Around the Life-Cycle: Deterministic Consumption-Investment Strategies

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Abstract

We study a classical continuous-time consumption-investment problem of a power utility investor with deterministic labour income with the important feature that the consumption-investment process is constrained to be deterministic. This is motivated by the design of modern pension schemes of defined contribution type where, typically, the savings rate is constant and the proportional investment in growth stocks is a function of age or time-to-retirement, a so-called life-cycle investment strategy. We derive and study the optimal behaviour corresponding to the optimal product design within this realistic family of products with deterministic decision profiles. We also propose a couple of suboptimal deterministic strategies inspired from the optimal stochastic strategy and compare the optimal stochastic control, the optimal deterministic control and these suboptimal deterministic controls in terms of certainty equivalents. The conclusion is that only little is lost by constraining to deterministic strategies and only little is lost by implementing the suboptimal simple explicit strategies rather than the optimal one we derive.

Keywords: power utility optimization; defined contribution; deterministic control; Black-Scholes market; Pontryagin's maximum principle; suboptimal strategies.

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1 Introduction

In saving plans where members bear all the investment risk, members are typically encouraged to take control of making the investment decisions. Most savers, however, have proved unwilling or unable to make appropriate choices and life-cycle strategies have evolved to ensure that plan members who do not make their own investment decisions have a reasonably appropriate risk/return profile as they progress through their savings career. Life-cycle strategies have become a well-established part of the savings investment landscape. Traditionally and generally, life-cycle investment strategies are based on an algorithm that links the investment risk to the number of years to retirement. Younger members with longer time to retirement invest more in equities, while more mature members with fewer years to retirement tend to gradually transfer their assets to bonds.

Asset allocation of life-cycle funds can be thought of as consisting of two parts. The growth part contains equities and other growth assets. The protection part matches the profile of retirement benefit liabilities. Assets are gradually transferred from the growth part to the protection part as the member ages. While the concept may seem straightforward, implementation around the world varies. In UK, asset allocation is determined relatively mechanically, driven by a predetermined formula, following a straight-line transition over a number of years and executed by the plan administrator. In contrast, in North America, the concept tends to be implemented via target-date funds. These are constructed with specific retirement dates in mind. The asset allocation of these funds is then changed at the discretion and judgement of the fund manager, but generally along lines of moving from predominantly growth to protection assets as the target date approaches. Plan members invest in funds whose target date is close to their intended retirement date. A number of different arguments are used to support younger savers investing more in growth assets, and hence taking more risk, than their old counterparts.

A popular argument is the idea of time horizon investment or time diversification. This argument is that which focuses on the link between the length of time an investment is held and the risk associated with that investment. This link may come from the perception that investment risk over a series of years is diversified and this diversification effect increases with the time horizon. Therefore one can accept a larger risk per year for investments made on a long time horizon than for those made on a small time horizon. The argument is reflected in the stylized fact that in the long run equities always outperform bonds. Though still widely used, it is actually difficult to find theoretical substantiation of this argument and, even practically, the hypothesis is not easy to underpin.

Another argument concerns the flexibility. Young investors have greater flexibility to recover from adverse market events than those nearer retirement. A young investor can adjust his savings pattern and retirement timing plans to make up for adverse market events without too great a lifestyle adjustment. Members nearer retirement have less time and flexibility to make up a shortfall. The idea is that the so-called human capital consisting of future labor income is partly controllable by working hours, professional education, and retirement timing. For young investors marginal changes in these variables can have huge impact on the human capital and therefore more easily make up for shortfalls in the investment than similar marginal changes do for more mature investors. The argument does find both theoretical and practical support but working with working hours, education, and retirement timing as controllable processes is a technically delicate task with somewhat subtle results.

While both of the arguments above are more frequently heard from practical advisors, a third argument is standard and simple from a theoretical point of view. It does not rely on either time diversification or flexibility in human capital. Rather it is based on the mere structure of the (non-controllable) human capital. At least for many saving plan members it makes sense to assume that the human capital looks more like a protection asset than like a growth asset. Labor income has more similarities with coupon payments from a bond than dividend and selling cash flows from an equity position. Taking constant proportions of the total assets in growth and protection assets as starting point and considering human capital as part of the total assets, the conclusion is that the young investor should add growth investments to the constant proportion in order to make up for the large position in protection assets indirectly held through future working life. Old investors should only add a little since the human capital vanishes as retirement approaches. To distinguish this argument from the argument in the former paragraph, it should be emphasized that this argument does not involve controlling human capital, but just recognizing its risk structure as counting it in as a source of risk to the total risk exposure competing with, and therefore ideally balanced off with, risk from financial assets.

An important question arising, in particular in connection with the last argument, is whether time-to-retirement is the only relevant parameter to take into account or whether also other criteria should be taken into account. In the next section we show that a theoretical substantiation of the argument shows that also wealth, including historical capital gains, is relevant as state process. Yet, for various reasons one may insist on designing the life-cycle investment strategies via the age parameter, exclusively. In fact, this discussion is the mere rationale for this paper. When, as we show in the next section, the optimal adapted strategy is stochastic (age and wealth dependent) since it reflects the historical development of prices, what is then the best strategy among the deterministic (only age dependent) ones? How can we technically calculate such an optimal strategy? Are there, possibly, other deterministic strategies easier to calculate based on a pragmatic view on the optimal stochastic strategy? And, finally, how do all these strategies compare? Taking a simple suboptimal deterministic strategy as starting point, how much do we gain from implementing the more complicated optimal deterministic one? And how much do we gain if we instead advance the product design and implement the optimal stochastic life-cycle investment strategy? These are the questions we raise and answer throughout the paper.

There exists a vast amount of financial literature where life-cycle investment turns out as part of the optimal solution in response to the existence of and, possibly, control over labor income. Here, we mention a few core references on the topic and refer the reader to these primary publications and to the references therein. Common for all references mentioned here is that the resulting strategy is never only depending on age but also on all other state processes in the system. Hakansson (1970) and Merton (1971) realized that deterministic income in the consumption-investment problem is similar to a bond position and can therefore be accounted for as such. Bodie et al. (1992) discuss how control over labor supply through working hours in the active years results in life-cycle investment because the flexibility in young ages allows a more risky asset position. Bodie et al. (2004) elaborate further on the argument by also varying the retirement age and introducing habit formation in preferences. Benzoni et al. (2007) consider the case where the labor income is cointegrated with the growth asset such that the argument that labor income is of coupon payment type is only partly true. Monographic overviews are given in Korn (1997), concerning mathematical techniques and methods applied to the area, and in Viceira (2007), concerning the economic scope of the methodology. The labor income argument only holds to the extent that one can borrow against future labor income and therefore dilutes if such borrowing is constrained, as shown by Dybvig and Liu (2010). Chai et al. (2011) contains, to the authors' knowledge, the so far most comprehensive model where both labor supply and retirement age are endogenously determined.

All articles in the previous paragraph are classical normative economic contributions to the literature deriving and studying optimal behavior within a family of models and specifications of preferences. For all these normative studies it is crucial that the more or less fixed payment received from labor income has no influence on the utility from consumption apart from influencing the admissible consumption patterns. This is in sharp contrast to the case where such fixed amounts occur directly in the utility function, in which case the optimal portfolios completely change structure into option-like holdings, see e.g. Korn (2005). Further, a couple of quite different references deserve attention in relation to our motivation in terms of design of pension saving products. Crane and Bodie (1999) consider the design of saving products by studying the various aspects of the distribution of retirement wealth. Bikker et al. (2012) examine how the realized investments of Dutch pensions funds fit to the life-cycle idea. Blake et al. (2013) study life-cycle investment in the light of so-called prospect theory.

A different and short stream of literature is concerned with consumption and investment being constrained to be deterministic. This idea is fundamental in Herzog et al. (2007), Geering et al. (2010) and Bäuerle and Rieder (2013) but the objectives and the technology used is substantially different from ours. The presentation in this paper has more similarities with Christiansen and Steffensen (2013) who essentially present and solve a problem similar to the one studied here but with a mean-variance objective function. While Christiansen and Steffensen (2013) use the Hamilton-Jacobi-Bellman approach, in the present paper we rather suggest to use Pontryagin's maximum principle. The former approach needs an infinite-dimensional state space, so that the Hamilton-Jacobi-Bellman equation is not a common partial differential equation but some kind of evolution equation for a functional on a function space. To our knowledge, solutions for this evolution equation are out of reach, so we focus on Pontryagin's maximum principle.

One may reasonably ask: If the optimal strategic investment strategy is known to be stochastically adapted to wealth, then why bother about deterministic strategies at all? Well, still age-but-not-wealth-dependent life-cycle strategies are crucial in practical pension asset management. They are simple to communicate and therefore have an advantage from a marketing point of view and they are (supposed to be) cheap to administrate compared to more involved alternatives. Some important marketed examples are the BlackRock LifePath Target Date Funds and the Vanguard Target Retirement Funds. Morningstar Fund Research (2012) reports that Target Date Funds have grown from 71 billion US dollars at the end of 2005 to approximately 378 billion dollars at the end of 2011.

Market presence makes life-cycle funds relevant for academic studies. Indeed, many authors have identified the welfare loss arising from suboptimal deterministic life-cycle investment strategies. However, they typically compare the stochastic investment plan with a so-called linear gliding path (i.e. an age-linear transfer from growth to protection assets) and find that the welfare loss is significant. A key reference is Cairns et al. (2006) but the theme is pushed by several challengers of the target fund idea, see e.g. Basu et al. (2009) and Martellini and Milhau (2010). More recently Bernard and Kwak (2016) criticised the linear life-cycle strategy for not being optimal for any utility function.

In contrast to the existing literature, we do not compare the stochastic plan with the linear deterministic plan but, instead, the optimal stochastic plan with the optimal deterministic plan. There, however, we find the welfare loss to be negligible. We show that it is negligible even for simpler suboptimal paths than the optimal deterministic one. We conclude that the appropriately designed life-cycle product is, indeed, a relevant product for pension savers to buy.

In Section 2 we discuss a stochastic control problem and propose some simple ways to control deterministically. In Section 3 we define the optimal deterministic control problem. Section4 derives a necessary condition for optimal solutions to our core problem by means of Pontryagin's maximum principle, and the solution is further characterized in Section 5. A brief discussion about the Hamilton-Jacobi-Bellman approach can be found in Section 6. Section 7 presents a numerical study and concludes.

2 Motivating discussion and suboptimal alternatives

In the introduction we discussed various arguments for working with life-cycle investment strategies. We claimed that the argument with strongest and simplest theoretical support is the idea that human capital shares similarities with a protection asset and supply of working power should therefore be balanced off by a position in growth stock. This hedging position should, however, become smaller and smaller as the investor approaches retirement and human capital falls away. The argument is here illustrated by a simple power utility optimization of consumption and/or terminal wealth in a Black-Scholes market with constant labor income. This also serves as the basic model in the rest of the paper.

On some finite time interval $[0,T]$, we assume that we have continuous income with rate a and continuous consumption with rate c. The deterministic initial wealth $x_0 > 0$ and the stochastic wealth $X(t)$ at $t > 0$ is distributed between a bank account with risk-free interest rate r and a stock or stock fund with price process

$$dS(t) = S(t)\alpha dt + S(t)\sigma dW(t), \quad S(0) = 1,$$

where $\alpha > r$ and $\sigma > 0$. We write $\pi(t)$ for the proportion of the wealth invested in stocks. We call $\pi(t)$ the investment strategy. The price process of a self-financing investment portfolio with strategy π satisfies

$$dI(t) = I(t)(r + (\alpha - r)\pi(t))dt + I(t)\sigma\pi(t)dW(t),$$

with explicit solution

$$I(t) = I(0)e^{\int_0^t dU},$$
$$dU(t) = (r + (\alpha - r)\pi(t) - \frac{1}{2}\sigma^2\pi(t)^2)dt + \sigma\pi(t)dW(t).$$

The wealth process $X(t)$ satisfies

$$dX(t) = X(t)(r + (\alpha - r)\pi(t))dt + (a(t) - c(t))dt + X(t)\sigma\pi(t)dW(t), \quad X(0) = x_0, \tag{1}$$

and has the explicit representation

$$X(t) = x_0 e^{\int_0^t dU} + \int_0^t (a(s) - c(s)) e^{\int_s^t dU} ds. \tag{2}$$

Writing $u(x)$ for the power utility function,

$$u(x) = \begin{cases} \frac{1}{\gamma} x^\gamma, & \gamma \in (-\infty, 0) \cup (0, 1), \\ \ln(x), & \gamma = 0, \end{cases}$$

we consider in this section the standard problem to maximize

$$G(\pi, c) := E\left[\int_0^T u(c(s)) e^{-\rho s} ds + u(X(T)) e^{-\rho T}\right] \tag{3}$$

over admissible strategies (π, c). The quantity γ is known as the risk aversion of the investor. The parameter $\rho \geq 0$ describes a preference for consuming today instead of tomorrow. Focusing here on the life-cycle investment strategy, till the next section we choose to fix the consumption rate at a specified level corresponding to the idea of a defined contribution saving plan. Then the consumption-investment problem introduced above is, essentially, just an investment problem with net income rate equal to the saving rate $a - c$ and the optimal investment strategy can be studied without taking the fixed consumption part of the objective function into account. This problem is a special case of the more delicate problem solved for also consumption, which we will study in the next section. The sophistication is, however, sufficient to illustrate important insight in the optimal life-cycle strategy, to inspire the development of simple suboptimal saving products, and to set the scene for the rest of the paper.

The resulting investment strategy of the dynamic control problem reads, see e.g. Merton (1971),

$$\pi^\diamond(t) = \frac{1}{1-\gamma} \frac{\alpha - r}{\sigma^2} \frac{X^{\pi^\diamond}(t) + h(t)}{X^{\pi^\diamond}(t)}, \tag{4}$$

where h is the deterministic process of human capital calculated as

$$h(t) = \int_t^T (a(s) - c(s)) e^{-r(s-t)} ds.$$

The superscript $\pi^\diamond$ of X emphasizes that the optimally invested wealth process is an argument in the strategy. This is typically unspoken, but here it is important to stress in order to distinguish this optimal strategy from several suboptimal ones proposed below. It is the appearance of X in the optimal strategy that makes it qualitatively much more involved than practical life-cycle strategies. Note that in this calculation the human capital is the present value of future earnings net of consumption, i.e. the present value of future retirement savings. For a relatively stable saving coefficient $a - c$, this human capital is decreasing in time. The simplest situation of this type to think of is a relatively stable income a from which a fixed proportion is saved. Considering $X^{\pi^\diamond}(t)$ as fixed, a decreasing human capital leads to a decreasing investment proportion in equity. Strongly increasing income or saving rates may actually lead to an increasing human capital, though.

Taking into account the dependence on X goes beyond the classical idea of life-cycle investment. Although we have now derived a strategy with life-cycle features but also with advancements in the design, it is natural to discuss deterministic strategies in the context of this stochastic control problem formulation. Namely, there may still be good reason for studying good life-cycle investment strategies only depending on time to retirement. Such investment patterns are easy to illustrate, to communicate and to implement and the loss of welfare by disregarding the wealth dependence may be negligible.

In the following Sections 3 to 8, which make up the core technical part of our paper, we attack the mathematically challenging problem of finding the best strategy among the deterministic ones to maximize the objective given in (3).

In the rest of this motivating section, we propose a couple of simple deterministic alternatives to the optimal deterministic one. What is common for them is that their structure relates directly to the optimal stochastic strategy and they are mathematically convenient. This makes us believe that they may perform relatively well, i.e. close to the optimal deterministic and maybe even the optimal stochastic ones, while at the same time being tractable to work with and communicate.

In Section 8 we compare some of the strategies specified in this section with the optimal stochastic one and the optimal deterministic one.

- The first deterministic approach is to replace the right hand side of (4) by its plain expectation. The formula becomes

$$\pi^1(t) = \frac{1}{1-\gamma}\frac{\alpha-r}{\sigma^2}E\left[\frac{X^\pi(t)+h(t)}{X^\pi(t)}\right]. \tag{5}$$

  On purpose, we have now decorated the wealth process with an arbitrary π rather than $\pi^\diamond$. Of course, since we are now looking for suboptimal strategies, we can propose any strategy plugged in on the right hand side of (5) that makes the formula tractable. The first idea to plug in $\pi^\diamond$ is appealing but this expectation, actually, does not exist. Another idea (difficult to implement) is to plug in π^1 on the right hand side to form a fixed point problem. One special case is, however, easy to handle. If we take $\pi = 0$, X on the right hand side in the formula becomes deterministic and the expectation operation becomes redundant. Note however that π^1 is not zero. The strategy is given by

$$\begin{aligned}
\pi^1(t) &= \frac{1}{1-\gamma}\frac{\alpha-r}{\sigma^2}\frac{X^0(t)+h(t)}{X^0(t)} \\
&= \frac{1}{1-\gamma}\frac{\alpha-r}{\sigma^2}\frac{x_0e^{rt}+\int_0^t e^{r(t-s)}\left(a(s)-c(s)\right)ds+h(t)}{x_0e^{rt}+\int_0^t e^{r(t-s)}\left(a(s)-c(s)\right)ds} \\
&= \frac{1}{1-\gamma}\frac{\alpha-r}{\sigma^2}\frac{x_0e^{rt}+\int_0^T e^{r(t-s)}\left(a(s)-c(s)\right)ds}{x_0e^{rt}+\int_0^t e^{r(t-s)}\left(a(s)-c(s)\right)ds}.
\end{aligned} \tag{6}$$

  The special case (6) of π^1 given in (5) forms one of the two suboptimal strategies that we carry on for comparisons at the end of the paper.

- The second deterministic approach is to replace X on the right hand side of (4) by its expectation. The formula becomes

$$\pi^2(t) = \frac{1}{1-\gamma}\frac{\alpha-r}{\sigma^2}\frac{E[X^\pi(t)]+h(t)}{E[X^\pi(t)]}. \tag{7}$$

  Again, we can take $\pi = \pi^\diamond$ on the right hand side as the obvious choice or $\pi = \pi^2$ to form a fixed point problem. Note that for the specific choice $\pi = 0$, π^2 and π^1 coincide. By choosing $\pi = \pi^\diamond$, it turns out to be easy to calculate the expectations on the right hand side since $X^{\pi^\diamond}(t) + h(t)$ follows a geometric Brownian motion. This is of course tractable since computational simplicity is important when choosing among suboptimal strategies. The dynamics of $X^{\pi^\diamond}(t) + h(t)$ are given by

$$\begin{aligned} d\left(X^{\pi^\diamond}(t)+h(t)\right) &= \left(r+\frac{1}{1-\gamma}\left(\frac{\alpha-r}{\sigma}\right)^2\right)\left(X^{\pi^\diamond}(t)+h(t)\right)dt \\ &\quad +\frac{1}{1-\gamma}\frac{\alpha-r}{\sigma}\left(X^{\pi^\diamond}(t)+h(t)\right)dW(t), \end{aligned} \tag{8}$$

  such that

$$E\left[X^{\pi^\diamond}(t)\right]+h(t) = (x_0+h(0))\,e^{\left(r+\frac{1}{1-\gamma}\left(\frac{\alpha-r}{\sigma}\right)^2\right)t}.$$

  Thus, we have that

$$\begin{aligned} \pi^2(t) &= \frac{1}{1-\gamma}\frac{\alpha-r}{\sigma^2}\frac{E\left[X^{\pi^\diamond}(t)\right]+h(t)}{E\left[X^{\pi^\diamond}(t)\right]} \\ &= \frac{1}{1-\gamma}\frac{\alpha-r}{\sigma^2}\frac{(x_0+h(0))\,e^{\left(r+\frac{1}{1-\gamma}\left(\frac{\alpha-r}{\sigma}\right)^2\right)t}}{(x_0+h(0))\,e^{\left(r+\frac{1}{1-\gamma}\left(\frac{\alpha-r}{\sigma}\right)^2\right)t}-h(t)}. \end{aligned} \tag{9}$$

  The special case (9) of π^2 given in (7) forms the other of the two suboptimal strategies that we carry on for comparisons at the end of the paper.

3 The optimization problem and existence of optimal solutions

The focus of this paper is to maximize (3) over deterministic investment strategies and deterministic consumption rates. In order to make sure that G is well-defined, the terminal wealth $X(T)$ and the consumption plan $c(t)$, $t \in [0,T]$, must be non-negative. Therefore we assume that

$$0 \le \underline{c}(t) \le c(t) \le \overline{c}(t) \le a(t), \quad t \in [0,T],$$

for some continuous functions $\underline{c}(t)$ and $\overline{c}(t)$ that describe the lower and upper consumption bounds. With the consumption never being greater than the income at any time t, the terminal wealth is (strictly) positive since we assumed that the initial wealth x_0 is (strictly) positive. Moreover, we assume that the investment strategy π and the consumption rate c are continuous on $[0,T]$, which implies that the stochastic differential equation (1) has a unique strong solution. Applying Lemma 10 and allowing the integral on $[0,T]$ in (3) to be improper, we can conclude that the mapping $G(\pi,c)$ is well-defined on

$$D := \left\{(\pi,c) \in \mathcal{C}([0,T]) \times \mathcal{C}([0,T]) : \underline{c}(t) \le c(t) \le \overline{c}(t), t \in [0,T]\right\},$$

where $\mathcal{C}([0,T])$ is the Banach space of continuous functions on $[0,T]$ equipped with the supremum norm.

Proposition 1 *There exists a constant $C < \infty$ such that $G(\pi,c) \le C$ for all $(\pi,c) \in D$.*

Proof. Since the optimal stochastic control $\pi^\diamond$ according to (4) leads to a utility that is greater than for any deterministic, continuous control π, it suffices to show that $c \mapsto G(\pi^\diamond, c)$ has a finite upper bound. By applying (8), we obtain that

$$G(\pi^\diamond, c) = \int_0^T u(c(s))e^{-\rho s}ds + E\left[u\left((x_0 + h(0))e^N\right)e^{-\rho T}\right]$$

for a random variable N that is normally distributed and that does not depend on the choice of c. Since $u(c(t)) \le u(\sup_{0\le t\le T}\overline{c}(t)) < \infty$ and

$$h(0) \le \int_0^T a(s)e^{-r(s-t)}ds < \infty,$$

we can conclude that there exists a finite K with $G(\pi,c) \le G(\pi^\diamond,c) \le K$ for all $(\pi,c) \in D$. ■

4 A Pontryagin maximum principle

In this section we give a necessary condition for optimal controls in D. With writing $F(t,x)$ for the cumulative distribution function of $X(t)$ at x, we have

$$E[u(X(T))] = \int E[u(X(T))|X(t) = x]F(t,dx).$$

The expectations indeed exists for each $(\pi,c) \in D$ because of Lemma 10. Using the decomposition

$$X(T) = X(t)e^{\int_t^T dU} + \int_t^T (a(s) - c(s))e^{\int_s^T dU}ds$$

and the fact that $X(t)$ and $\int_s^T dU$ are stochastically independent for all $s \ge t$, we obtain that

$$E[u(X(T))|X(t) = x] = E[u(X^{t,x}(T))] =: g(t,x), \tag{10}$$

where

$$X^{t,x}(T) := xe^{\int_t^T dU} + \int_t^T (a(s) - c(s))e^{-\int_s^T dU}ds.$$

Proposition 2 *For each $(\pi, c) \in D$ the function $g : [0,T] \times (0,\infty) \to \mathbf{R}$ is continuously differentiable in t, twice continuously differentiable in x, and satisfies the Kolmogorov backward equation*

$$\begin{aligned} \partial_t g(t,x) &= -\big(xr + x(\alpha - r)\pi(t) + a(t) - c(t)\big)\partial_x g(t,x) - \frac{1}{2}x^2\sigma^2\pi(t)^2\partial_{xx}g(t,x), \\ g(T,x) &= u(x). \end{aligned} \tag{11}$$

Furthermore, for each positive integer k we have

$$\partial_x^k g(t,x) = E\Big[\partial_x^k u(X^{t,x})e^{k\int_t^T dU}\Big]. \tag{12}$$

Proposition 2 corresponds to the Feynman-Kac Theorem. Instead of just referring to the literature, we present a direct proof in section 9, because the partial differential equation (11) is degenerated as $\pi(t)$ can be zero. Our degenerated case is not very well covered in the literature, in particular with the coefficients $(xr + x(\alpha - r)\pi(t) + a(t) - c(t))$ and $x^2\sigma^2\pi(t)^2$ and the terminal condition $u(x)$ being unbounded and the latter two being non-Lipschitz in x on $\mathbf{R}$.

Theorem 3 *Suppose that $(\pi^*, c^*) \in D$ is optimal, i.e. $G(\pi^*, c^*) = \sup_{(\pi,c)\in D} G(\pi, c)$. Then for each $t \in [0,T]$ we have*

$$\begin{aligned} (\pi^*(t), c^*(t)) = \operatorname*{argmax}_{(\pi(t),c(t))\in\mathbf{R}\times[\underline{c}(t),\overline{c}(t)]} \Big\{ &\int e^{-\rho T}(\partial_x g^*(t,x))\big(xr + x(\alpha - r)\pi(t) + a(t) - c(t)\big)F^*(t,dx) \\ &+ \int e^{-\rho T}\frac{1}{2}(\partial_{xx}g^*(t,x))\big(x^2\sigma^2\pi(t)^2\big)F^*(t,dx) + e^{-\rho t}u(c(t))\Big\}. \end{aligned} \tag{13}$$

Proof of Theorem 3. We write $X^{t+\epsilon,X(t)}(T)$ for the wealth function that is interrupted on the interval $(t, t+\epsilon]$,

$$X^{t+\epsilon,X(t)}(T) = x_0 e^{\int_{(0,T]\setminus(t,t+\epsilon]} dU} + \int_{(0,T]\setminus(t,t+\epsilon]} (a(u) - c(u))e^{\int_{(s,T]\cap(t,t+\epsilon]} dU} ds.$$

Let l and k, $l \le k$, be non-negative integers. We write $u^{(k)}(x)$ for the k-th derivative of the power utility function $u(x)$. With the help of the Taylor expansion for $u^{(k)}(x)$ as given in the proof of Lemma 11, we can show that

$$\begin{aligned} u^{(k)}(X^{t+\epsilon,X(t)}(T))X(t)^l e^{k\int_{t+\epsilon}^T dU} =& u^{(k)}(X(T))X(t)^l e^{k\int_t^T dU} \\ &+ u^{(k)}(X(T))X(t)^l e^{k\int_{t+\epsilon}^T dU}\Big(1 - e^{k\int_t^{t+\epsilon} dU}\Big) \\ &+ u^{(k+1)}(\Xi)\big(X^{t+\epsilon,X(t)}(T) - X(T)\big)X(t)^l e^{k\int_{t+\epsilon}^T dU} \end{aligned} \tag{14}$$

for some random variable Ξ between $X^{t+\epsilon,X(t)}(T)$ and $X(T)$. We now show that the expectations of the second and third addend on the right hand side vanish with order $\mathcal{O}(\epsilon^{1/2})$. At first we apply Hölder's inequality repeatedly to derive for the absolutes of the second and third addends the upper bound

$$\begin{aligned} &E\Big[u^{(k)}(X(T))^6\Big]^{1/6} E\Big[X(t)^{6l}\Big]^{1/6} E\Big[e^{6k\int_{t+\epsilon}^T dU}\Big]^{1/6} E\Big[\Big(1 - e^{k\int_t^{t+\epsilon} dU}\Big)^2\Big]^{1/2} \\ &+ E\Big[u^{(k+1)}(Z)^6\Big]^{1/6} E[X(t)^{6l}\Big]^{1/6} E\Big[e^{6k\int_{t+\epsilon}^T dU}\Big]^{1/6} E\Big[\big(X^{t+\epsilon,X(t)}(T) - X(T)\big)^2\Big]^{1/2}, \end{aligned} \tag{15}$$

where Z denotes the lower bound $\Xi \geq x_0 e^{\int_0^t dU} e^{\int_{t+\epsilon}^T dU} e^{(\int_t^{t+\epsilon} dU)_-} =: Z$. Recall that $u^{(k+1)}(x)$ is proportional to $x^{-\delta}$ for some $\delta > 0$, so that $u^{(k+1)}(\Xi)^6 \leq u^{(k+1)}(Z)^6$, where the latter term is proportional to the independent factors $x_0^{-4\delta} e^{-4\delta \int_0^t dU} e^{-4\delta \int_{t+\epsilon}^T dU} e^{-4\delta (\int_t^{t+\epsilon} dU)_-}$. Therefore, the first, second, and third expectation in both lines of (15) are finite because of Lemma 9 and Lemma 10. By applying Lemma 9 and then using the expansion $\exp\{y\} = 1 + \mathcal{O}(y)$ for bounded arguments y, we can show that the fourth expectation on the right hand side of the first line of (15) has an upper bound of $\mathcal{O}(\epsilon^{1/2})$. The fourth expectation in the second line of (15) equals

$$E\Big[\Big(X(t) e^{\int_{t+\epsilon}^T dU}\Big(1 - e^{\int_t^{t+\epsilon} dU}\Big) - e^{\int_{t+\epsilon}^T dU} \int_t^{t+\epsilon} (a(s) - c(s)) e^{\int_s^{t+\epsilon} dU}\Big)^2\Big]^{1/2}.$$

By expanding the inner square, taking the expectation separately for the (independent) factors corresponding to $(0, t]$, $(t, t+\epsilon]$, $(t+\epsilon, T]$, and then applying Lemma 9 and Lemma 10, we get an upper bound of order $\mathcal{O}(\epsilon^{1/2})$. All in all, using (12) we can conclude that

$$\begin{aligned} \int x^l \partial_x^k g(t+\epsilon, x) F(t, dx) &= E\Big[u^{(k)}(X^{t+\epsilon, X(t)}(T)) X(t)^l e^{k \int_{t+\epsilon}^T dU}\Big] \\ &= E\Big[u^{(k)}(X(T)) X(t)^l e^{k \int_t^T dU}\Big] + \mathcal{O}(|\epsilon|^{1/2}) \\ &= \int x^l \partial_x^k g(t+\epsilon, x) F(t, dx) + \mathcal{O}(|\epsilon|^{1/2}) \end{aligned} \tag{16}$$

for non-negative integers k and l, $k \geq l$. In all estimates above we can choose upper bounds that hold uniformly in t, so (16) holds in fact uniformly in t. Together with equation (11), we also obtain that

$$\int \partial_t g(t+\epsilon) F(t, dx) = \int \partial_t g(t) F(t, dx) + \mathcal{O}(|\epsilon|^{1/2}) \tag{17}$$

for $\epsilon > 0$ and uniformly in t. Analogously to the case $\epsilon > 0$, one can show that (16) and (17) are also true if we replace $+\epsilon$ on the left hand side by $-\epsilon$.

We now show (13) by indirect evidence. Let $\mathcal{T} \subset [0, T]$ be the set of times where $u(c^*(t)) = -\infty$, which could theoretically happen when $c^*(t)$ touches zero and $\gamma \leq 0$. The complementary set $\mathcal{T}^c = [0, T] \setminus \mathcal{T}$ is dense in $[0, T]$. Otherwise we could find a small ball where $u(c^*(t)) = -\infty$, which implies $G(\pi^*, c^*) = -\infty$ and is a contradiction to the optimality of (π^*, c^*). Suppose now that (13) is not satisfied at an arbitrary but fixed time $t_0 \in \mathcal{T}^c \cap (0, T)$. Then there exists a small ball $(t_0 - \epsilon, t_0 + \epsilon) \subset [0, T]$ where $(\pi^*(t), c^*(t))$ is different from the argmax (and c^* different from zero), since $(\pi^*(t), c^*(t))$ is continuous and (13) is continuous according to Proposition 5. Let $0 < \epsilon_n \leq \epsilon$ be a decreasing sequence that converges to zero. We choose a sequence $(\pi_n, c_n) \in D$ in such a way that $(\pi^n(t), c^n(t)) = (\pi^*(t), c^*(t))$ on $[0, T] \setminus (t_0 - \epsilon_n, t_0 + \epsilon_n)$ and such that $(\pi^n(t_0), c^n(t_0))$ equals the argmax (13) at t_0. For this sequence we get

$$\begin{aligned} &G(\pi^*, c^*) - G(\pi^n, c^n) \\ &= \int g^*(t_0 - \epsilon, x) F^*(t - \epsilon, dx) - \int g^n(t_0 - \epsilon, x) F^*(t_0 - \epsilon, dx) + \int_0^T e^{-\rho s}\big(u(c^*(s)) - u(c^n(s))\big) ds \\ &= \int \int_{t_0 - \epsilon}^{t_0 + \epsilon} (\partial_t g^n(s, x) - \partial_t g^*(s, x)) ds F^*(t_0 - \epsilon, dx) + \int_{t_0 - \epsilon_n}^{t_0 + \epsilon_n} e^{-\rho s}\big(u(c^*(s)) - u(c^n(s))\big) ds \end{aligned}$$

since $F^*(t_0-\epsilon,dx)=F_n(t_0-\epsilon,dx)$ and $g^*(t_0+\epsilon,x)=g^n(t_0+\epsilon,x)$ for all x. By applying (17) (recall that (17) holds uniformly in t), we can show that

$$\begin{aligned}G(\pi^*,c^*)-G(\pi_n,c_n)=&2\epsilon_n\int(\partial_t g_n(t_0,x)-\partial_t g^*(t_0,x))\,ds\,F^*(t_0,dx)+\mathcal{O}(|\epsilon_n|^{3/2})\\&+2\epsilon_n e^{-\rho s}\big(u(c^*(t_0))-u(c_n(t_0))\big)ds+r(\epsilon_n),\end{aligned}$$

where $r(\epsilon_n)/\epsilon_n\to 0$. Furthermore, because of (11) and (17) we have

$$G(\pi^*,c^*)-G(\pi_n,c_n)=2\epsilon_n\Big(h^*(t_0,\pi^*(t_0),c^*(t_0))-h^*(t_0,\pi^n(t_0),c^n(t_0))\Big)+\mathcal{O}(|\epsilon_n|^{3/2})+r(\epsilon_n),$$

where $h^*(t,\pi(t),c(t))$ is the term in curly brackets on the right hand side of (13). By construction, $h^*(t_0,\pi^*(t_0),c^*(t_0))-h^*(t_0,\pi^n(t_0),c^n(t_0))<-\delta<0$ for each n. As $\mathcal{O}(|\epsilon_n|^{3/2})+r(\epsilon_n)$ vanishes faster than ϵ_n, there must exists an n for which $G(\pi^*,c^*)-G(\pi_n,c_n)$ is negative. However, this is a contradiction to the fact that $G(\pi^*,c^*)$ is maximal.

All in all, we showed that (13) holds on $\mathcal{T}^c\cap[0,T]$. As the latter set is dense in $[0,T]$ and $(\pi^*(t),c^*(t))$ and (13) are both continuous (see Proposition 5), we can conclude that (13) must hold on all of $[0,T]$. ∎

5 Stochastic characterization of optimal controls

The necessary condition (13) can be rewritten in terms of our stochastic model.

Lemma 4 *The argmax in* (13) *is equivalent to*

$$\pi^*(t)=-\frac{\int(\partial_x g^*(t,x))x(\alpha-r)F^*(t,dx)}{\int(\partial_{xx}g^*(t,x))x^2\sigma^2F^*(t,dx)},\tag{18}$$

$$c^*(t)=\Big(\underline{c}(t)\vee\Big(\int e^{-\rho(T-t)}(\partial_x g^*(t,x))F^*(t,dx)\Big)^{1/(\gamma-1)}\Big)\wedge\overline{c}(t).\tag{19}$$

Proof. In order to find the argmax in (13), we calculate the partial derivatives with respect to $\pi(t)$ and $c(t)$ and set them equal to zero,

$$\begin{aligned}0&=\int e^{-\rho T}(\partial_x g^*(t,x))x(\alpha-r)F^*(t,dx)+\int e^{-\rho T}(\partial_{xx}g^*(t,x))x^2\sigma^2\pi F^*(t,dx),\\0&=-\int e^{-\rho T}(\partial_x g^*(t,x))F^*(t,dx)+e^{-\rho t}c^{\gamma-1}.\end{aligned}\tag{20}$$

Solving the equations (20) leads to the unconstrained extremum, which is indeed a maximum since the matrix of the second-order partial derivatives (Hessian matrix) is negative definite. To see that, note that the non-diagonal entries of the Hessian matrix are zero and that the diagonal entries are

$$\int e^{-\rho T}(\partial_{xx}g^*(t,x))x^2\sigma^2F^*(t,dx)\quad\text{and}\quad(\gamma-1)e^{-\rho t}c^{\gamma-2}.$$

The second term is negative since $\gamma<1$ and the first term is negative since

$$\partial_{xx}g^*(t,x)=(\gamma-1)\,E\Big[\Big(e^{\int_t^T dU^*}\Big)^2\Big(xe^{\int_t^T dU^*}+\int_t^T(a(u)-c^*(u))e^{\int_u^T dU^*}du\Big)^{\gamma-2}\Big]$$

is negative for all $x > 0$. As (18) does not depend on c and since the right hand side of (13) is concave in c, the maximum under the constraint $0 \leq \underline{c}(t) \leq c \leq \overline{c}(t) \leq a(t)$ has the form as stated in the proposition. ■

Proposition 5 *The argmax in* (13) *is equivalent to*

$$\pi^*(t) = \frac{\alpha - r}{\sigma^2(1-\gamma)} \frac{E\Big[X^*(T)^{\gamma-1} X^*(t) e^{\int_t^T dU^*}\Big]}{E\Big[X^*(T)^{\gamma-2}\Big(X^*(t) e^{\int_t^T dU^*}\Big)^2\Big]}, \tag{21}$$

$$c^*(t) = \Big(\underline{c}(t) \vee \Big(e^{-\rho(T-t)} E\Big[X^*(T)^{\gamma-1} e^{\int_t^T dU^*}\Big]\Big)^{1/(\gamma-1)}\Big) \wedge \overline{c}(t). \tag{22}$$

In particular, the right hand sides of (21) *and* (22) *are continuous in* t.

Note that the denominator in (21) is never zero, because $X^*(T)$ and $X^*(t)e^{\int_t^T dU^*}$ are (strictly) positive.

Proof. By applying (12), we can show that

$$\begin{aligned} \int (\partial_x g^*(t,x)) F^*(t,dx) &= E\Big[(X^*(T))^{\gamma-1} e^{\int_t^T dU^*}\Big], \\ \int (\partial_x g^*(t,x)) x F^*(t,dx) &= E\Big[(X^*(T))^{\gamma-1} X^*(t) e^{\int_t^T dU^*}\Big], \\ \int (\partial_{xx} g^*(t,x)) x^2 F^*(t,dx) &= (\gamma-1) E\Big[(X^*(T))^{\gamma-2}\Big(X^*(t) e^{\int_t^T dU^*}\Big)^2\Big]. \end{aligned} \tag{23}$$

Plugging these results into equations (18) and (19) leads to (21) and (22). Starting from the equation

$$\begin{aligned} u^{(k)}(X^*(T)) X^*(t+\epsilon)^l e^{k\int_{t+\epsilon}^T dU^*} =& u^{(k)}(X^*(T)) X^*(t)^l e^{k\int_t^T dU^*} \\ &+ u^{(k)}(X^*(T)) X^*(t+\epsilon)^l e^{k\int_{t+\epsilon}^T dU^*}\Big(1 - e^{k\int_t^{t+\epsilon} dU^*}\Big) \\ &+ u^{(k)}(X^*(T))\big(X^*(t+\epsilon)^l - X^*(t)^l\big) e^{k\int_t^T dU^*} \end{aligned}$$

for $k = 1, 2$ and $l = 0, 1, 2$ with $l \leq k$ and using the line of arguments that follow equation (14), after some algebra we can show that

$$E\Big[u^{(k)}(X^*(T)) X^*(t+\epsilon)^l e^{k\int_{t+\epsilon}^T dU^*}\Big] \longrightarrow E\Big[u^{(k)}(X^*(T)) X^*(t)^l e^{k\int_t^T dU^*}\Big], \quad \epsilon \to 0.$$

Hence, all three terms in (23) are continuous in t, and so are the right hand sides of (21) and (22). ■

At time $t = T$ all investment strategies that we discussed so far are equal,

$$\pi^*(T) = \frac{\alpha - r}{\sigma^2(1-\gamma)} = \pi^\diamond(T) = \pi^1(T) = \pi^2(T),$$

where $\pi^\diamond$ is the optimal stochastic investment strategy according to (4) and π^1 and π^2 are the suboptimal investment strategies from (6) and (9). If there is no income and no consumption ($a = \underline{c} = \overline{c} = 0$), then all investment strategies are equivalent,

$$\pi^*(t) = \frac{\alpha - r}{\sigma^2(1-\gamma)} = \pi^\diamond(t) = \pi^1(t) = \pi^2(t), \quad t \in [0, T].$$

6 Remarks on the Hamilton-Jacobi-Bellman approach

In the literature dealing with optimal stochastic control, the optimal consumption and investment problem is solved by choosing the wealth process as state variable, see e.g. Merton (1971). In order to solve the deterministic control problem, the idea is here to use the probability density function of the wealth as state variable rather than the wealth itself. However, that implies that the state space is some function space, which leads to major difficulties.

For simplicity we assume that $X(t)$ has a density function $f(t,x) = \partial_x F(t,x)$. Let us define a value function by

$$\begin{aligned} E\Big[e^{-\rho T} u(X(T))\Big] + \int_t^T e^{-\rho s} u(c(s))ds &= \int e^{-\rho T} g(t,x) f(t,x)dx + \int_t^T e^{-\rho s} u(c(s))ds \\ &=: V(t, f(t,\cdot)), \qquad t > 0. \end{aligned} \tag{24}$$

Here the probability density function $f(t,\cdot)$ is the state variable at time t, and the functional V is a mapping from $[0,T] \times L_1(\mathbf{R})$ into the real numbers. Frechet differentiation with respect to t on the left hand side and on the right hand side of equation (24) yields

$$-e^{-\rho T} u(c(t)) = \frac{d}{dt} V(t, f(t,\cdot)) = \partial_t V(t, f(t,\cdot)) + D_f V(t, f(t,\cdot)) f_t(t,\cdot),$$

where $D_f V(t, f(t,\cdot))$ denotes the Frechet derivative of V at $(t, f(t,\cdot))$ with respect to the second argument. By applying (25) we obtain the evolution equation

$$0 = \partial_t V(t, f(t,\cdot)) + D_f V(t, f(t,\cdot)) \mathcal{A}_t f(t,\cdot) + e^{-\rho t} u(c(t))$$

with operator $\mathcal{A}_t$ given by

$$\mathcal{A}_t h(x) = -\partial_x \Big(\big(xr + x(\alpha - r)\pi + a(t) - c\big) h(x) \Big) + \frac{1}{2} \partial_{xx} \big(x^2 \sigma^2 \pi^2 h(x) \big).$$

Since $g(T,x) = u(x)$, we can derive the terminal condition

$$V(T, f(T,\cdot)) = \int e^{-\rho T} u(x)\, f(T,x) dx.$$

Following the Hamilton-Jacobi-Bellman concept, we have to solve this evolution equation for all possible state variables $f(t,\cdot)$ simultaneously, and we have to (locally) minimize the derivative $\partial_t V$ in order to maximize the value function V. Here, the resulting Hamilton-Jacobi-Bellman equation has the form

$$\begin{aligned} 0 &= \partial_t V(t,h) + \max_{\pi,c} \big\{ D_f V(t, f(t,\cdot)) \mathcal{A}_t h + e^{-\rho t} u(c) \big\}, \\ V(T,h) &= \int e^{-\rho T} u(x)\, h(x) dx. \end{aligned}$$

However, solving this evolution equation is extremely difficult, and it is not clear whether a solution exists at all. Therefore we recommend to use the Pontryagin maximum principle.

7 Numerical methods and results and conclusion

Here we discuss the numerical calculation of deterministic optimal controls.

Lemma 6 *If $(\pi, c) \in D$, then the Kolmogorov forward equation*

$$\begin{aligned} \partial_t F(t,x) &= -\partial_x\Big(\big(xr + x(\alpha - r)\pi(t) + a(t) - c(t)\big)F(t,x)\Big) + \frac{1}{2}\partial_{xx}\big(x^2\sigma^2\pi(t)^2 F(t,x)\big) \\ F(0,x) &= \mathbf{1}_{x \geq x_0} \end{aligned} \tag{25}$$

for the wealth process X has a unique weak solution.

Proof. The existence of a weak solution follows from Theorem 2.5 in Manita and Shaposhnikov (2015). Suppose there are two weak solutions F and $\tilde{F}$ of the partial differential equation. Then the difference $F - \tilde{F}$ is a weak solution of the same partial differential equation but with initial condition $F(0,x) - \tilde{F}(0,x) = 0$ for all $x \in \mathbf{R}$. Theorem 2.7 in Manita and Shaposhnikov (2015) says that we necessarily have $F(t,x) - \tilde{F}(t,x) = 0$ for all $t > 0$, i.e. F and $\tilde{F}$ are equivalent. ■

Formulas (18) and (19) give useful characterizations of optimal controls. In order to find numerical solutions, we read (18) and (19) as fixed-point equations and do the following iterative calculations:

1. We choose a starting investment strategy $\pi^{(0)}$ and a starting consumption rate $c^{(0)}$.
2. On the basis of $\pi^{(0)}$ and $c^{(0)}$ we solve the Kolmogorov forward equation (25) to obtain $F^{(0)}(t,x)$ and we solve the Kolmogorov backward equation (11) to obtain $g^{(0)}(t,x)$.
3. We define a new investment strategy $\pi^{(1)}$ and consumption rate $c^{(1)}$ via the equations (18) and (19).
4. By repeating the steps 2 and 3 iteratively, we obtain the sequence $(\pi^{(j)}, c^{(j)})$, $j = \{0, 1, 2, ...\}$.

For the following numerical examples, we have implemented this fixed-point approach. If the sequence $(\pi^{(j)}, c^{(j)})$, $j = \{0, 1, 2, ...\}$ converges uniformly to a limit, then we have a likely candidate for an optimal control strategy. We cannot exactly proof that it is indeed optimal. In the numerical examples below, we repeated the algorithm for many different starting values. We always ended up with the same limit, which makes it quite likely that our numerical result is in fact optimal.

Consider an employee who is $T = 20$ years away from retirement and has a constant income of $a(t) = 100$ (the reader could think of e.g. the unit thousand Euro) and a present wealth of $x_0 = 200$. Assume that the bank account yields interest with rate $r = 0.04$ and that the stock fund process has the parameters $\alpha = 0.06$ and $\sigma = 0.2$. (With these parameters a log investor with no income would place 50% of his current wealth into stocks.)

Example 7 *Suppose that the employee needs at least* 70% *of his income to live a decent life and is obliged to save at least* 10% *of his income for his post-retirement life. Hence, the employee's* *consumption rate is bounded by*

$$\begin{aligned} \overline{c} &= 90\% \cdot a = 90 \\ \underline{c} &= 70\% \cdot a = 70. \end{aligned}$$

The preference parameter of consuming today instead of tomorrow is set to $\rho = 0.1$*, and we consider three different values of gamma,* -2*,* -1*, and* -0.85*.*

Applying the iteration method above leads in Example 7 to an optimal consumption rate of $c^*(t) = 90$ and optimal investment rates $\pi^*(t)$ as shown in Figure 1. We tried different starting

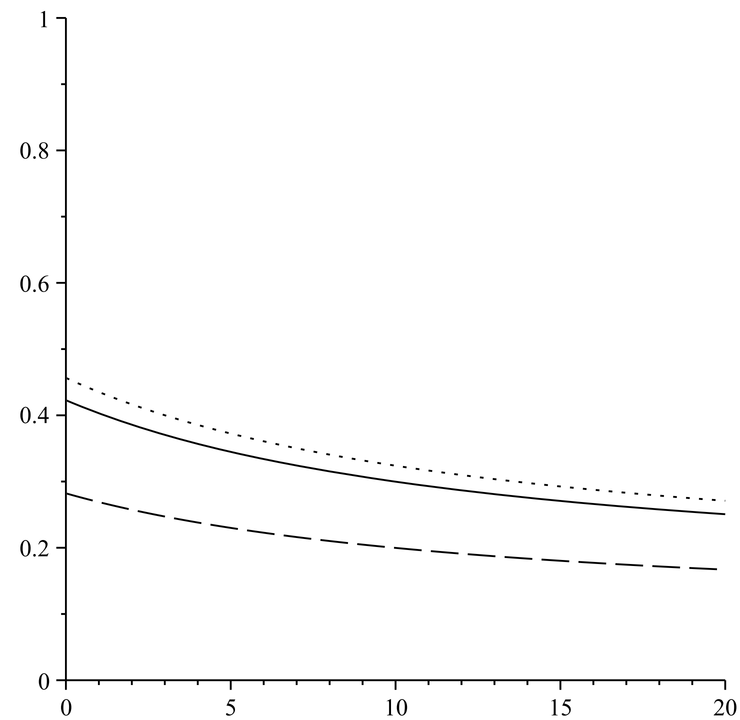

Figure 1: Optimal investment rate $\pi^*(t)$ in Example 7 for $\gamma = -2$ (dash), -1 (solid), -0.85 (dot)

values for the iteration and always obtained the same limit. This is a strong indication that the numerical solution is indeed a global maximum.

Following the discussion in Section 2, we now study examples where the consumption rate is fixed and only the investment is controlled.

Example 8 *We take the same parameters as in Example 7 but* $\gamma = -1$ *and*

$$\overline{c} = c(t) = \underline{c} = 0, 50, 80, 100.$$

Figure 2 shows the optimal investment rates $\pi^*(t)$ for Example 8. Note that $c(t) = 100$ means that the human capital $h(t)$ is zero, and thus the optimal stochastic control (4) is deterministic and equals the optimal deterministic control (equals 1/4).

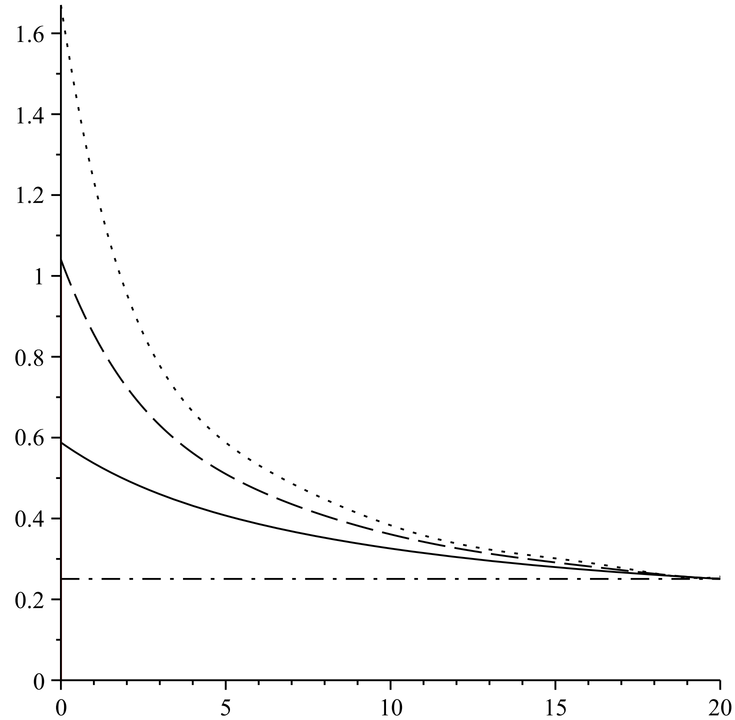

Figure 2: Optimal investment rate $\pi^*(t)$ in Example 8 for consumption rate $c(t) = 0$ (dot), 50 (dash), 80 (solid), 100 (dashdot)

In Figure 3 we compare the optimal deterministic investment strategy for fixed consumption $c(t) = 0$ with the suboptimal approximations (6) and (9). Although the human capital is relatively large for $c(t) = 0$, the optimal, suboptimal and local cost minimizing investment strategies are amazingly similar.

Table 1 shows the increase in initial capital x_0 that we need for the suboptimal strategies in order to achieve the same utility as for the optimal strategy. We see that the losses due to acting suboptimal are very small. Table 1 also shows the decrease in initial capital needed if we wanted the same utility as we get from the optimal deterministic strategy, but had access to stochastic investment. The monetary gain from using stochastic investment control is astonishingly small.

8 Conclusion

We conclude by emphasizing the key output of the paper. Standard stochastic control theory gives a state (wealth) dependent solution to the consumption-investment problem. Still, standard marketed products offer a deterministic life-cycle investment strategy. We derive a representation of the best of this kind. In terms of certainty equivalents, this best deterministic strategy performs very well for realistic parameters. But so do two much simpler deterministic sub-optimal alternatives. Given the efforts it takes to deal with the optimal solution, it may be appropriate to implement

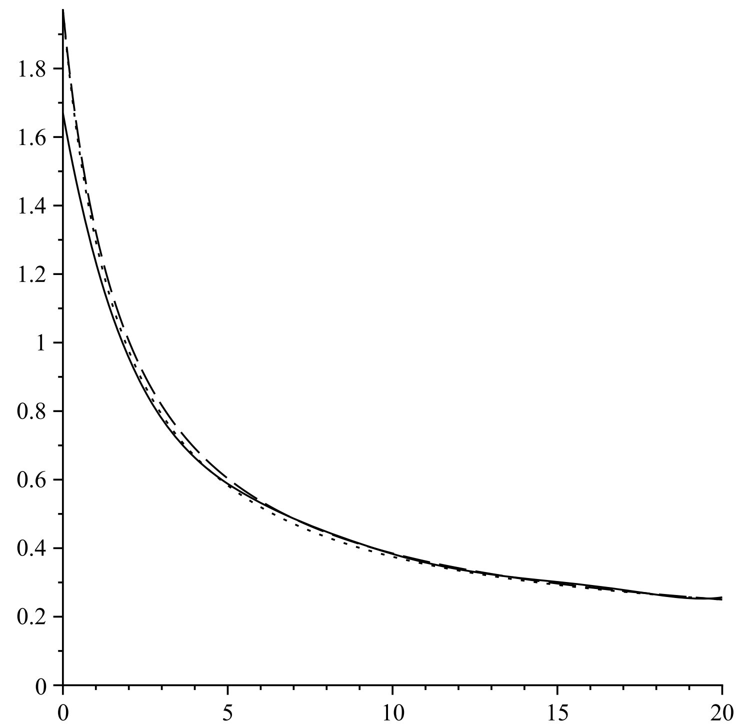

Figure 3: Optimal investment rate (solid), suboptimal strategy 1 (dash) and suboptimal strategy 2 (dot) in Example 8 for $c(t) = 0$

the suboptimal alternatives. Given the simplicity of these non-linear profiles of the growth asset proportion as function of age, they may be good alternatives to the more standard linear profiles. In particular, note the 'rule-of-thumb' style of e.g. π^1. Invest the preference dependent Merton proportion multiplied by a ratio which is simply the present value of total savings divided by the present value of past savings. Such a rule-of-thumb could even work well for much more realistic situations with stochastic investment environment and non-hedgeable income (in case one would have to estimate the present value of total savings). Such an analysis could be the topic for future research based on the ideas and suboptimal strategies presented in this paper.

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savings rate: $a(t) - c(t)$	optimal stochastic control: equivalent x_0	suboptimal strategy 1: equivalent x_0	suboptimal strategy 2: equivalent x_0
0	200.00 -0.00	200.00 $+0.00$	200.00 $+0.00$
20	200.00 -0.46	200.00 $+0.01$	200.00 $+0.01$
50	200.00 -1.29	200.00 $+0.02$	200.00 $+0.02$
100	200.00 -2.85	200.00 $+0.18$	200.00 $+0.13$

Table 1: Equivalent initial wealth with respect to the optimal deterministic investment in Example 8

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9 Appendix: tools and proofs

Lemma 9 *For all $(\pi, c) \in D$, $k > 0$, and $0 \leq s < t \leq T$, we have*

$$E\Big[e^{\left(k\int_s^t dU\right)_-}\Big] \leq E\Big[e^{k\int_s^t dU}\Big] = \exp\Big\{k\int_s^t \Big(r + (\alpha - r)\pi(u) + \frac{1}{2}(k-1)\sigma^2\pi(u)^2\Big)\,du\Big\},$$

$$E\Big[e^{-\left(k\int_s^t dU\right)_-}\Big] \leq E\Big[1 + e^{-k\int_s^t dU}\Big] = 1 + \exp\Big\{k\int_s^t \Big(-r - (\alpha - r)\pi(u) + \frac{1}{2}(k+1)\sigma^2\pi(u)^2\Big)\,du\Big\},$$

where $(\cdot)_-$ denotes the negative part of an argument.

Proof. The formula follows from the fact that $k\int_s^t dU$ is normally distributed with an expectation and a variance of

$$k\int_s^t \Big(r + (\alpha - r)\pi(u) - \frac{1}{2}\sigma^2\pi(u)^2\Big)\,du \quad \text{and} \quad k^2\int_s^t \sigma^2\pi(u)^2 du,$$

respectively. ■

Lemma 10 *Let $0 \leq t \leq \tau \leq T$ and let M be a set of investment strategies and consumption rates (π, c) that is uniformly bounded with respect to the supremum norm.*

(a) If $k \leq 0$, then $\sup_M E[|X^{t,x}(\tau)|^k] \leq x^k C$ for some $C = C(k) < \infty$,

(b) If $k > 0$, then $\sup_M E[|X^{t,x}(\tau)|^k] \leq (1 + x^{2n})C$ for some $C = C(k) < \infty$, where n is the smallest positive integer for which $k \leq 2n$.

Proof. Suppose that $k \in [0, 2]$. Then $|y|^k \leq 1 + y^2$, so we just need to consider the case $k = 2$, which follows from Theorem 4.5.4 in Kloeden and Platen (1999). Now let $k \leq 0$. Since (2) and $a(s) - c(s) \geq 0$ imply that

$$0 \leq E\left[|X^{t,x}(T)|^k\right] \leq E\left[x^k e^{k\int_t^T dU}\right],$$

it suffices to show that the random variables $\exp\{k\int_t^T dU\}$ are uniformly integrable. Indeed, we can show uniform integrability by applying Lemma 9 and using the fact that the (π, c) are uniformly bounded with respect to the supremum norm. ■

Lemma 11 *For each integer $k \geq 0$, $(t, x) \in (0, T] \times \mathbf{R}$, and $(\pi, c) \in D$ the term that*

$$E\Big[u^{(k)}(X^{t-\epsilon,x}(T))e^{k\int_{t-\epsilon}^T dU}\Big] + \mathcal{O}(\epsilon^2)$$

equals the sum of (27), (28), and (29). Here, $u^{(k)}(x)$ is the k-th derivative of the power utility function $u(x)$. Furthermore,

$$E\Big[u^{(k)}(X^{t,x+\epsilon}(T))e^{k\int_t^T dU}\Big] = E\Big[u^{(k)}(X^{t,x}(T))e^{k\int_t^T dU}\Big] + \epsilon\, E\Big[u^{(k+1)}(X^{t,x}(T))e^{(k+1)\int_t^T dU}\Big] + \mathcal{O}(\epsilon^2).$$

Proof. From Taylor's theorem we know that

$$u^{(k)}(y) = u^{(k)}(y_0) + u^{(k+1)}(y_0)(y - y_0) + \frac{1}{2}u^{(k+2)}(y_0)(y - y_0)^2 + \frac{1}{6}u^{(k+3)}(\xi)(y - y_0)^3$$

for some ξ between y_0 and y. By identifying y and y_0 with $X^{t-\epsilon,x}(T)$ and $X^{t,x}(T)$ such that

$$y - y_0 = xe^{\int_t^T dU}\Big(e^{\int_{t-\epsilon}^t dU} - 1\Big) + \int_{t-\epsilon}^t (a(s) - c(s))e^{\int_s^T dU} ds =: Z_1 + Z_2,$$

we obtain

$$\begin{aligned} u^{(k)}(X^{t-\epsilon,x}(T)) =& u^{(k)}(X^{t,x}(T)) + u^{(k+1)}(X^{t,x}(T))(Z_1 + Z_2) \\ &+ \frac{1}{2}u^{(k+2)}(X^{t,x}(T))(Z_1 + Z_2)^2 + \frac{1}{6}u^{(k+3)}(\Xi)(Z_1 + Z_2)^3 \end{aligned}$$

for some random variable Ξ that is between $X^{t-\epsilon,x}(T)$ and $X^{t,x}(T)$. So we can write

$$\begin{aligned} u^{(k)}(X^{t-\epsilon,x}(T))e^{k\int_{t-\epsilon}^T dU} =& u^{(k)}(X^{t,x}(T))e^{k\int_{t-\epsilon}^T dU} + u^{(k+1)}(X^{t,x}(T))e^{k\int_{t-\epsilon}^T dU}(Z_1 + Z_2) \\ &+ u^{(k+2)}(X^{t,x}(T))e^{k\int_{t-\epsilon}^T dU} Z_1^2 + R \end{aligned} \tag{26}$$

for some remainder R. Since the increments dU are independent on disjunct intervals, the random variable $\exp\{\int_{t-\epsilon}^t dU\}$ is independent of $X^{t,x}(T)$ and $\int_t^T dU$. Thus, the first addend on the right hand side of (26) has the expectation

$$E\Big[u^{(k)}(X^{t,x}(T))e^{k\int_t^T dU}\Big]E\Big[e^{k\int_{t-\epsilon}^T dU}\Big].$$

By applying Lemma 9 for the second factor and using the fact that $\exp\{y\} = 1 + y + \mathcal{O}(y^2)$ if y is bounded, we obtain the equivalent formula

$$E\Big[u^{(k)}(X^{t,x}(T))e^{k\int_t^T dU}\Big]\Big(1 + k\int_{t-\epsilon}^t \Big(r + (\alpha - r)\pi(u) + \frac{k-1}{2}\sigma^2\pi(u)^2\Big)du + \mathcal{O}(\epsilon^2)\Big). \tag{27}$$

Using similar arguments, the second addend on the right hand side of (26) has the expectation

$$E\Big[u^{(k+1)}(X^{t,x}(T))e^{(k+1)\int_t^T dU}\Big]E\Big[x\Big(e^{(k+1)\int_{t-\epsilon}^t dU} - e^{k\int_{t-\epsilon}^t dU}\Big) + \int_{t-\epsilon}^t (a(s) - c(s))e^{k\int_s^t dU} ds\Big]$$

which, by applying Lemma 9 and using Taylor's expansion for the exponential function, can be rewritten to

$$E\Big[u^{(k+1)}(X^{t,x}(T))e^{(k+1)\int_t^T dU}\Big]\Big(\int_{t-\epsilon}^t \Big(x\big(r + (\alpha - r)\pi(u) + k\sigma^2\pi(u)^2\big) + a(u) - c(u)\Big)du + \mathcal{O}(\epsilon^2)\Big) \tag{28}$$

Analogously, for the third addend on the right hand side of (26) we get an expectation of

$$E\Big[\frac{1}{2}u^{(k+2)}(X^{t,x}(T))e^{(k+2)\int_t^T dU}\Big]\Big(x^2\int_{t-\epsilon}^t \sigma^2\pi(u)^2 du + \mathcal{O}(\epsilon^2)\Big). \tag{29}$$

The expectation of the remainder R is of the form $\mathcal{O}(\epsilon^2)$. To see that, at first notice that $a(s) - c(s) \geq 0$ implies that

$$X^{t,x} \geq xe^{\int_t^T dU} \geq 0 \quad \text{and} \quad X^{t-\epsilon,x} \geq xe^{\int_{t-\epsilon}^T dU} \geq 0,$$

so that $\Xi = \theta X^{t,x} + (1-\theta)X^{t-\epsilon,x}$ for some random variable $0 \le \theta \le 1$ has a lower bound of

$$\Xi \ge x e^{\int_t^T dU}\Big(\theta + (1-\theta)e^{\int_{t-\epsilon}^t dU}\Big) \ge x e^{\int_t^T dU} e^{(\int_{t-\epsilon}^t dU)_-} =: Z_3,$$

where $(\cdot)_-$ denotes the negative part of a term. As $u^{(k+3)}(y)$ is proportional to $y^{-\delta}$ for some $\delta > 2$, the absolute of $u^{(k+3)}(\Xi)$ has an upper bound that is proportional to the absolute of $u^{(k+3)}(Z_3)$. That means that the absolute of R has an upper bound of the form

$$|R| \le \frac{1}{2}|u^{(k+2)}(X^{t,x})| e^{k\int_{t-\epsilon}^T dU}\big(2Z_1Z_2 + Z_2^2\big) + \frac{1}{2}|u^{(k+3)}(Z_3)| e^{k\int_{t-\epsilon}^T dU}\big(Z_1 + Z_2\big)^3.$$

Now we take the expectation on both hand sides and apply Hölder's inequality several times in order to obtain

$$\begin{aligned} E[|R|] \le &\frac{1}{2}E\Big[(u^{(k+2)}(X^{t,x}))^4\Big]^{1/4} E\Big[e^{4k\int_{t-\epsilon}^T dU}\Big]^{1/4} E\Big[\big(2Z_1Z_2 + Z_2^2\big)^2\Big]^{1/2} \\ &+ \frac{1}{2}E\Big[(u^{(k+3)}(Z_3))^4\Big]^{1/4} E\Big[e^{4k\int_{t-\epsilon}^T dU}\Big]^{1/4} E\Big[\big(Z_1^3 + 3Z_1^2Z_2 + 3Z_1^1Z_2^2 + Z_2^3\big)^6\Big]^{1/2}. \end{aligned}$$

The first two expectations in each line have finite upper bounds because of Lemma 10 and Lemma 9, using the fact that $u^{(k+3)}(Z_3)^4$ is proportional to the indendent factors $x^{\gamma-k-3}$, $e^{4(\gamma-k-3)\int_t^T dU}$, and $e^{4(\gamma-k-3)(\int_{t-\epsilon}^t dU)_-}$. The other two expectations are of order $\mathcal{O}(\epsilon^2)$. To see that, first expand the polynomials, then split the factors into independent sub factors that correspond to the intervals $[t-\epsilon, t]$ and $[t, T]$, and take the expectations. For the factors corresponding to $[t-\epsilon, t]$ we get with Lemma 9 and Taylor's expansion for the exponential function the order $\mathcal{O}(\epsilon^2)$. The factors corresponding to $[t, T]$ have finite upper bounds, which can be verified with the help of Lemma 9. So, all in all, obtain that $E\Big[u^{(k)}(X^{t-\epsilon,x}(T))e^{k\int_{t-\epsilon}^T dU}\Big]$ equals the sum of (27), (28), (29), and $\mathcal{O}(\epsilon^2)$.

The proof for the expansion of $E\Big[u^{(k)}(X^{t,x+\epsilon}(T))e^{k\int_t^T dU}\Big]$ is analogous: First, apply Taylor's expansion of $u^{(k)}$ for $y - y_0 = \epsilon\, e^{\int_t^T dU}$, and then show that the expectations of the Taylor terms of second and higher order are of order $\mathcal{O}(\epsilon)$. ∎

Proof of Proposition 2. By applying Lemma 11, we get for $g(t,x) = E\big[u(X^{t,x}(T))\big]$ that the partial derivative $\partial_t g(t,x)$ exists and has the form

$$\begin{aligned} \partial_t g(t,x) &= \lim_{\epsilon \downarrow 0} \frac{g(t,x) - g(t-\epsilon, x)}{\epsilon} \\ &= E\Big[u'(X^{t,x}(T))e^{\int_t^T dU}\Big]\Big(x(r + (\alpha - r)\pi(t)) + a(t) - c(t)\Big) \\ &\quad + E\Big[u''(X^{t,x}(T))e^{2\int_t^T dU}\Big]x^2\sigma^2\pi(t)^2. \end{aligned} \tag{30}$$

Applying Lemma 11 again, but this time on $E\Big[u'(X^{t,x}(T))e^{\int_t^T dU}\Big]$ and $E\Big[u''(X^{t,x}(T))e^{2\int_t^T dU}\Big]$, we can see that $\partial_t g(t,x)$ is indeed continuous in t and x. The existence and continuity of $\partial_x^k g(t,x)$ and its representation (12) follows by induction in $k = 0, 1, 2, \ldots$, where in each step the expansion according to Lemma 11 for $\partial_x^k g(t, x+\epsilon)$ is used.

By plugging (12) for $k = 1, 2$ into (30), we finally arrive at the Kolmogorov backward equation (11). The terminal condition follows from $g(T,x) = E[u(X^{T,x}(T))] = E[u(x)] = u(x)$. ∎