Around the Life-Cycle:

Deterministic

Consumption-Investment Strategies

Mogens Steffensen

Based on joint works with Marcus Christiansen

IAA Life Colloquium, Barcelona, 24.10.17

The human capital motivation

$$dX(t) = \underbrace{a(t)\,dt}_{\text{labor income}} + \underbrace{X(t)\left(\left(r+\pi(t)(\alpha-r)\right)dt+\pi(t)\,\sigma\right)dW(t)}_{\text{capital gains}}$$

$$\max_{\pi} E\left[\frac{1}{1-\gamma}X^{1-\gamma}(T)\right]$$

$$\pi(t, X(t)) = \frac{1}{\gamma}\,\frac{\alpha-r}{\sigma^2}\,\frac{\overbrace{X(t)}^{\text{financial wealth}} + \overbrace{h(t)}^{\text{human capital}}}{X(t)}$$

Deterministic Mean-Variance Optimal Consumption and Investment (in *Stochastics*)

$$\max_{\pi \text{ deterministic}} \{E[X(T)] - \gamma Var[X(T)]\}$$

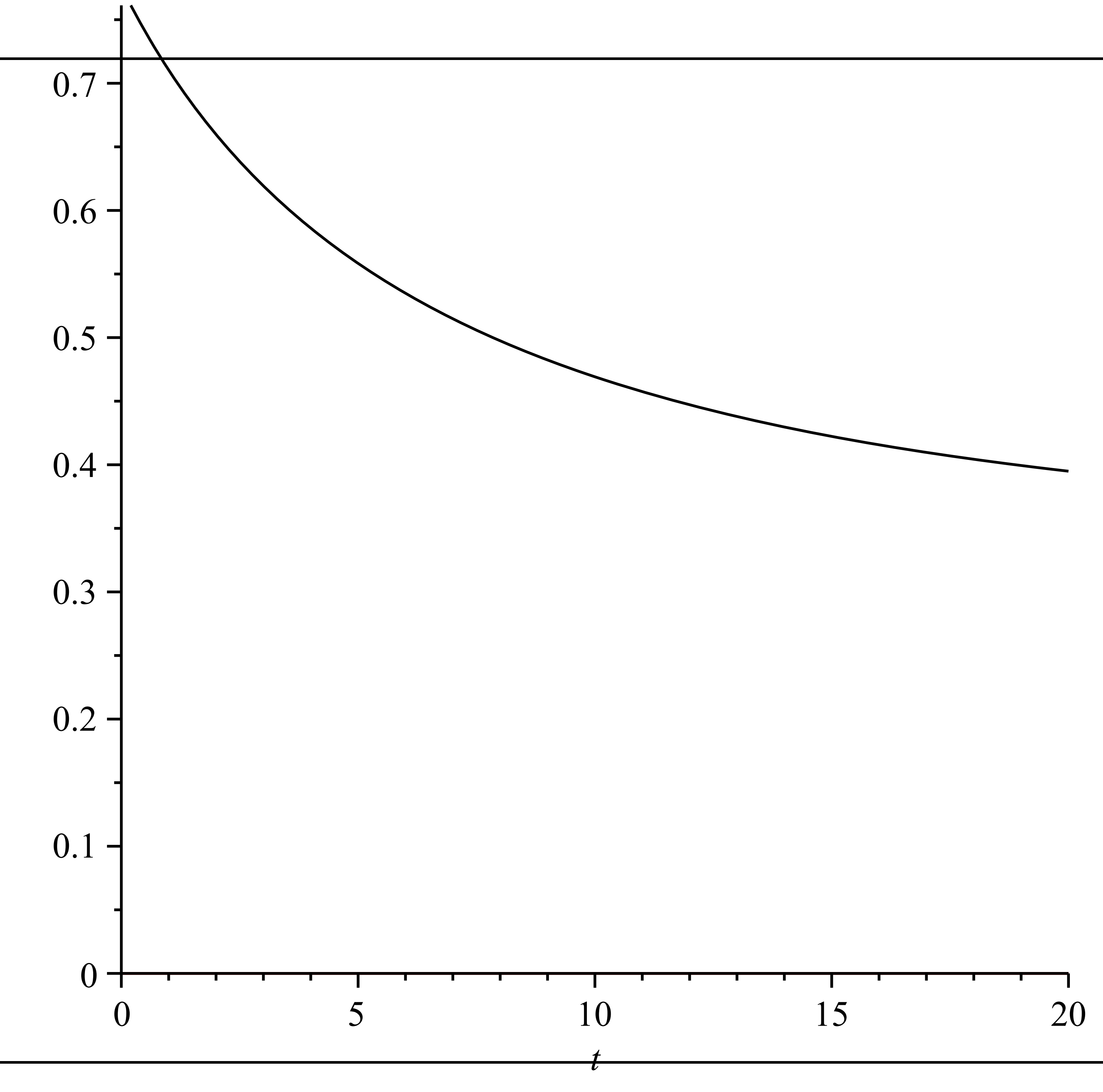
Optimal deterministic strategy: 7-dimensional forward (5) -backward (2) ODE (forward and backward linked by π)

Quite different from (from the long list of problems that we do not solve, these appear to be the most relevant to point out):

$$\max_{\pi \text{ constant}} \{E[X(T)] - \gamma Var[X(T)]\} \text{ (Markowitz (1952))}$$

$$\max_{\pi \text{ adapted and } E[X]=K} \left\{E_{t,x}[X(T)] - \gamma E_{t,x}\left[(X(T) - K)^2\right]\right\} \text{ (Korn (1997))}$$

$$\max_{\pi \text{ adapted and consistent}} \left\{E_{t,x}[X(T)] - \gamma Var_{t,x}[X(T)]\right\} \text{ (Basak and Ch.(2010))}$$


0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
5
10
15
20
t

Around the Life-Cycle: Deterministic Consumption-Investment Strategies (under revision)

$$dX(t) = a(t)\,dt + X(t)\left((r + \pi(t)(\alpha - r))\,dt + \pi(t)\,\sigma\right)dW(t) - \underbrace{c(t)\,dt}_{\text{consumption}}$$

$$\max_{\pi:\text{adapted}} E\left[\int_t^T e^{-\rho(s-t)}\frac{1}{1-\gamma}c^{1-\gamma}(s)\,ds + e^{-\rho(T-t)}\frac{1}{1-\gamma}X^{1-\gamma}(T)\right]$$

$$\begin{aligned}
\pi(t, X^{\pi}(t)) &= \frac{1}{\gamma}\,\frac{\alpha - r}{\sigma^2}\,\frac{\overbrace{X^{\pi}(t)}^{\text{financial wealth}} + \overbrace{h(t)}^{\text{human capital=future premiums}}}{X^{\pi}(t)} \\
h(t) &= \int_t^T e^{-r(s-t)}\left(a(s) - c(s)\right)ds
\end{aligned}$$

A suboptimal deterministic idea, difficult to find:

$$\pi\left(t, X^{\pi}(t)\right) = \frac{1}{\gamma}\,\frac{\alpha - r}{\sigma^2}\,E\left[\frac{X^{\pi}(t) + h(t)}{X^{\pi}(t)}\right]$$

Suboptimal deterministic strategy 1:

$$\pi(t) = \frac{1}{\gamma}\,\frac{\alpha - r}{\sigma^2}\,\frac{X^{0}(t) + h(t)}{X^{0}(t)}$$

Suboptimal deterministic strategy 2:

$$\pi(t) = \frac{1}{\gamma}\,\frac{\alpha - r}{\sigma^2}\,\frac{E\left[X^{\pi}(t) + h(t)\right]}{E\left[X^{\pi}(t)\right]}$$

Optimal detereministic strategy: 2-dimensional forward (1) -backward (1) PDE (forward and backward linked by π)

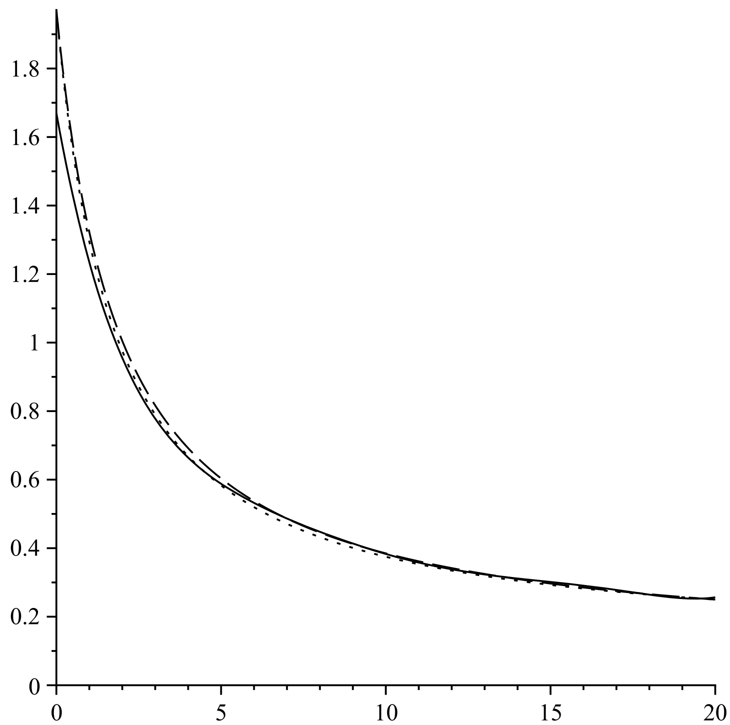


Figure 3: Optimal investment rate (solid), suboptimal strategy 1 (dash) and suboptimal strategy 2 (dot) in Example 8 for $c(t) = 0$

the suboptimal alternatives. Given the simplicity of these non-linear profiles of the growth asset proportion as function of age, they may be good alternatives to the more standard linear profiles. In particular, note the 'rule-of-thumb' style of e.g. π^1. Invest the preference dependent Merton proportion multiplied by a ratio which is simply the present value of total savings divided by the present value of past savings. Such a rule-of-thumb could even work well for much more realistic situations with stochastic investment environment and non-hedgeable income (in case one would have to estimate the present value of total savings). Such an analysis could be the topic for future research based on the ideas and suboptimal strategies presented in this paper.

References

Basu, A., Byrnes, A., Drew, M.E. (2009). Dynamic Lifecycle Strategies for Target Date Retirement Funds. *Available at SSRN 1302586.*

Bernard, C., Kwak, M. (2016). Dynamic preferences for popular investment strategies in pension funds. *Scandinavian Actuarial Journal* 5:398-419.

Blake, D., Wright, D., Zhang, Y. (2013). Target-driven investing: Optimal investment strategies

Stochastic to optimal deteministic certainty equivalence loss

$$0 - 2\%$$

Optimal deterministic to suboptimal deterministic (1 and 2) certainty equivalence loss

$$0 - 2 \text{ er } (0/000)$$