Investigation of Dependency between Short Rate and Transition Rate on Pension Buy-outs

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The Outline of the Study

1 Preliminaries

2 The Model Framework
   - Liability Model
   - Financial Market Model
   - Application

3 Conclusion
The main purpose of the study

Introduce a general valuation expression for pension buy-outs

1. as an extension of Arık et al. (2017),
2. under the dependence assumption between short rate and transition rates,
3. in a continuous Markovian setting.
What is a Pension Buy-out?

- **Pensioners** contribute to the **DB pension scheme**, which provides a **benefit** to the pensioners.

- The **Buy-out insurer** is responsible for the **liability** and is paid by the contributions.

- The payment from the contributors to the insurer is denoted by \( P_{\text{buyout}} \).
The Model Framework

\[ P_{\text{buyout}}(t) = \sum_{t_i > t}^{t_M} \mathbb{E}^Q \left[ e^{-\int_t^{t_i} r(s)ds} \max\{PA(t_i^+) - (PA(t_i) - N(t_i) \cdot C), 0\} \mid \mathcal{I}_t \right] \]

- \( r(\cdot) \): the stochastic short rate for \( t \geq 0 \).
- \( L(\cdot) \) is the liability of the pension scheme as defined in Definition 1.
- \( a(\cdot) \) is the fair price of an immediate life annuity deal.
- \( B(\cdot) \) is the fair price of a zero coupon bond.
- \( PA(t_i^+) \) is the value of the pension portfolio at the beginning of time \( t_{i+1} \).
Liability Model

Definition

*(Liability Process)*  The pension liability process $L(t)$ at time $t$, $t \in [0, T]$, is determined as

$$L(t) = N(t) \times a(t, x),$$

where

- $N(t)$: the number of survivors in the model at time $t$ (determined according to the force of mortality rate dynamics).
- For the valuation of $a(t, x)$, see the next slides.
An Illness-Death Model for a DB Pension Scheme

Figure 1: The illness-death model for a hypothetical DB pension scheme
Correlated Transition and Short Rates

1. Suppose \( \eta(t) = \mu_{t}^{01}, \mu(t) = \mu_{t}^{2} \) and \( \mathbb{X} \) is a 3-dimensional affine process:

\[
(\eta(t), \mu(t), r(t))' = c(t) + \Gamma(t)\mathbb{X}(t),
\]

where \( c : \mathbb{R}_+ \rightarrow \mathbb{R}^3 \) and \( \Gamma : \mathbb{R}_+ \rightarrow \mathbb{R}^{3 \times 3} \).

2. Hence

\[
\begin{pmatrix}
  d\eta(t) \\
  d\mu(t) \\
  dr(t)
\end{pmatrix} = \begin{pmatrix}
  c_1(t) \\
  c_2(t) \\
  c_3(t)
\end{pmatrix} dt + \begin{pmatrix}
  0 & 0 & \Gamma_{13} \\
  \Gamma_{21} & 0 & \Gamma_{23} \\
  0 & \Gamma_{32} & \Gamma_{33}
\end{pmatrix} \begin{pmatrix}
  dX_1(t) \\
  dX_2(t) \\
  dX_3(t)
\end{pmatrix}
\]

3. The relevant SDEs are as

\[
\begin{align*}
  d\eta(t) &= c_1(t)dt + \Gamma_{13}dX_3(t), \\
  d\mu(t) &= c_2(t)dt + \Gamma_{21}dX_1(t) + \Gamma_{23}dX_3(t), \\
  dr(t) &= c_3(t)dt + \Gamma_{32}dX_2(t) + \Gamma_{33}dX_3(t).
\end{align*}
\]
How to Derive Possible Transition Probabilities in Figure 1?

- Chapman-Kolmogorov equations are held

\[ t+h p^j_i = \sum_{k \in S} h p^k_{i+t} \times t p^i_k, \]

where \( i, j \in S \) and \( S = \{0, 1, 2\} \).

- Hence, Kolmogorov forward differential equations are

\[
\begin{align*}
\frac{d}{dt} (t p_{x}^{00}) &= t p_{x}^{01} \mu_{t}^{10} - t p_{x}^{00} [\mu_{t}^{01} + \mu_{t}^{02}] \\
\frac{d}{dt} (t p_{x}^{01}) &= t p_{x}^{00} \mu_{t}^{01} - t p_{x}^{01} [\mu_{t}^{10} + \mu_{t}^{12}] \\
\frac{d}{dt} (t p_{x}^{02}) &= t p_{x}^{00} \mu_{t}^{02} + t p_{x}^{01} \mu_{t}^{12} \\
\frac{d}{dt} (t p_{x}^{10}) &= t p_{x}^{11} \mu_{t}^{10} - t p_{x}^{10} [\mu_{t}^{01} + \mu_{t}^{02}] \\
\frac{d}{dt} (t p_{x}^{11}) &= t p_{x}^{10} \mu_{t}^{01} - t p_{x}^{11} [\mu_{t}^{10} + \mu_{t}^{12}] \\
\frac{d}{dt} (t p_{x}^{12}) &= t p_{x}^{10} \mu_{t}^{02} + t p_{x}^{11} \mu_{t}^{12}.
\end{align*}
\]
The value of pension portfolio right after possible adjustments at the end of time \( t_i \) where \( t_i = t + i \) for \( i = 1, 2, \ldots \)

The value of pension portfolio at time \( t_i \)

\[
P_{\text{buyout}}(t) = \frac{\sum_{t_{i:j} > t}^{t_M} E^Q \left[ e^{-\int_t^{t_i} r(s) ds} \max\{\text{PA}(t_{i:j}^+) - (\text{PA}(t_{i:j}) - N(t_{i:j}) \times C), 0\} \right]}{L(t)}
\]

Here,

\[
\text{PA}(t_{i:j}^+) = \max\{\text{PA}(t_{i:j}) - N(t_{i:j}) \times C, L(t_{i:j})\}.
\]
The value of pension assets under $\mathbb{P}$

$$dA_k(t) = A_k(t)[\alpha_k \, dt + \sigma_k \, dW_k(t)]$$  \hspace{1cm} (3)

- $\text{Cov}(A_k(t), A_l(t)) = \rho_{kl} \sigma_k \sigma_l$.
- $\text{Cov}(W_k(t), W_l(t)) = \rho_{kl} \, t$, $k = 1, 2, 3$; $l = 1, 2, 3$; $k \neq l$.

1. $\rho_{kl}$: the correlation coefficient between assets $k$ and $l$.
2. $\alpha_k$: the drift term for asset $k$ where $k = 1, 2, 3$.
3. $\sigma_k$: the instantaneous volatility
The value of the pension portfolio under $\mathbb{Q}$

$$d \log PA(t) = \left( r - \frac{1}{2} \sigma^2_W \right) dt + \sum_{k=1}^{3} \pi_k(t) \sigma_k dW_k^Q(t)$$  \hspace{1cm} (4)

1. $r$ is the risk free rate (stochastic).
2. $\pi(t) = [\pi_1(t), \pi_2(t), \pi_3(t)]$ are the weights of the assets in the portfolio.
3. $\sigma^2_W = \sum_{k,l=1}^{3} \pi_k(t) \pi_l(t) \rho_{kl} \sigma_k \sigma_l$.
4. Moreover,

$$dW^P_k(t) = dW^Q_k(t) - \left( \frac{\sum_{1}^{3} \pi_k(t) \alpha_k - r}{\sum_{1}^{3} \pi_k(t) \sigma_k} \right) dt.$$
Application

Assumptions

1. No annual contributions. No pension gap at inception.
2. No inflation risk and \( \pi_{UK} = [0.10, 0.85, 0.05] \).
3. Initial state is state 0. Benefits are payable for state 0.
4. No differential mortality; \( \mu_0^2 = \mu_1^2 = \mu(t) \).
5. \( \mu_0^{10} = 0.1 \mu_0^{01} \) and \( \mu_1^{01} = \eta(t) \).
6. Exponential jump sizes with mean \( j \).
7. \( x = 65, N(0) = 100, C = 60000, dt = 1/252 \)

Parameter set for the application

<table>
<thead>
<tr>
<th>Process</th>
<th>Parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>OU process, ( X_1 )</td>
<td>( \bar{x} = 0, \gamma = -0.078282, \sigma_1 = 0.002271, x_1(0) = 0.01820 )</td>
</tr>
<tr>
<td>CIR model, ( X_2 )</td>
<td>( \kappa = 0.2, \theta = 0.04, \sigma_2 = 0.1, x_2(0) = 0.04 )</td>
</tr>
<tr>
<td>CP process, ( X_3 )</td>
<td>( \lambda = 0, 0.0001, 0.001, j = 0.1, 0.01 )</td>
</tr>
</tbody>
</table>
Application (Continued)

- The plan funds are assumed to be invested in the S&P UK stock total return index $A_1(t)$, the Merrill Lynch UK Sterling corporate bond index $A_2(t)$ and the 3-month UK cash total return index $A_3(t)$.

**Simulation Approach**

1. Generate transition rates for illness-death model based on the continuous Markov process and calculate the liability process,
2. Generate asset processes and pension asset portfolio,
3. Apply the main formula to determine buy-out premiums depending on Monte Carlo simulation under various sample paths.
Main Scenario

Definition

*Short rate and mortality rate dynamics under measure $\mathbb{Q}$*

\[ 
\begin{align*}
\frac{d\mu(t)}{dt} &= dX_1(t) + dX_3(t) \\
\frac{dr(t)}{dt} &= dX_2(t) - dX_3(t) \\
\frac{d\eta(t)}{dt} &= dX_3(t),
\end{align*} \]

by choosing $\Gamma$ as

\[ 
\begin{pmatrix}
0 & 0 & \Gamma_{13} \\
\Gamma_{21} & 0 & \Gamma_{23} \\
0 & \Gamma_{32} & \Gamma_{33}
\end{pmatrix} = \begin{pmatrix}
0 & 0 & 1 \\
1 & 0 & 1 \\
0 & 1 & -1
\end{pmatrix}. 
\]

\[ 
\begin{align*}
\frac{dX_1(t)}{dt} &= \gamma(\bar{x}_t - X_1(t))dt + \sigma_1 dW_1(t) \\
\frac{dX_2(t)}{dt} &= \kappa(\theta - X_2(t))dt + \sigma_2 \sqrt{X_2(t)} dW_2(t) \\
\frac{dX_3(t)}{dt} &= dJ(t),
\end{align*} \]

where $W_1(t)$ and $W_2(t)$ are independent Wiener processes under measure $\mathbb{Q}$. 
Table 1: Actuarial fair prices of the buy-out deal under 10000 MC iterations based on different levels of $\lambda$ and $j$

<table>
<thead>
<tr>
<th>$P_{\text{buyout}}(0)$</th>
<th>$j = 0.1$</th>
<th>$j = 0.01$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda = 0$</td>
<td>0.317292</td>
<td>0.317292</td>
</tr>
<tr>
<td>$\lambda = 0.0001$</td>
<td>0.329216</td>
<td>0.317266</td>
</tr>
<tr>
<td>$\lambda = 0.001$</td>
<td>0.603008</td>
<td>0.317144</td>
</tr>
</tbody>
</table>
Confidence Interval for Buy-out Premiums

**Definition**

*Definition of 95% confidence interval for buy-out premiums* \( P_{\text{buyout}}(0) \)

The confidence interval for \( P_{\text{buyout}}(0) \) is calculated as follows:

\[
P_{\text{buyout}}(0)^- = \frac{\sum_{t_i=1}^{t_M} PV_{\text{payoff}}(t_i)^-}{L(0)}
\]

\[
P_{\text{buyout}}(0)^+ = \frac{\sum_{t_i=1}^{t_M} PV_{\text{payoff}}(t_i)^+}{L(0)},
\]

where \( P_{\text{buyout}}(0)^- \) and \( P_{\text{buyout}}(0)^+ \) show the lower and upper bounds of the confidence interval respectively. Here, \( PV_{\text{payoff}}(t_i) = E^Q \left[ e^{-\int_0^{t_i} r(s)ds} H(t_i) \right] \).

\[
PV_{\text{payoff}}(t_i)^- = \mu_{\text{payoff}}(t_i) - 1.96[\sigma_{\text{payoff}}(t_i)/\sqrt{N}]
\]

\[
PV_{\text{payoff}}(t_i)^+ = \mu_{\text{payoff}}(t_i) + 1.96[\sigma_{\text{payoff}}(t_i)/\sqrt{N}]
\]
Application (Continued)

Figure 2: Calculated buy-out premiums with the corresponding confidence intervals when $\lambda = 0$ and $\lambda = 0.0001$
Summary

1. A different setting based on a continuous time Markov process to obtain buy-out premiums,
2. Affine term structure as a method for modeling dependent short rate and transition rates,
3. Analytical solution is the next step.
References

- Buchardt, K., 2014, Dependent Interest and Transition Rates in Life Insurance.
- Deshmukh, S., 2012, Multiple Decrement Models in Insurance.
Thanks for your attention.