The Life Cycle Model with Recursive Utility: Defined benefit vs defined contribution.

Knut K. Aase *

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Abstract

We analyze optimal consumption and portfolio choice when the agent takes the market as given, with applications to pension insurance. First, we recall the results for the conventional additive and separable expected utility model. We demonstrate that this model is not viable in a temporal context. As a consequence, when confronted with real data this theory meets with several conflicting implications. Second, we consider recursive utility, which in contrast has an axiomatic underpinning in a temporal context. All the anomalies of the expected utility approach that we are aware of, disappear with recursive utility. This we find striking, as it calls for a shift in paradigm. The alternative theory explains why people prefer stable pension plans, like defined benefit, to a mere mutual fund, it removes anomalies in the conventional, optimal portfolio choice theory when confronted with market data, and it solves the celebrated equity premium puzzle. Since the recursive model fits market data much more convincingly than the conventional model, this leaves more credibility to the former representation, and more weight to recommendations based on it.

KEYWORDS: The life cycle model, recursive utility, consumption smoothing, consumption puzzles, defined benefit, defined contribution
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*The Norwegian School of Economics, 5045 Bergen.

1 Introduction

In the the standard life cycle model with additive and separable utility one can not separate risk aversion from the intertemporal elasticity of substitution in consumption. Since these are different aspects of an individual's preferences this is clearly a weakness with the standard model.

We consider the consumer in the life cycle model having recursive utility, in a continuous-time setting. Here we focus on optimal consumption, and its consequences for optimal pension insurance. In the paper Aase (2016b) it is shown that with recursive utility, instead of the standard expected utility in a dynamic setting, both market risk premiums and the real equilibrium interest rate explain market and consumption data over the past 100 years or so, with reasonable parameters of the utility function. This has consequences for the life cycle model, and for all of economic dynamics for that matter, where now optimal consumption finally becomes consistent with aggregate market data.

The recursive utility customer finds it optimal to smooth market shocks to a larger extent than the conventional expected utility model predicts. One question is then how this can be accomplished in the real world. This is of great importance when analyzing pensions and life insurance contracts, where insuring consumers against adverse shocks in the market ought to be a main issue. After the financial crisis in 2008, insurers are inclined to pass all, or most of the financial risk to its customers, presenting them with mainly defined contribution, or unit linked pension plans. The lessons from the present paper for the insurance industry is clear: To provide the kind of consumption smoothing that consumers of the last century seem to prefer, which clearly points in the direction of defined benefit rather than defined contribution pension plans.

The paper is organized as follows. In Section 2 we present the model of the financial market, recursive utility is introduced in Section 3, in Section 4 we discuss optimal consumption. In Section 5 we discuss pensions, Section 6 treats the associated optimal portfolio choice, and Section 7 concludes.

2 The Financial Market

We consider a consumer/insurance customer who has access to a securities market, as well as a credit market and pension and life insurance contracts. This section draws form the lessons learnt from the literature on derivative securities following the analysis of options in the paper by Black and Scholes (1973).

The securities market can be described by the vector ν_t of expected returns of N given risky securities in excess of the risk-less instantaneous return r_t, and σ_t is an $N \times N$ matrix of diffusion coefficients of the risky asset prices, normalized by the asset process, so that $\sigma_t \sigma_t'$ is the instantaneous covariance matrix for asset returns. Both ν_t and σ_t are assumed to be progressively measurable stochastic processes. Here N is also the dimension of the Brownian motion B.

We assume that the cumulative return process R_t^n is an an ergodic, stochastic process for each n, where $dX_t^n = X_t^n dR_t^n$ for $n = 1, 2, \ldots, N$, and X_t^n is the cum dividend price process of the nth risky asset.

Underlying is a probability space $(\Omega, \mathcal{F}, P)$ and an increasing information filtration $\mathcal{F}_t$ generated by the N-dimensional Brownian motion, and satisfying the usual conditions. Each price process $X_t^{(n)}$ is a continuous stochastic process, and we suppose that $\sigma^{(0)} = 0$, so that $r_t = \mu_0(t)$ is the risk-free interest rate, also a stochastic process. T is the finite horizon of the economy. The state price deflator $\pi(t)$ is given by

$$\pi_t = \xi_t e^{-\int_0^t r_s\, ds}, \tag{1}$$

where the 'density' process ξ has the representation

$$\xi_t = \exp\Big(- \int_0^t \eta_s' \cdot dB_s - \frac{1}{2} \int_0^t \eta_s' \cdot \eta_s ds \Big). \tag{2}$$

Here $\eta(t)$ is the market-price-of-risk for the discounted price process $X_t e^{-\int_0^t r_s ds}$, defined by

$$\sigma(\omega, t)\eta(\omega, t) = \nu(\omega, t), \quad (\omega, t) \in \Omega \times [0, T], \tag{3}$$

where the nth component of ν_t equals $(\mu_n(t) - r_t)$, the excess rate of return on security n, $n = 1, 2, \cdots, N$. From Ito's lemma it follows from (2) that

$$d\xi_t = -\xi_t \; \eta_t' \cdot dB_t, \tag{4}$$

and, from (1) it follows that

$$d\pi_t = -r_t \pi_t dt - \pi_t \eta_t' dB_t. \tag{5}$$

The density ξ_t is assumed to be a martingale (Novikov's condition suffices). We assume there is a bounded market price of risk. Under these assumptions there exists an equivalent martingale measure for X and there is no (well-behaved) arbitrage in this market (see e.g., Duffie (2001)).

The agent is represented by an endowment process e (income) and a utility function $U : L_+ \rightarrow \mathbb{R}$, where

$$L = \{c : c_t \text{ is progressively measurable, } \mathcal{F}_t\text{-adapted and } E(\int_0^T c_t^2 \, dt) < \infty\}.$$

L_+, the positive cone of L, is the set of consumption rate processes. The specific form of the function U is specified in the next section.

For a price π_t of the consumption good, the problem is to solve

$$\sup_{c \in L_+} U(c), \tag{6}$$

subject to the budget constraint

$$E\Big\{ \int_0^\tau \pi_t c_t \, dt \Big\} \leq E\Big\{ \int_0^\tau \pi_t e_t \, dt \Big\} := w. \tag{7}$$

The quantity π_t is also known as the "state price deflator", or the Arrow-Debreu prices in units of probability. State prices reflect what the representative consumer is willing to pay for an extra unit of consumption; in particular is π_t high in "times of crises" and low in "good times".

The present situation is known as a *temporal* problem of choice. In such a setting it is far from clear that the time additive and separable form expected utility is the natural representation of preferences. For example, derived preferences do not satisfy the substitution axiom (see e.g., Mossin (1969), Kreps (1988)). This is the axiom that gives additivity in probability of the utility function. If this property does not hold for any time t, it certainly does not help to add the representation across time. Also, the resulting model does not explain aggregate market data (e.g., the equity premium puzzle).

When there is no market uncertainty, i.e., $\xi_t = 1$ for all $t \in [0, T]$, the Ramsey (1928) model applies. This model does not encounter this problem with the axioms, but has of course problems with realism [1].

The consumer's problem is, for each initial wealth level w, to solve

$$\sup_{(c,\varphi)} U(c) \tag{8}$$

subject to an intertemporal budget constraint

$$dW_t = \big(W_t(\varphi_t' \cdot \nu_t + r_t) - c_t\big)dt + W_t \varphi_t' \cdot \sigma_t dB_t, \qquad W_0 = w. \tag{9}$$

Here $\varphi_t' = (\varphi_t^{(1)}, \varphi_t^{(2)}, \cdots, \varphi_t^{(N)})$ are the fractions of total wealth held in the risky securities.

A more detailed description of the steps leading to the problem (8) with the dynamic constraint (9) can be found in Duffie (2001), Ch 9, p 206. See also Aase (2015b), Section 2. That this problem is equivalent to problem (6)-(7) when markets are complete, is shown in Pliska (1986) and Cox and Huang (1989), among others.

[1] Also, the *timeless* problem with two time points only, uncertainty only on the last time and no consumption choice at the first, does not have this problem with the axioms.

3 Recursive utility

3.1 Introduction

We now introduce recursive utility. Here we use the framework established by Duffie and Epstein (1992a-b) and Duffie and Skiadas (1994) which elaborate the foundational work by Kreps and Porteus (1978) of recursive utility in dynamic models. Recursive utility leads to the separation of risk aversion from the elasticity of intertemporal substitution in consumption, within a time-consistent model framework.

The recursive utility $U : L \rightarrow \mathbb{R}$ is defined by two primitive functions: $f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ and $A : \mathbb{R} \rightarrow \mathbb{R}$. The function $f(c_t, V_t)$ corresponds to a felicity index, and A corresponds to a measure of absolute risk aversion of the Arrow-Pratt type for the agent. In addition to current consumption c_t, the function f also depends on utility V_t at time t, a stochastic process with volatility $\tilde{\sigma}_V(t) := Z_t$ at each time t.

The utility process V for a given consumption process c, satisfying $V_T = 0$, is given by the representation

$$V_t = E_t\Big\{ \int_t^T \big(f(c_s, V_s) - \frac{1}{2}A(V_s)\,\tilde{\sigma}_V(s)'\tilde{\sigma}_V(s)\big)ds\Big\}, \quad t \in [0, T] \tag{10}$$

If, for each consumption process c_t, there is a well-defined utility process V, the stochastic differential utility U is defined by $U(c) = V_0$, the initial utility. The pair (f, A) generating V is called an aggregator.

The utility function U is monotonic and risk averse if $A(\cdot) \geq 0$ and f is jointly concave and increasing in consumption.

As for the last term in (10), recall the Arrow-Prat approximation to the certainty equivalent of a mean zero risk X. It is $-\frac{1}{2}A(\cdot)\sigma^2$, where σ^2 is the variance of X, and $A(\cdot)$ is the absolute risk aversion function.

In the discrete time world the starting point for recursive utility is that future utility at time t is given by $V_t = g(c_t, m(V_{t+1}))$ for some function $g : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$, where m is a certainty equivalent at time t (see e.g, Epstein and Zin (1989)). If h is a von Neumann-Morgenstern index, then $m(V) = h^{-1}(E[h(V)])$. The passage to the continuous-time version in (10) is explained in Duffie and Epstein (1992b).

The preference ordering represented by recursive utility is assumed to satisfy: Dynamic consistency, in the sense of Johnsen and Donaldson (1985); Independence of past consumption; and State independence of time preference (see Skiadas (2009a)).

Unlike expected utility theory in a *timeless* situation, i.e., when consumption only takes place at the end, in a *temporal* setting where the agent consumes in every period, derived preferences do not satisfy the substitution axiom (e.g., Mossin (1969), Kreps (1988)). Thus additive Eu-theory in a dynamic context, i.e., in situations where a financial market is utilized by the agents to smooth consumption across time and states of the world, has no axiomatic underpinning, unlike recursive utility (Kreps and Porteus (1978), Chew and Epstein (1991)). It is notable that one of the four central axioms in the latter theory, *recursivity*, is essentially identical to the notion of consistency the sense of Johnsen and Donaldson (1985). Kreps and Porteus (1978) call this *temporal consistency*.

3.2 The specification we work with

Stochastic differential utility disentangles intertemporal substitution from risk aversion: In the case of deterministic consumption, $\tilde{\sigma}_V(t) = 0$ for all t. Hence risk aversion A is then irrelevant, since it multiplies a zero variance. Thus certainty preferences, including the willingness to substitute consumption across time, are determined by f alone. Only risk attitudes are affected by changes in A for f fixed. In particular, if

$$\tilde{A}(\cdot) \geq A(\cdot)$$

where U and $\tilde{U}$ are utility functions corresponding to (f, A) and $(f, \tilde{A})$ respectively, then $\tilde{U}$ is more risk averse than U in the sense that any consumption process c rejected by U in favor of some deterministic process $\bar{c}$ would also be rejected by $\tilde{U}$.

We work with the Kreps-Porteus utility, with aggregator with the CES specification

$$f(c, v) = \frac{\delta}{1-\rho} \frac{c^{(1-\rho)} - v^{(1-\rho)}}{v^{-\rho}} \quad \text{and} \quad A(v) = \frac{\gamma}{v}. \tag{11}$$

The parameter $\delta \geq 0$ is the agent's impatience rate, $\rho \geq 0$, $\rho \neq 1$ is the time preference and $\gamma \geq 0$, $\gamma \neq 1$, is the relative risk aversion. The parameter $\psi = 1/\rho$ is the elasticity of intertemporal substitution in consumption, referred to as the EIS-parameter. The higher the value of the parameter ρ is, the more aversion the agent has towards consumption fluctuations *across time* in a deterministic world. The higher the value of γ, the more aversion the agent has to consumption fluctuations, due to the different states of the world that can occur. Clearly these two properties of an individual's preferences are different. In the conventional model $\gamma = \rho$. The discrete-time analog of (11) is the one given by Epstein and Zin (1989-91).

Recursive utility has an *ordinally equivalent* specification. When the aggregator (f, A) is given corresponding to the utility function U, there exists a strictly increasing and smooth function $\varphi(\cdot)$ such that the ordinally equivalent $U_1 = \varphi \circ U$ has the aggregator (f_1, A_1) where

$$f_1(c,v) = ((1-\gamma)v)^{-\frac{\gamma}{1-\gamma}} f(c, ((1-\gamma)v)^{\frac{1}{1-\gamma}}), \qquad A_1 = 0.$$

The connection is

$$U_1 = \frac{1}{1-\gamma} U^{1-\gamma}.$$

This is the specification Duffie and Epstein (1991) work with, where f_1 has the CES-form

$$f_1(c,v) = \frac{\delta}{1-\rho} \frac{c^{(1-\rho)} - ((1-\gamma)v)^{\frac{1-\rho}{1-\gamma}}}{((1-\gamma)v)^{\frac{\gamma-\rho}{1-\gamma}}}, \qquad A_1(v) = 0. \tag{12}$$

Is is emphasized in the above reference that the reduction to a normalized aggregator $(f_1, 0)$ does not mean that intertemporal utility is risk neutral, or that this representation has lost the ability to separate risk aversion from substitution. The corresponding utility U_1 retains the essential features, namely that of partly disentangling intertemporal elasticity of substitution from risk aversion. However, we can not claim any more that f_1 alone determines the willingness to substitute consumption across time.

The version (12) was analyzed in Aase (2015b) in the life cycle model, and by Duffie and Epstein (1992a,b) in the rational expectations equilibrium model. Similarly Schroder and Skiadas (1999) analyzed various versions of recursive utility with $A = 0$ related to the life cycle model. In the present paper we analyze the version (11) directly, using the stochastic maximum principle.

As can be seen, this version explains the separation of risk aversion from time substitution, but is also the version which is the most demanding to work with. The method we use, the stochastic maximum principle, allows for state dependence and a non-Markovian structure of the economy. This is more difficult to handle using dynamic programming.

3.3 The first order conditions

In the following we find the solution to the consumer's problem. For any of the versions $i = 1, 2$ formulated in the previous section, the problem is to solve

$$\sup_{c \in L_+} U(c)$$

subject to the budget constraint

$$E\Big\{ \int_0^T c_t \pi_t dt \Big\} \leq E\Big\{ \int_0^T e_t \pi_t dt \Big\}.$$

Here $V_t = V_t^{\tilde{c}}$ and $Z(t) := \tilde{\sigma}_V(t)$ is the solution of the backward stochastic differential equation (BSDE)

$$\begin{cases} dV_t = -\tilde{f}(t, c_t, V_t, \tilde{\sigma}_V(t))\, dt + Z(t)\, dB_t \\ V_T = 0, \end{cases} \tag{13}$$

where

$$\tilde{f}(t, c_t, V_t, Z(t)) = f(c_t, V_t) - \frac{1}{2} A(V_t)\, Z(t)' Z(t).$$

Notice that (13) covers both the versions (11) and (12).

Existence and uniqueness of solutions of the BSDE (13) is proven in Duffie and Lions (1992) for the Epstein-Zin specification.

For $\alpha > 0$ define the Lagrangian

$$\mathcal{L}(c; \alpha) = U(c) - \alpha\, E\Big(\int_0^T \pi_t (c_t - e_t) dt \Big).$$

Important is that the volatility $Z(t) := \tilde{\sigma}_V(t)$ is exogenously given as part of the preferences. Below we shall se that there is a connection between this volatility and two other volatilities, namely that of the agent's wealth and optimal consumption.

Because of the generality of the problem, we utilize the stochastic maximum principle (see Pontryagin (1972), Bismut (1978), Kushner (1972), Bensoussan (1983), Peng (1990), and Øksendal and Sulem (2013)) : We are then given a system of forward/backward stochastic differential equations (FBSDE) consisting of the simple FSDE $dX(t) = 0; X(0) = 0$ and the BSDE (13)[2]. The objective function is

$$\mathcal{L}(c; \alpha) = V_0^c - \alpha\, E\Big(\int_0^T \pi_t (c_t - e_t) dt \Big) \tag{14}$$

where α is the Lagrange multiplier. The Hamiltonian for this problem is

$$H(t, c, v, z, y) = y_t\, \tilde{f}(t, c_t, v_t, z_t) - \alpha\, \pi_t (c_t - e_t) \tag{15}$$

where y_t is the adjoint variable. It is given by

$$\begin{cases} dY_t = Y(t) \big(\frac{\partial \tilde{f}}{\partial v}(t, c_t, V_t, Z(t))\, dt + \frac{\partial \tilde{f}}{\partial z}(t, c_t, V_t, Z(t))\, dB_t \big) \\ Y_0 = 1. \end{cases} \tag{16}$$

[2]Here the process X is used in the general formulation, and must be set equal to zero in the application at hand; it is not the return on a risky asset.

where we use the notation $Z(t) = \tilde{\sigma}_V(t)$, and z as the generic variable. If c^* is optimal we therefore have

$$Y_t = \exp\Big(\int_0^t \{ \frac{\partial \tilde{f}}{\partial v}(s, c_s^*, V_s, Z(s)) - \frac{1}{2}\big(\frac{\partial \tilde{f}}{\partial z}(s, c_s^*, V_s, Z(s))\big)^2 \} \, ds \\ + \int_0^t \frac{\partial \tilde{f}}{\partial z}(s, c_s^*, V_s, Z(s)) \, dB(s) \Big) \quad a.s. \tag{17}$$

Maximizing the Hamiltonian with respect to c gives the first order equation

$$y \, \frac{\partial \tilde{f}}{\partial \tilde{c}}(t, c^*, v, z) - \alpha \, \pi = 0$$

or

$$\alpha \, \pi_t = Y(t) \frac{\partial \tilde{f}}{\partial \tilde{c}}(t, c_t^*, V(t), Z(t)) \quad \text{a.s. for all } t \in [0, T]. \tag{18}$$

Notice that the state price deflator π_t at time t depends, through the adjoint variable Y_t, an unbounded variation process, on the entire, optimal paths $(c_s, V_s, Z_s))$ for $0 \leq s \leq t$. (One of the the strengths of the stochastic maximum principle is that the Hamiltonian is allowed to depend on the state.)

Sufficient conditions for the existence of a unique solution to the stochastic maximum principle are the same as those giving existence and uniqueness of a solution to the BSDE (13)

When $\gamma = \rho$ then $Y_t = e^{-\delta t}$ for the aggregator (**??**) of the conventional model, so the state price deflator is a Markov process, and dynamic programming is appropriate. If $\gamma \neq \rho$ on the other hand, we use the stochastic maximum principle in the continuous time model of this paper.

3.4 The derivation of the optimal consumption

Here we present the analysis for the basic version of recursive utility (11). From the above we have the following first order conditions for this version

$$\alpha \pi_t = Y_t \, f_c(c_t^*, V_t), \tag{19}$$

since $\tilde{f}_c = f_c$ for the version (11). Since $f_c(c, v) = \delta c^{-\rho} v^{\rho}$, it follows that the optimal consumption can be written

$$c_t^* = \Big(\frac{\alpha \pi_t}{\delta \, Y_t}\Big)^{-\frac{1}{\rho}} V_t. \tag{20}$$

Using the notation $Z(t) = V_t \sigma_V(t)$, the dynamics of the stochastic processes involved are as follows.

$$dV_t = \Big(-\frac{\delta}{1-\rho}\frac{(c_t^*)^{1-\rho} - V_t^{1-\rho}}{V_t^{-\rho}} + \frac{1}{2}\gamma V_t \sigma_V'(t)\sigma_V(t)\Big) dt + V_t\sigma_V(t)\, dB_t, \quad (21)$$

for $0 \leq t \leq T$, where $V_T = 0$. This is the backward stochastic differential equation. The dynamics of the adjoint variable is

$$dY_t = Y_t\Big(\Big\{ -\frac{\delta}{1-\rho}(1-\rho(c_t^*)^{1-\rho}V_t^{\rho-1}) + \frac{1}{2}\gamma\sigma_V'(t)\sigma_V(t)\Big\}\, dt \\ - \gamma\sigma_V(t)\, dB_t\Big), \quad (22)$$

for $0 \leq t \leq T$, where $Y_0 = 1$. Here we have used

$$f_v(c,v) := \frac{\partial f(c,v)}{\partial v} = -\frac{\delta}{1-\rho}(1-\rho c^{1-\rho}v^{\rho-1}).$$

Equation (22) is the adjoint equation. Finally the dynamics of the state price deflator is

$$d\pi_t = -r_t\pi_t\, dt - \pi_t\eta_t\, dB_t, \quad (23)$$

where η_t is the market-price-of-risk.

Based on this we can derive the dynamics of the optimal consumption. For this we need the following partial derivatives:

$$\frac{\partial c(\alpha\pi_t, V_t, Y_t)}{\partial \pi} = -\frac{1}{\rho}\Big(\frac{c_t^*}{\pi_t}\Big), \quad \frac{\partial c(\alpha\pi_t, V_t, Y_t)}{\partial v} = \frac{c_t^*}{V_t},$$

$$\frac{\partial c(\alpha\pi_t, V_t, Y_t)}{\partial y} = \frac{1}{\rho}\Big(\frac{c_t^*}{Y_t}\Big), \quad \frac{\partial^2 c(\alpha\pi_t, V_t, Y_t)}{\partial \pi^2} = \frac{1}{\rho}\Big(\frac{1}{\rho}+1\Big)\Big(\frac{c_t^*}{\pi_t^2}\Big),$$

$$\frac{\partial^2 c(\alpha\pi_t, V_t, Y_t)}{\partial y^2} = \frac{1}{\rho}\Big(\frac{1}{\rho}-1\Big)\Big(\frac{c_t^*}{Y_t^2}\Big), \quad \frac{\partial^2 c(\alpha\pi_t, V_t, Y_t)}{\partial v^2} = 0,$$

$$\frac{\partial^2 c(\alpha\pi_t, V_t, Y_t)}{\partial \pi\partial v} = -\frac{1}{\rho}\Big(\frac{c_t^*}{\pi_t V_t}\Big), \quad \frac{\partial^2 c(\alpha\pi_t, V_t, Y_t)}{\partial \pi\partial y} = -\frac{1}{\rho^2}\Big(\frac{c_t^*}{\pi_t Y_t}\Big),$$

and

$$\frac{\partial^2 c(\alpha\pi_t, V_t, Y_t)}{\partial v\partial y} = \frac{1}{\rho}\frac{c_t^*}{Y_t V_t}.$$

By the multidimensional version of Ito's lemma we can now calculate the dynamics of the optimal consumption as follows:

$$dc_t^* = \frac{\partial c}{\partial \pi}d\pi_t + \frac{\partial c}{\partial v}dV_t + \frac{\partial c}{\partial y}dY_t + \frac{1}{2}\frac{\partial^2 c}{\partial \pi^2}d\pi_t^2 + \frac{1}{2}\frac{\partial^2 c}{\partial v^2}dV_t^2 + \frac{1}{2}\frac{\partial^2 c}{\partial y^2}dY_t^2 \\ + \frac{\partial^2 c}{\partial \pi\partial v}d\pi_t dV_t + \frac{\partial^2 c}{\partial \pi\partial y}d\pi_t dY_t + \frac{\partial^2 c}{\partial v\partial y}dV_t dY_t. \quad (24)$$

The stochastic representation for the consumption growth rate is given by

$$\frac{dc_t^*}{c_t^*} = \mu_c(t)\, dt + \sigma_c(t)\, dB_t. \tag{25}$$

We now use the representations for the processes π_t, V_t and Y_t given above. After a fair amount of routine calculations, the result is

$$\mu_c(t) = \frac{1}{\rho}\,(r_t - \delta) + \frac{1}{2}\,\frac{1}{\rho}\,(1 + \frac{1}{\rho})\,\eta_t'\eta_t - \frac{(\gamma - \rho)}{\rho^2}\,\eta_t'\,\sigma_V(t) \\ + \frac{1}{2}\frac{(\gamma - \rho)\gamma(1 - \rho)}{\rho^2}\sigma_V'(t)\sigma_V(t) \tag{26}$$

and

$$\sigma_c(t) = \frac{1}{\rho}\,\Big(\eta_t + (\rho - \gamma)\sigma_V(t)\Big). \tag{27}$$

Here $\sigma_V(t)$ and V_t exist as a solution to the system of the backward stochastic differential equation for V.

When $\rho = \gamma$ (or $\gamma = 1/\psi$), the optimal consumption dynamics for the conventional model results.

By the Doleans-Dade formula it follows that

$$c_t^* = c_0\, e^{\int_0^t (\mu_c(s) - \frac{1}{2}\sigma_c(s)'\sigma_c(s))ds + \int_0^t \sigma_c(s) dB_s} \tag{28}$$

where $\mu_c(t)$ and $\sigma_c(t)$ are as determined above. This gives a characterization of the optimal consumption in terms of the primitives of the model.

From (20) and the fact that the recursive utility function we work with is homogeneous of degree one, there is a one-to-one correspondence between c_0 and α. Given a suitable integrability condition, for each $c_0 > 0$ there corresponds a unique α_0 that satisfies the budget constraint with equality. Under these assumptions we have, by The Saddle Point Theorem, a complete characterization of the optimal consumption in terms of the primitives of the model.

4 Some properties of the optimal consumption

Since the agent takes the market as given, it is of interest to study how shocks to the state price π affect the optimal consumption. Towards this end it is convenient to rewrite the expression for the optimal consumption in terms of the state price. Using the dynamics of π in (5) we can write (28) as follows

$$c_t^* = c_0 \pi_t^{-\frac{1}{\rho}} e^{\int_0^t (-\frac{\delta}{\rho} + \frac{1}{2\rho}(\gamma - \rho)(1-\gamma)\sigma_V'(s)\sigma_V(s))ds + \frac{1}{\rho}(\rho - \gamma)\int_0^t \sigma_V(s)\, dB_s}. \tag{29}$$

In terms of the density process ξ the expression is

$$c_t^* = c_0 \xi_t^{-\frac{1}{\rho}} e^{\int_0^t (\frac{1}{\rho}(r_s - \delta) + \frac{1}{2\rho}(\gamma-\rho)(1-\gamma)\sigma_V'(s)\sigma_V(s))ds + \frac{1}{\rho}(\rho-\gamma)\int_0^t \sigma_V(s)\, dB_s}, \qquad (30)$$

where

$$\pi_t = e^{-\int_0^t r_s ds} \xi_t = e^{-\int_0^t r_s ds} e^{-\int_0^t \eta_s dB_s - \frac{1}{2}\int_0^t \eta_s' \eta_s ds}.$$

It is shown in Aase (2015b) that the same consumption dynamics as given in (25)-(30) result for the ordinally equivalent specification (12). Thus both asset pricing implications and optimal consumption are unaffected by a monotone transformation of recursive utility, satisfying some regularity conditions. For example is the expected optimal consumption at time t as of time zero given by

$$E(c_t^*) = c_0 E\{e^{\int_0^t (\mu_c(s) ds}\}. \qquad (31)$$

When $\rho = \gamma$, or $\gamma = 1/\psi$, the optimal consumption dynamics for the conventional model results. As a direct comparison with (29) and (30) the conventional model gives

$$c_t^* = c_0 \pi_t^{-\frac{1}{\rho}} e^{-\frac{\delta}{\rho} t} = c_0 \xi_t^{-\frac{1}{\rho}} e^{\int_0^t \frac{1}{\rho}(r_s - \delta) ds} \quad (\text{when } \rho = \gamma) \qquad (32)$$

Comparing to the corresponding expressions for the conventional model ($\rho = \gamma$) we notice several important differences. Recall that the state price reflects what the consumer is willing to pay for an extra unit of consumption. In particular, with the conventional model in mind, it has been convenient to think of π_t as high in "times of crises" and low in "good times". Consider for example a "shock" to the economy via the state price π_t. It is natural to think of this as stemming from a shock to the term $\int_0^t \eta_s dB_s$ via the process B. Assuming η positive, this lowers the state price, and seen in isolation, increases optimal consumption. This is as for the conventional model. However, a shock from B has also an effect on the last factor in (29). Assuming σ_V positive, the direction of this shock depends on the sign of $(\rho - \gamma)$. When the individual prefers early resolution of uncertainty to late ($\gamma > \rho$), this shock has the opposite effect on c_t^*. As a consequence the individual wants to smooth shocks to the economy. More precisely, it is *optimal* for the consumer to smooth consumption this way provided he/she prefers early resolution of uncertainty to late. This is obviously important when discussing consumption smoothing. It seems like some of the conventional wisdom has to be rewritten in the presence of recursive utility.

A shock to the interest rate has (in isolation) the same effect on the recursive consumer as predicted by the conventional model.

4.1 The consumption puzzle

Recursive utility separates time preference from risk preference, and permits the individual to care about the time when uncertainty is resolved, unlike the conventional, additive and separable expected utility representation. At each time t this agent cares about future utility in addition to current consumption.

Part of the asset pricing and consumption puzzles is related to the following question: How can the per capita consumption be so smooth at such a relatively large expected growth rate as indicated by market data?

The *first* major problem with the conventional life cycle model is to explain the smooth path of the consumption growth rate in society. The volatility of this growth rate in (27) can be made arbitrarily small when $\eta_t \approx (\gamma - \rho)\sigma_V(t)$. In contrast, for the conventional model only the first term on the right-hand side is present. For the estimated value of η_t, this requires a very large value of γ to match the low estimate for the consumption growth rate volatility. In the recursive model this can be readily explained.

The *second* major problem with the conventional model is to explain the relatively large estimate of the expected growth rate of consumption in society for plausible values of the parameters. For the estimated value of η_t, and the large value of γ required to match the low estimated volatility, this requires a very low, even *negative*, value of the impatience rate δ in the conventional model to match the estimate of the (conditional) expected consumption growth rate.

With the growth rate given by (26) instead, this is different. Here ρ takes the place of γ in the two first terms, which are present also in the conventional model. Thus it is consumption substitution, not risk aversion that is the correct interpretation here. Furthermore, by inspection of the the forth term on the right-hand side in (26), it is clear that a large consumption growth rate is possible. Two obvious cases are when $\gamma > \rho$ and $\rho < 1$, and $\gamma < \rho$ and $\rho > 1$, depending of course on the term $\sigma_V(t)$. In the latter case also the third term on the right-hand side of (26) can be sufficiently large. In the former this term puts a limit on how much larger than ρ the risk aversion γ can be in order to match the estimated value of the growth rate.

In the next section we show that the quantity $\sigma_V(t)$ is connected to the volatility of the wealth portfolio in equilibrium. However, the same result holds also here, and is given by (see Aase (2016b))

$$\sigma_W(t) = (1 - \rho)\sigma_V(t) + \rho\sigma_c(t). \tag{33}$$

The expected utility model is too simple to capture this important property of the individual.

From (33) we notice that we may 'invert' this relationship to obtain

$$\sigma_V(t) = \frac{1}{1-\rho}\big(\sigma_W(t) - \rho\sigma_c(t)\big)$$

Provided $\sigma_W(t) > \rho\sigma_c(t)$ for any t, and $\rho < 1$, it is the case that $\sigma_V(t) > 0$ for all t, for example. This is typically in accordance with observations and calibrations of this model, as we later demonstrate.

Recall that the market-price-of-risk parameter $\eta > 0$. If $\gamma > \rho$, the recursive utility agent has preference for *early resolution* of uncertainty, see Figure 1. We summarize as follows:

Proposition 1 *Assume the preferences are such that σ_V is positive for all t, and the market price of risk η is positive. The individual with recursive utility will then prefer to smooth market shocks provided the consumer prefers early resolution of uncertainty to late ($\gamma > \rho$). If $\sigma_V(t)$ is negative for all t and $\rho > \gamma$, the same conclusion follows.*

Combining (27) with the above equation (33) we obtain the following relationship

$$\sigma_c(t) = \frac{1-\rho}{\rho(1-\gamma)}\eta_t + \frac{\rho-\gamma}{\rho(1-\gamma)}\sigma_W(t). \tag{34}$$

The corresponding relationship for the expected utility model is simply $\sigma_c(t) = \frac{1}{\gamma}\eta_t$. The interpretation of the latter is that when the market price of risk changes, the volatility of the optimal consumption of the agent changes accordingly, in units of risk aversion.

With recursive utility this is seen to be different. Equation (34) can be interpreted to mean that the agent uses wealth to stabilize the volatility of the optimal consumption in response to changes in the market conditions, something the expected utility maximizer is unable to do. The expected utility model is simply too simplistic to capture this important property of the individual. Here it is tempting to quote Albert Einstein: "A problem should be studied through the simplest possible model, not simpler."

Where the expected utility consumer just follows is the wake of others, the contents of the *mutuality principle* in the present situation, living each period as if it were the last one, the recursive individual displays a more sophisticated behavior under market uncertainty as time elapses. It may depend on whether the individual has preference for early, or late resolution of uncertainty. With these two possibilities the mutuality principle does not hold for recursive utility, who both takes expectations about the future into account when making consumption and investment decisions, and also remembers the past. All this is of importance for decision making, including deciding on optimal pensions. Some of the conventional wisdom has to be rewritten in presence of recursive utility.

The investment strategy that attains the optimal consumption of the agent is presented in Section 9. It is here we can see in more detail how the agent uses wealth to smooth consumption. The recursive agent does not behave myopically, in contrast looks at several periods at the time. When times are good the agent consumes less than the myopic agent, invests more for the future, and can hence enjoy higher consumption than the expected utility maximizer when times are bad.

5 Some properties of the optimal pension

Returning to the life cycle model, let us briefly consider pensions. Towards this end, let T_x be the remaining life time of a person who entered into a pension contract at age x. Let $[0, \tau]$ be the support of T_x. The single premium of an annuity paying one unit per unit of time is given by the formula

$$\bar{a}_x^{(r)} = \int_0^\tau e^{-rt} \frac{l_{x+t}}{l_x} dt, \tag{35}$$

where r is the short term interest rate, and $P(T_x > t) := \frac{l_{x+t}}{l_x}$ in actuarial notation, where l_x os the decrement function. The single premium of a "temporary annuity" which terminates after time n is

$$\bar{a}_{x:\bar{n}|}^{(r)} = \int_0^n e^{-rt} \frac{l_{x+t}}{l_x} dt. \tag{36}$$

Consider the following income process e_t:

$$e_t = \begin{cases} y, & \text{if } t \leq n; \\ 0, & \text{if } t > n \end{cases} \tag{37}$$

Here y is a constant, interpreted as the consumer's salary when working, and n is the time of retirement for a pension insurance buyer, who initiated a pension insurance contract in age x. Equality in the budget constraint can then be written

$$E\Big(\int_0^\tau (e_t - c_t^*) \pi_t P(T_x > t) dt \Big) = 0.$$

which is *The Principle of Equivalence.*

For the standard Eu-model, the optimal life time consumption ($t \in [0, n]$) and pension ($t \in [n, \tau]$) is

$$c_t^* = y \, \frac{\bar{a}_{x:\bar{n}|}^{(r)}}{\bar{a}_x^{(\tilde{r})}} \exp\big\{ (\frac{1}{\gamma}(r - \delta) + \frac{1}{2\rho}\eta^2)t + \frac{1}{\gamma}\eta B_t \big\}, \tag{38}$$

provided the agent is alive at time t (otherwise $c_t^* = 0$). The initial value c_0 is then

$$c_0 = y \frac{\bar{a}^{(r)}_{x:\bar{n}|}}{\bar{a}^{(\tilde{r})}_x}$$

where

$$\tilde{r} = r - \frac{1}{\rho}(r - \delta) + \frac{1}{2}\frac{1}{\gamma}\,(1 - \frac{1}{\gamma})\,\eta'\eta. \tag{39}$$

The premium intensity p_t at time t while working is given by $p_t = y - c_t^*$, an $\mathcal{F}_t$- adapted process. This shows that the same conclusions hold for the optimal pension as with optimal consumption with regard to the sensitivity of stock market uncertainty, e.g., the mutuality principle holds for pensions with expected utility.

This model may be taken as support for unit linked pension insurance, or, defined contribution (DC)-plans. Here all the financial risk resides with the customers.

Optimal pensions in the the life cycle model with recursive utility goes as follows: The optimal life time consumption ($t \in [0, n]$) and pension ($t \in [n, \tau]$) is

$$c_t^* = y\,\frac{\bar{a}^{(r)}_{x:\bar{n}|}}{\bar{a}^{(\hat{r})}_x}\,\exp\big\{(\frac{1}{\rho}(r - \delta) + \frac{1}{2\rho}\eta^2 + \frac{1}{2\rho}(\gamma - \rho)(1 - \gamma)\sigma_V^2)t \\ + \frac{1}{\rho}(\eta + (\rho - \gamma)\sigma_V)B_t\big\}, \tag{40}$$

provided the agent is alive at time t (otherwise $c_t^* = 0$). Here

$$\hat{r} = r - \frac{1}{\rho}(r - \delta) + \frac{1}{2}\frac{1}{\rho}\,(1 - \frac{1}{\rho})\,\eta'\eta + \frac{1}{\rho}\,(\frac{1}{\rho} - 1)\,(\rho - \gamma)\,\eta\sigma_V \\ - \frac{1}{\rho}(\gamma - \rho)\Big(\frac{1}{\rho}(\gamma - \rho) + \frac{1}{2}(1 - \gamma)\Big)\sigma_V^2. \tag{41}$$

The premium intensity is given by the F_t-adapted process $p_t := y - c_t^*$. As can be seen, the optimal pension with recursive utility is being "smoothened" in the same manner as the optimal consumption, summarized in Proposition 1.

A positive shock to the economy via the term B_t increases the optimal pension benefits via the term ηB_t, which may be mitigated, or strenghtend by the term $(\rho - \gamma)\sigma_V B_t$, depending on its sign. When $(\gamma > \rho)$, then $\sigma_V(t) > 0$ and shocks to the economy are smoothened in the optimal pension with RU. This indicates that the pensioner in this model can be considerably more sophisticated than the one modeled in the conventional way when $\rho = \gamma$. We summarize as follows:

Proposition 2 *Under the same assumptions as in Proposition 1, the individual with recursive utility will prefer a pension plan that smoothens market shocks provided the consumer prefers early resolution of uncertainty to late ($\gamma > \rho$).*

This result points in the direction of defined benefit pension plan rather than a defined contribution plan, since the inequality $\gamma > \rho$ is likely to hold for most people.

Recall the theory of syndicates. In a Pareto optimum, the risk tolerance of the syndicate is equal to the sum of the risk tolerances of all its members. An insurance company can be interpreted as a syndicate, in particular a mutual company. Clearly the risk carrying capacity of a life and pension insurance company is much larger than than of any of its policy holders. This lesson seems to have been forgotten these days.

6 Portfolio choice with recursive utility

We now address the optimal investment strategy that the recursive utility consumer will use in order to obtain the optimal consumption.

Consider an agent with recursive utility who takes the market introduced in Section 2 as given. In this setting we now analyze optimal portfolio choice. We then have the following result:

Proposition 3 *The optimal portfolio fractions in the risky assets are given by*

$$\varphi(t) = \frac{1-\rho}{\gamma-\rho}(\sigma_t \sigma_t')^{-1}\nu_t - \frac{\rho(1-\gamma)}{\gamma-\rho}(\sigma_t \sigma_t')^{-1}(\sigma_t \sigma_c^*(t)), \tag{42}$$

assuming $\gamma \neq \rho$.

<u>Proof:</u> First we recall the dynamics of the optimal consumption for the individual investor under consideration. The volatility $\sigma_{c^*}(t)$ has been shown in (27) of Section 3.4 to be

$$\sigma_{c^*}(t) = \frac{1}{\rho}\Big(\eta_t + (\rho-\gamma)\sigma_V(t)\Big), \tag{43}$$

where $\sigma_t \eta_t = \nu_t$ is the market-price-of-risk given in Section 2. Also, the volatility of utility is given by

$$\sigma_V(t) = \frac{1}{1-\rho}(\sigma_W(t) - \rho\sigma_{c^*}), \tag{44}$$

as shown in Aase (2016b), and indicated previously, where $\sigma_M(t)$ is the volatility of the agent's wealth portfolio. The dynamics of the wealth is given in (9) of Section 2, implying that $\sigma_W(t) = \sigma_t' \varphi_t$ (see Section 6). This leads to a single equation for φ_t, and the solution is given by the formula (42) when $\gamma \neq \rho$. $\square$

Notice that this proof only works for recursive utility when $\gamma \neq \rho$. Unless this is the case, the term with $\sigma_V(t)$ in (43) drops out, and this is the term that contains $\sigma_W(t)$ (see (44)), meaning that the term containing the portfolio rations $\varphi(t)$ drops out of the equation when $\gamma = \rho$. In the latter case, however, the answer is well known from the literature.

At the beginning of each period, the agent allocates a certain proportion of wealth to immediate consumption, and then invests the remaining amount in the available securities for future consumption. Accordingly, the the optimal portfolio choice will depend on consumption, unlike for the expected utility model.

The growth rate of consumption, and its conditional variance, is known once consumption is determined. This is achieved by the agent using his/her preferences faced with the various market opportunities.

The agent first determines the optimal consumption growth rate and then the optimal portfolio choice, in such a manner that the relationship in Proposition 3 holds.

The optimal fractions with recursive utility depend on both risk aversion and time preference as well as the market-price-of-risk and risk premiums. They also depend, although only indirectly, on the volatility of utility $\sigma_V(t)$ and various covariances with the market portfolio. And, unlike the expected utility mode, they depend on the consumption decision taken before the optimal portfolio choice is made.

As can be seen from the relationship (20), the agent determines optimal consumption based on preferences, future utility V_t, and the quantity Y_t which depends on preferences and the past consumption and utility history of the agent, which can be seen from equation (17). If done optimally, equation (42) holds.

In each period the agent both consumes and invests for future consumption. Compared to the expected utility consumer, the recursive agent consumes less in good times, and then invests more for future consumption, and vice versa in bad times. This is how this agent averages consumption across time and states of the world in a more efficient manner than the conventional expected utility agent.

As we have seen, the recursive utility maximizer considers more than one period at the time, which allows for a smoother consumption path than the expected utility maximizer can achieve, who is just myopic.

Based on the conventional, pure demand theory of this paper, by assuming a relative risk aversion of around two, the optimal fraction in equity is 119% follows from the standard formula $\varphi = \frac{1}{\gamma}(\sigma_t \sigma_t')^{-1}\nu_t$ (see Mossin (1968), Merton (1971), Samuelson (1969)), using the summary statistics of Table 1, and assuming one single risky asset, represented by the index itself. In contrast, depending upon estimates, the typical American household holds between 6% to 20% in equity during the period of this data set. Conditional on participating in the stock market, this number increases to about 40% in financial assets. Recent estimates predict close to 60%, including indirect holdings via pension funds invested in the stock market. In the above application this formula reduces to $\varphi = \frac{1}{\gamma}(\sigma_R \sigma_R')^{-1}(\mu_R - r)$, where R is the risky asset, here assumed equal to the market M. Notice that here $\gamma = \rho$.

One could object to this that the conventional model is consistent with a value for γ around 26 only. Using this value instead, the optimal fraction in equity is down to around 9%, which in isolation seems reasonable enough. However, such a high value for the relative risk aversion is considered implausible, as discussed before.

As an illustration of the general, recursive utility based formula of Proposition 3, consider the standard situation with one risky and one risk-free asset, letting the S&P-500 index represent the risky security, and employ the data of Table 1 in the same manner as indicated above. Here we may use the estimate for the optimal consumption growth rate in Table 1, as a consistency check of the theory.

With these assumptions, the recursive model explains an average of 13 per cent in risky securities for the following values of the preference parameters: $\gamma = 2.6$ and $\rho = .90$. Given participation in the stock market, when $\varphi = .40$, this is consistent with $\gamma = 2.2$ and $\rho = .76$. If $\varphi = .60$, this may correspond to $\gamma = 2.0$ and $\rho = .66$, etc., a potential resolution of this puzzle.

The standard formula based on expected utility also results when there is no intermediate consumption. In this situation the expected utility model has a better axiomatic support, since consumption only takes place at the last point in time (see e.g., Mossin (1969)). According to Mossin (1968), his excuse for leaving out intermediate consumption was that of "taking one thing at the time". However, taking this case seriously for the moment, in this situation $\sigma_c^*(t) = 0$ in our model, and the corresponding recursive formula takes the form

$$\varphi(t) = \frac{1-\rho}{\gamma-\rho}(\sigma_t \sigma_t')^{-1}\nu_t. \tag{45}$$

From this we notice that when $\rho = 0$, the standard expected utility formula results. This points to the requirements under which this latter formula obtains in the 'bigger' model: The agent must be neutral with respect to consumption substitution across time.

Notice also how the formula (45) explains the empirical facts mentioned above: For any value of $\gamma > \rho$, the factor $(1-\rho)/(\gamma-\rho)$ can be as small as we please by setting ρ close enough to 1 from below. So this formula can, e.g., explain 13% in equities for very reasonable values of γ and ρ, where $\gamma > \rho$ and $\rho < 1$.

In addition to the insurance industry, other interesting applications would be to management of funds that invests public wealth to the benefits of the citizens of a country, or the members of a society. If these organizations happen to be large enough for an estimate of the volatility of the consumption growth rate of the group is available, the applicability of the present model becomes particularly simple, since we can then use the expression (42). This is of course a formal use of this result. One such example is the Norwegian Government Pension Fund Global (formerly the Norwegian Petroleum Fund).

7 Summary

For the conventional model with additive and separable utility risk aversion and intertemporal elasticity of substitution in consumption sometimes play conflicting roles when discussing optimal consumption. We propose to look at a wider class of utility functions, recursive utility, to sort out some of these problems.

We show that the recursive utility customer finds it optimal to smooth market shocks to a larger extent than the conventional model predicts. One question is then how this can be accomplished in the real world. This is of great importance when analyzing pensions and life insurance contracts, where insuring consumers against adverse shocks in the market should be a main issue. After the financial crisis in 2008, insurers seem eager to pass all or most of the financial risk to its customers, presenting them with mainly defined contribution pension plans, or unit linked plans. The lessons from the present paper for the insurance industry is clear: To provide the kind of consumption smoothing that consumers of the last century seem to prefer, which points in the direction of defined benefit rather than defined contribution, or unit linked pension plans.

It follows from our model how aggregate consumption in society can be as smooth as implied by data, and at the same time be consistent with the relatively large, observed growth rate. Since the recursive model fits market data much more convincingly than the conventional model, this leaves more credibility to the former representation, and weight to our recommendation.

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