t_i).C), 0\} | \mathcal{I}_t \right]}{L(t)}$$

- Here,

$$\mathbf{PA(t_i^+)} = \max\{\mathbf{PA(t_i)} - \mathbf{N(t_i)} \times \mathbf{C}, \mathbf{L(t_i)}\}.$$

Financial Market Model

- The value of pension assets under $\mathbb{P}$

$$dA_k(t) = A_k(t)[\alpha_k \, dt + \sigma_k \, dW_k(t)] \tag{3}$$

- $\mathrm{Cov}(A_k(t), A_l(t)) = \rho_{kl}\sigma_k\sigma_l$.
- $\mathrm{Cov}(W_k(t), W_l(t)) = \rho_{kl} \, t, \;\; k = 1,2,3; \;\; l = 1,2,3; \;\; k \neq l$.
  1. ρ_{kl} : the correlation coefficient between assets k and l.
  2. α_k : the drift term for asset k where $k = 1,2,3$.
  3. σ_k : the instantaneous volatility

Financial Market Model (Continued)

- The value of the pension portfolio under $\mathbb{Q}$

$$d\log PA(t) = \left(r - \frac{1}{2}\sigma_W^2\right)dt + \sum_{k=1}^{3}\pi_k(t)\sigma_k dW_k^{\mathbb{Q}}(t) \quad (4)$$

1. r is the risk free rate (stochastic).
2. $\pi(t) = [\pi_1(t), \pi_2(t), \pi_3(t)]$ are the weights of the assets in the portfolio.
3. $\sigma_W^2 = \sum_{k,l=1}^{3}\pi_k(t)\pi_l(t)\rho_{kl}\sigma_k\sigma_l$.
4. Moreover,

$$dW_k^{\mathbb{P}}(t) = dW_k^{\mathbb{Q}}(t) - \left(\frac{\sum_1^3 \pi_k(t)\alpha_k - r}{\sum_1^3 \pi_k(t)\sigma_k}\right)dt.$$

Application

Assumptions

1. No annual contributions. No pension gap at inception.
2. No inflation risk and $\pi_{UK} = [0.10, 0.85, 0.05]$.
3. Initial state is state 0. Benefits are payable for state 0.
4. No differential mortality; $\mu_t^{02} = \mu_t^{12} = \mu(t)$.
5. $\mu_t^{10} = 0.1\mu_t^{01}$ and $\mu_t^{01} = \eta(t)$.
6. Exponential jump sizes with mean j.
7. $x = 65,\ N(0) = 100,\ C = 60000,\ dt = 1/252$

Parameter set for the application

OU process, X_1	$\bar{x} = 0,\ \gamma = -0.078282,\ \sigma_1 = 0.002271,\ x_1(0) = 0.01820$
CIR model, X_2	$\kappa = 0.2,\ \theta = 0.04,\ \sigma_2 = 0.1,\ x_2(0) = 0.04$
CP process, X_3	$\lambda = 0, 0.0001, 0.001,\ j = 0.1, 0.01$

Application (Continued)

- The plan funds are assumed to be invested in the **S&P** ***UK stock total return index*** $A_1(t)$**, the** ***Merrill Lynch UK Sterling corporate bond index*** $A_2(t)$ **and the** ***3-month UK cash total return index*** $A_3(t)$.

Simulation Approach

1. Generate transition rates for illness-death model based on the continuous Markov process and calculate the liability process,
2. Generate asset processes and pension asset portfolio,
3. Apply the main formula to determine buy-out premiums depending on Monte Carlo simulation under various sample paths.

Main Scenario

Definition

(Short rate and mortality rate dynamics under measure $\mathbb{Q}$)

$$\begin{aligned} d\mu(t) &= dX_1(t) + dX_3(t) \\ dr(t) &= dX_2(t) - dX_3(t) \\ d\eta(t) &= dX_3(t), \end{aligned} \tag{5}$$

by choosing Γ *as*

$$\begin{pmatrix} 0 & 0 & \Gamma_{13} \\ \Gamma_{21} & 0 & \Gamma_{23} \\ 0 & \Gamma_{32} & \Gamma_{33} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}. \tag{6}$$

$$\begin{aligned} dX_1(t) &= \gamma(\bar{x}_t - X_1(t))dt + \sigma_1 dW_1(t) \\ dX_2(t) &= \kappa(\theta - X_2(t))dt + \sigma_2 \sqrt{X_2(t)}dW_2(t) \\ dX_3(t) &= dJ(t), \end{aligned} \tag{7}$$

where $W_1(t)$ *and* $W_2(t)$ *are independent Wiener processes under measure* $\mathbb{Q}$.

Buy-out Premiums

Table 1: Actuarial fair prices of the buy-out deal under 10000 MC iterations based on different levels of λ and j

$P_{\text{buyout}}(0)$	$j = 0.1$	$j = 0.01$
$\lambda = 0$	0.317292	0.317292
$\lambda = 0.0001$	0.329216	0.317266
$\lambda = 0.001$	0.603008	0.317144

Confidence Interval for Buy-out Premiums

Definition

(Determination of 95% confidence interval for buy-out premiums $P_{buyout}(0)$)
The confidence interval for $P_{buyout}(0)$ is calculated as follows:

$$P_{buyout}(0)_{-} = \frac{\sum_{t_i=1}^{t_M} PV_{payoff}(t_i)_{-}}{L(0)}$$
$$P_{buyout}(0)^{-} = \frac{\sum_{t_i=1}^{t_M} PV_{payoff}(t_i)^{-}}{L(0)}, \tag{8}$$

where $P_{buyout}(0)_{-}$ and $P_{buyout}(0)^{-}$ show the lower and upper bounds of the confidence interval respectively. Here, $PV_{payoff}(t_i) = E^{\mathbb{Q}}\left[e^{-\int_0^{t_i} r(s)ds} H(t_i)\right]$.

$$PV_{payoff}(t_i)_{-} = \mu_{payoff}(t_i) - 1.96[\sigma_{payoff}(t_i)/\sqrt{N}]$$
$$PV_{payoff}(t_i)^{-} = \mu_{payoff}(t_i) + 1.96[\sigma_{payoff}(t_i)/\sqrt{N}]$$

Application (Continued)

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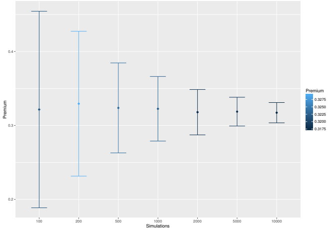
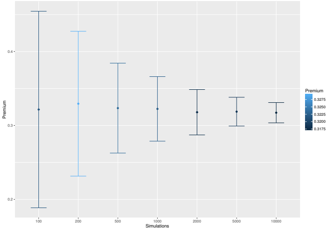


(a) $\lambda = 0$



(b) $\lambda = 0.0001, j = 0.01$

Figure 2: Calculated buy-out premiums with the corresponding confidence intervals when $\lambda = 0$ and $\lambda = 0.0001$

Summary

1. A different setting based on a continuous time Markov process to obtain buy-out premiums,
2. Affine term structure as a method for modeling dependent short rate and transition rates,
3. Analytical solution is the next step.

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Thanks for your attention.