The Influence of FX Risk on Credit Spreads

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Abstract

We analyze the connections between the credit spreads that the same credit risk commands in different currencies. We show that the empirically observed differences in these credit spreads are mostly driven by the dependency between the default risk of the obligor and the exchange rate. In our model there are two different channels to capture this dependence: First, the diffusions driving FX and default intensities may be correlated, and second, an additional jump in the exchange rate may occur at the time of default. The differences between the default intensities under the domestic and foreign pricing measures are analyzed and closed-form prices for a variety of securities affected by default risk and FX risk are given (including CDS). In the empirical part of the paper we find that a purely diffusion-based correlation between the exchange rate and the default intensity is not able to explain the observed differences between JPY and USD CDS rates for a set of large Japanese obligors. The data implies a significant additional jump in the FX rate at default.

1 Introduction

In modern debt markets, many large debtors issue debt in more than one currency, e.g. a large Japanese obligor may find it advantageous to issue debt in USD, or a European obligor in JPY. Furthermore, since the advent of liquid markets for credit default swaps (CDS) there are markets for credit protection in currencies different from the obligors "home" currency, even if the obligor has not issued bonds in that currency. (There is

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demand for this in order to hedge loan exposures or OTC derivatives transactions.) In particular, CDS protection on many international corporations is now available in their home currency and EUR and USD.

Given this situation, it is natural to ask about the correct *relative pricing* of the credit risk in the different currencies, i.e.: How should the credit spread be adjusted (either on bonds or on CDS) if a different currency is used? And: What information about likely FX movements in crisis events can we imply from the relative difference of CDS spreads in different currencies?

We aim to answer this question in two steps. First, we analyse the connections between local and foreign currency credit spreads on a theoretical basis in an intensity-based framework and highlight the effects that we can expect to encounter. We find that the essential feature driving differences between credit spreads in different currencies is the dependency between default risk and FX risk. If default risk and FX risk are independent (in a sense which will be made precise later on), credit spreads in different currencies should not differ.

In order to capture empirically observed differences, we model dependency between spreads in two different ways. First, there may be correlation between the diffusions driving FX and default intensities, and second, an additional jump in the exchange rate may occur at the time of default, i.e. the default "causes" a devaluation of the currency. We give closed-form solutions for CDS rates and defaultable bond prices in a model which encompasses both cases using an affine jump-diffusion (AJD) model.

In the empirical part of the paper these models are estimated using a historical database of CDS rates in Japanese yen (JPY) and US dollars (USD) on a set of major Japanese corporate obligors. Besides being relatively clean, liquid and standardised, CDS data has the additional advantage that the recovery rates on USD-denominated and JPY-denominated CDS will be identical by definition (*by documentation*, to be more precise). Thus, spread differences in CDS rates cannot be caused by the effects of different legal regimes and bond specification which may affect corporate bond data.

In the pure diffusion hypothesis (i.e. without jumps in the FX rate), we first estimate the model parameters using USD CDS spreads and the JPY/USD rate, without using the JPY CDS spread. In the second step, we then calculate the JPY CDS spread which would hold if the model were correct and compare it to the empirically observed JPY CDS spreads. In all cases, we can strongly reject the hypothesis that the empirically observed JPY CDS spreads are noisy observations of the model predictions, the predicted spread difference is only a small fraction of the observed spread difference. Consequently, we reject the pure diffusion model, there must be jumps in the exchange rate at default.

In many cases, the implied jump in FX rates at default of the obligors seems quite large and is difficult to explain unless we assume that the obligor's default was caused by a major macroeconomic crisis. The techniques used in this paper are also useful for a number of other applications like the pricing of counterparty risk and the pricing of sovereign default risk. Some of these applications are also pointed out in the final section, and we give prices for some more exotic FX-related credit derivatives.

Sparked by the Asian crisis in the late 1990's (and the Peso crisis earlier on) there is a large literature on sovereign default risk, banking crises and currency crises in emerging markets[1]. While some of the techniques used in the present paper can also be applied to these situations, this paper has a different focus than the domestic debt of a sovereign obligor which is special because (at least in theory) the sovereign could always repay that debt by printing more domestic currency. Similarly, this paper also differs in focus from papers which empirically investigate sovereign credit spreads, like e.g. Duffie et al. (2003) and Singh (2003).

In this paper our main focus are multinational corporations which face a quite different situation from sovereigns. Here, the foreign sovereign often has negligible default risk compared to the corporate (unless it is an emerging market), and the exchange rate floats freely. Furthermore, the available data is significantly different: CDS on large corporations are routinely quoted in the major currencies (USD, EUR, and also JPY, GBP, CHF). Despite this focus on corporate risk, the techniques of this paper can be used to back out market implied information about the sovereign, in particular values for the expected currency devaluation upon a default of the corporate. The same criteria apply to the CDS spreads of sovereigns in different *foreign* (to the sovereign) currencies (e.g. USD and EUR-denominated CDS on Brazil) which also fall within the scope of this paper. Finally, it is not difficult to extend the model to cover corporates which are based in a country with non-negligible default risk.

In a related paper Jankowitsch and Pichler (2003), the authors address the question of the construction of corporate credit spread curves from corporate *bond* prices in different currencies. In their sample, the authors find strong evidence against the assumption of independence of corporate bond credit spreads and exchange rates. Our paper differs from Jankowitsch and Pichler (2003) in several respects: First, we provide a full theoretical model which is able to capture the stochastic dependency between default intensities and exchange rates and to replicate the observed spread differences. Second, we base our analysis on CDS which are significantly better suited to the empirical and theoretical analysis of these questions as it avoids the issues caused by differences in recovery rates in different currencies.
Another related paper is Warnes and Acosta (2002) who extend the classical Merton (1974) firm's value approach to incorporate debt in a foreign currency and provide closed-form solutions for debt prices under the assumption of constant interest-rates in both countries.

The rest of the paper is structured as follows: To set the stage for the rest of the paper, we recapitulate in the next section the payoff mechanics of credit default swaps (CDS) and show that the delivery option in the protection leg of the CDS makes the effective recovery rate currency-independent. In section 3 we then set up the mathematical background to the general FX model under default-risk which follows in section 4. To provide a concrete specification for the empirical estimation in section 5, we also specify an affine jump-diffusion (AJD) version of the model in sections 3 and 4. Furthermore, section 4 contains the presentation of change-of-measure techniques that apply to the valuation of *payoffs at stopping times* when the numeraire asset jumps at this stopping time, an analysis of the the relationship between the default intensities of the obligor under the domestic and the foreign pricing measures, and some concrete pricing results for basic defaultable securities (zero-coupon bonds and CDS) under a variety of recovery assumptions. As our focus is

[1]See e.g. Kaminsky and Reinhart (1999), Reinhart (2002), Bulow and Rogoff (1989) or other papers listed on N. Roubini's `http://www.stern.nyu.edu/globalmacro/`

on the difference between domestic and foreign CDS rates, we discuss in section 4.3 the effects which we expect to influence this quantity.
Section 5 contains an empirical analysis of the AJD-model using historical CDS data on a number of large Japanese obligors. We show that there is a persistent, significant and rather large difference between CDS rates in JPY and USD which cannot be explained by a purely diffusion-based dependency between default intensity and FX rate alone. Thus, we conclude that the market must be pricing an implicit devaluation at default into these CDS spreads. Finally, in section 6 the empirical results are discussed and it is shown how the techniques introduced in sections 3 and 4 can be applied to other default-sensitive FX derivatives.

2 CDS in Multiple Currencies

Credit default swaps (CDS) are derivative instruments which allow the trading of payoffs contingent on the occurrence of a *credit event.* Single-name CDS are the most important class of credit derivatives transactions, Patel (2003) finds that in 2003 they accounted for around 72.5% of the entire credit derivatives market in terms of notional outstanding which was equivalent to a notional of around 1'671 bn USD.

In many cases, the liquidity of the CDS market has surpassed the liquidity of the market for the bonds of the underlying obligor. This trading volume and liquidity has been made possible by the standardisation of the documentation for CDS transactions which has been proposed by the International Swap Dealers Association (ISDA) (see `http://www.isda.org` and (ISDA) (1999)). In particular when it comes to the analysis of the default risk of any given obligor in two or more different currencies (and thus in two different jurisdictions) this standardisation is essential: Bonds in domestic and foreign currency are typically issued in different jurisdictions and therefore are governed by different legal rules which has a significant impact on the resulting recovery rates of the bonds (see e.g. Davydenko and Franks (2004)). CDSs referencing the obligor on the other hand will be governed by the same standardised ISDA documentation even if they are denominated in different currencies, in particular they will have the same recovery rates. Thus, for the purposes of this paper we consider CDS to be more standardised and more easily comparable than the underlying corporate bonds.

We now present a quick summary of the payoff mechanics of an ISDA-standard CDS with physical settlement in order to explain why the recovery rate of a CDS is typically independent from the currency of its denomination:

Being an over-the-counter traded derivative, a CDS is a contract between two counterparties: the protection buyer and the protection seller. The protection buyer makes the payments of the *fee leg* of the CDS, the protection seller pays the *protection leg.* In order to define these payment streams, the following data is specified in every CDS:

- the *notional amount* N, and the currency c of the notional amount,
- the *maturity* date T,
- the *CDS rate* $\overline{s}$,
- the *reference credit* (i.e. the obligor whose credit risk is traded)
- the applicable (precise) definition of the *credit event*, and

- the *set of deliverable obligations.*

It is important to note that both the definition of the credit event and the list of the deliverable obligations usually do *not* depend on the currency of the CDS. The ISDA-definition of a credit event includes bankruptcy, failure to make due payments on bonds or loans ("failure to pay"), repudiation or moratorium, cross-acceleration, obligation default, distressed restructuring and credit events upon mergers. These events apply globally to the reference obligor and are in most cases objectively verifiable and independent from local legal rules. The set of deliverable obligations contains most bonds and loans issued by the reference credit irrespective of their currency, excluding special cases such as subordinated bonds or bonds with unusual maturity dates but including all major bond issues.

The Fee Leg. The protection buyer makes fee payments to the protection seller at regular intervals until the CDS matures or until a credit event occurs. The fee payments are made in the currency of the CDS and are calculated as [daycount fraction]×[CDS rate]×[Notional] $= \Delta t \cdot \overline{s} \cdot N$.

The Protection Leg. At the credit event, the protection buyer chooses a portfolio from the set of deliverable obligations such that the total notional amount of the portfolio is N. If any obligations in the portfolio are denominated in other currencies than the CDS reference currency c, then the notional amount of these obligations is converted into the reference currency using the actual exchange rate of the day. The protection buyer then delivers this portfolio to the protection seller who has to pay the full notional N for it.

Clearly, although they will trade at a significant discount to par, not all deliverable obligations will trade at the same price. The protection buyer has a *delivery option:* The protection buyer will choose to deliver those bonds which trade at the highest discount to their par value, the *cheapest-to-deliver* bonds. Interestingly, the choice of the cheapest-to-deliver bond is *independent* of the currency in which that bond is denominated, it only depends on the relative discount of the bond to its par value.

To illustrate this let us assume the bond that we want to deliver trades at a discount of q, i.e. at a price of $(1-q)$ per unit 1.00 of notional in its currency $\tilde{c} \neq c$. If one unit of $\tilde{c}$ is worth X units of c at the time of the credit event, then in order to reach a portfolio of notional N in currency c, we have to buy a portfolio of notional N/X in the bond's currency $\tilde{c}$. Thus the delivery portfolio costs $(1-q) \cdot X \cdot N/X = (1-q) \cdot N$ in the CDS's currency. This portfolio is put to the protection seller for a payment of N in c which yields a net value of the protection payment of $N \cdot q$ in c.

This value does not depend on the exchange rate X any more. Thus, in order to maximise the value of the protection payment, the protection buyer will choose to deliver a portfolio of those bonds which have the lowest $(1-q)$, irrespective of the exchange rate for the currency of denomination of this bond. In particular, *the effective recovery rate for the CDS will be independent from the currency in which the CDS is denominated.*

This does not mean that CDS which are denominated in different currencies are identical. Consider two CDS on the same reference credit with the same deliverable obligations but denominated in different currencies c (with notional N), and $\tilde{c}$ (with notional $\tilde{N}$). At default, the first CDS will pay off $N \cdot q$ in currency c, and the other will pay off $\tilde{N} \cdot q$ in currency $\tilde{c}$. Thus, the *amount* of protection that the $\tilde{c}$-CDS provides in currency c

depends on the exchange rate at the time of default. This is the relationship which we are going to explore in this paper.

3 Mathematical Framework

3.1 Set-Up

Our model is set in a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{Q})$ with finite time horizon $T^* < \infty$. The filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T^*]}$ is assumed to satisfy the usual conditions, $\mathcal{F}_{T^*} = \mathcal{F}$ and is generated by an N-dimensional Brownian motion (BM) W_t and a K-dimensional purely discontinuous process J_t with jumps $\Delta J_t \in \mathcal{Z} = (0,1]^K$. We represent J_t using its associated jump measure $\mu(dt, dz)$ on $[0,T^*] \times \mathcal{Z}$ as $J_t := \int_0^t \int_{\mathcal{Z}} z \cdot \mu(dz, ds)$; the compensator measure of $\mu(dz, dt)$ under $\mathbb{Q}$ is denoted with $\nu(dz, dt)$. We assume $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{Q})$ has the predictable representation property with respect to W and $\mu - \nu$. The transpose of a matrix M is denoted by M', and if x is a vector, then $\mathrm{diag}(x)$ is a diagonal matrix with the elements of x on its diagonal. Standard arithmetical functions, integrals and comparisons of vectors are meant componentwise, except in the case of multiplications we will use matrix multiplications. l denotes the Lebesgue measure on $\mathbb{R}$. Regarding time points we always assume $0 \le t \le T \le T^*$.

Assumption 1.

(i) $N_t := \int_0^t \int_{\mathcal{Z}} \mu(dz, ds)$ *is a counting process, i.e.* $N_t < \infty$ $\mathbb{Q}$*-a.s. for all* t*. We denote the time of the first jump by the stopping time*

$$\tau := \inf\{t; N_t > 0\} \tag{1}$$

(with the convention $\inf \emptyset = \infty$*).*

(ii) There exist a (nonnegative) $\mathbb{F}^W$*-adapted (hence predictable) processes* λ *and an* $\mathbb{F}^W$*-adapted (predictable) function*[2] F *on* $\mathcal{Z}\times[0,T^*]$ *such that the predictable compensator* ν *of* μ *satisfies*

$$\mathbf{1}_{\{t\le\tau\}}\nu(dz, dt) = \mathbf{1}_{\{t\le\tau\}} dF_t(z)\lambda_t dt$$

and F_t *is a distribution function on* $\mathcal{Z}$*.*

From this assumption follows that τ is a totally inaccessible stopping time, and $\int_0^{t\wedge\tau} \lambda_s ds$ is the predictable compensator of $\mathbf{1}_{\{t<\tau\}}$, i.e. $1_{\{t\le\tau\}}\lambda_t$ is the predictable intensity of $\mathbf{1}_{\{t<\tau\}}$. Note that we do not make any assumption about the form of the predictable compensator after τ apart from $\int_0^t \int_{\mathcal{Z}} \nu(dz, ds) < \infty$ $\mathbb{Q}$-a.s. for all t.[3]

We define a family of conditional Laplace transforms $\mathcal{L}_t : \mathbb{R}_+^K \to \mathbb{R}_+$ indexed by $t \in [0,T^*]$

$$\mathcal{L}(t;\cdot):\ u \longmapsto \int_{\mathcal{Z}} \mathbf{1}_{\{t\le\tau\}}(1-z)^u dF_t(z) := \int_{\mathcal{Z}} \mathbf{1}_{\{t\le\tau\}}\Big(\prod_{k=1}^K (1-z_k)^{u_k}\Big) dF_t(z)$$

[2]See e.g. Jacod and Shiryaev (1988) for the definition of predictable functions.

[3]This is a consequence of $N_{T^*} < \infty$ $\mathbb{Q}$-a.s.

suppressing the dependence on $\omega \in \Omega$. Intuitively, $\mathcal{L}$ is the Laplace transform of the jump $\log(1-\Delta J_t)$ conditional on the event $\{\tau = t\}$. Sometimes we will use a version of $\mathcal{L}$ extended to all values $u \in \mathbb{R}^K$ with $\mathcal{L}(t;u) < \infty$ $\mathbb{Q} \times l$-a.s.

Further we define a continuous time-homogeneous N-dimensional Markov process Y as the unique strong solution of a stochastic differential equation (SDE)

$$dY_t = \gamma(Y_t)dt + \sigma(Y_t)dW_t, \qquad Y(0) = Y_0 \tag{2}$$

with affine functions $\gamma(\cdot)$ and $\sigma\sigma'(\cdot)$.[4] The process Y is called an *affine diffusion* (AD). We say, the process $W + J$ is a *jump diffusion* (JD); and, if Y is an AD, λ is affine in Y and $\mathcal{L}(t;u)$ is exponentially affine[5], in Y_t for every u then we call $Y + J$ an *affine jump diffusion* (AJD).

3.2 Hypothesis $\mathbb{H}$ and its Implications

In the set-up we only assumed $\mathbb{F} = \mathbb{F}^W \vee \mathbb{F}^J$. In addition we assume that hypothesis

$\mathbb{H}$: *Every $\mathbb{F}^W$-martingale is an $\mathbb{F}$-martingale.*

is satisfied. See Jeanblanc and Rutkowski (2000) for details on hypothesis $\mathbb{H}$. In presence of hypothesis $\mathbb{H}$ the computation of the conditional expectation of a *"defaultable claim"*, i.e. an $\mathcal{F}_T$-measurable random variable Z with $Z\mathbf{1}_{\{\tau \le T\}} = 0$, can be reduced to a conditional expectation of a related $\mathcal{F}_T^W$-measurable random variable. The following lemma gives the precise result.

Lemma 2. *Let g_T be $\mathcal{F}_T^W$-measurable with $\mathbf{E}[\,|g_T|\,] < \infty$. If hypothesis $\mathbb{H}$ holds, then*

$$\mathbf{E}\left[\, \mathbf{1}_{\{T<\tau\}} g_T \,\middle|\, \mathcal{F}_t \right] = \mathbf{1}_{\{t<\tau\}} \mathbf{E}\left[\, e^{-\int_t^T \lambda_s ds} g_T \,\middle|\, \mathcal{F}_t \right]. \tag{3}$$

It should be noted that in particular $\mathbf{E}\left[e^{-\int_t^T \lambda_s ds} g_T \,\middle|\, \mathcal{F}_t \right] = \mathbf{E}\left[e^{-\int_t^T \lambda_s ds} g_T \,\middle|\, \mathcal{F}_t^W \right]$ by hypothesis $\mathbb{H}$. With Fubinis theorem we immediately derive the corollary below.

Corollary 3. *Let hypothesis $\mathbb{H}$ hold and g be $\mathbb{F}^W$-adapted with $\mathbf{E}\left[\int_0^T |g_s| ds \right] < \infty$. Then*

$$\mathbf{E}\left[\int_t^T \mathbf{1}_{\{s\le\tau\}} g_s \, ds \,\middle|\, \mathcal{F}_t \right] = \mathbf{1}_{\{t<\tau\}} \int_t^T \mathbf{E}\left[e^{-\int_t^s \lambda_u du} g_s \,\middle|\, \mathcal{F}_t \right] ds.$$

And the property of the predictable compensator yields the following result.

Corollary 4. *Let $\mathbb{H}$ hold and f be an $\mathbb{F}^W$-adapted (hence predictable) function with*

$$\mathbf{E}\left[\int_0^T \int_{\mathcal{Z}} \mathbf{1}_{\{t\le\tau\}} |f(z,t)| dF_t(z) \lambda_t dt \right] < \infty. \tag{4}$$

[4] Indeed as shown by Duffie and Kan (1996), under light technical assumptions given in section 3.4, SDE (2) admits a unique strong solution on $(\Omega, \mathcal{F}, \mathbb{Q})$.

[5] I.e. there exist real functions α and β such that $\mathcal{L}(t;u) = e^{\alpha(u)+\beta(u)'Y_t}$.

Define $G_t := \int_{\mathcal{Z}} f(z,t) dF_t(z)$. *Then*

$$\mathbf{E}\left[\int_t^T \int_{\mathcal{Z}} \mathbf{1}_{\{s\le\tau\}} f(z,s)\mu(dz,ds) \,\middle|\, \mathcal{F}_t \right] = \mathbf{1}_{\{t<\tau\}} \int_t^T \mathbf{E}\left[e^{-\int_t^s \lambda_u du} G_s \lambda_s \,\middle|\, \mathcal{F}_t \right] ds.$$

The proofs can be found in the appendix.

3.3 Jump Diffusions under Change of Measure

The distributional properties of the processes Y and J under different probability measures will be at the heart of this article. We give a general form of Girsanov's theorem which is valid for the probability space under consideration. For the proof see e.g. Jacod and Shiryaev (1988).

Theorem 5 (Girsanov's Theorem for JD). *Let* L *be an* $\mathbb{F}$*-martingale under* $\mathbb{Q}$ *with*

$$\frac{dL_t}{L_{t-}} = \phi(t)' dW_t + \int_{\mathcal{Z}} \big(\Phi(z,t) - 1\big)(\mu - \nu)(dz,dt), \qquad L_0 = 1$$

for a predictable process ϕ *and a predictable function* $\Phi \ge 0$*. Then, the probability measure* $\widetilde{\mathbb{Q}}$ *on* $(\Omega, \mathcal{F})$*, defined by*

$$\left.\frac{\mathrm{d}\widetilde{\mathbb{Q}}}{\mathrm{d}\mathbb{Q}}\right|_{\mathcal{F}_t} = L_t, \qquad t \le T^* \tag{5}$$

is absolutely continuous wrt. $\mathbb{Q}$ *(*$\widetilde{\mathbb{Q}} \ll \mathbb{Q}$*) and it holds that:*

(i) The process $\widetilde{W}_t = W_t - \int_0^t \phi(s) ds$ *is a BM under* $\widetilde{\mathbb{Q}}$*.*

(ii) The predictable compensator $\widetilde{\nu}$ *of* μ *under* $\widetilde{\mathbb{Q}}$ *satisfies* $\mathbf{1}_{\{t\le\tau\}}\widetilde{\nu}(dz,dt) = \mathbf{1}_{\{t\le\tau\}} d\widetilde{F}_t(z)\widetilde{\lambda}_t dt$*, where*

$$\widetilde{\lambda}(t) = \lambda(t) \int_{\mathcal{Z}} \Phi(z,t) dF_t(z)$$

and $\widetilde{F}_t$ *is a distribution on* $\mathcal{Z}$ *for all* $t \in [0, T^*]$ *with*

$$d\widetilde{F}_t(z) = \begin{cases} \frac{\Phi(z,t)}{\int_{\mathcal{Z}} \Phi(z,t) dF_t(z)} dF_t(z) & \text{if } \int_{\mathcal{Z}} \Phi(z,t) dF_t(z) > 0, \\ dF_t(z) & \text{otherwise.} \end{cases}$$

Note that by *(ii)*, $\mathbf{1}_{\{t\le\tau\}}\widetilde{\lambda}_t$ is the predictable intensity of $\mathbf{1}_{\{t<\tau\}}$ under $\widetilde{\mathbb{Q}}$. The following corollary is a straightforward implication of Girsanov's theorem.

Corollary 6. *Let* $\widetilde{\mathbb{Q}} \sim \mathbb{Q}$*. Then* $\langle \frac{\mathrm{d}\widetilde{\mathbb{Q}}}{\mathrm{d}\mathbb{Q}}, N \rangle_{t\wedge\tau} = 0$ $\mathbb{Q}$*-a.s. for all* t*, if and only if*

$$\mathbf{1}_{\{t\le\tau\}}\widetilde{\lambda}(t) = \mathbf{1}_{\{t\le\tau\}}\lambda(t) \qquad \mathbb{Q}\times l\text{-a.e.}. \tag{6}$$

I.e. the indicator $\mathbf{1}_{\{t<\tau\}}$ has the same intensity under two equivalent measures $\widetilde{\mathbb{Q}} \sim \mathbb{Q}$ if $\langle \frac{\mathrm{d}\widetilde{\mathbb{Q}}}{\mathrm{d}\mathbb{Q}}, N \rangle_{t\wedge\tau} = 0$ a.s. However, this does not imply that $\widetilde{\nu} = \nu$.

Proof of Corollary 6. Let $L := \frac{d\widetilde{\mathbb{Q}}}{d\mathbb{Q}}$ be as in theorem 5. If $\widetilde{\mathbb{Q}} \sim \mathbb{Q}$, then $L_t > 0$ $\mathbb{Q}$-a.s. for all t. By theorem 5 the predictable covariation of L and $\mathbf{1}_{\{t<\tau\}}$ is given by

$$\langle L, \mathbf{1}_{\{\cdot<\tau\}}\rangle_t = \langle L, N\rangle_{t\wedge\tau} = \int_0^{t\wedge\tau}\int_{\mathcal{Z}} L_{s-}\big(\Phi(z,s)-1\big)\nu(dz,ds) = \int_0^{t\wedge\tau} L_{s-}(\widetilde{\lambda}-\lambda)(s)ds.$$

hence $\mathbf{1}_{\{\,\cdot\,\leq\tau\}}L_-(\widetilde{\lambda}-\lambda) = 0$ $\mathbb{Q}\times l$-a.e. if and only if $\langle L, N\rangle_{t\wedge\tau} = 0$ $\mathbb{Q}$-a.s for all t. and the claim follows. □

In the sequel of the paper we use the abbreviation $\widetilde{\mathbf{E}}[\,\cdot\,] := \mathbf{E}[L_T\,\cdot]$ when taking expectations under $\widetilde{\mathbb{Q}}$. The following lemma states, how the expected value of stochastic integrals wrt. μ acts under a measure $\widetilde{\mathbb{Q}} \ll \mathbb{Q}$.

Lemma 7 (Stochastic Integrals wrt. μ). *Let L and $\widetilde{\mathbb{Q}}$ be as in theorem 5 and f be a predictable function such that $\int_0^T\int_{\mathcal{Z}}|f(z,s)|\mu(dz,ds)$ is $\widetilde{\mathbb{Q}}$-integrable.*[6] *Define $h_t := \int_0^t\int_{\mathcal{Z}} f(z,s)\mu(dz,ds)$. If $\mathbf{E}\left[\int_0^T h_{t-}^2\, d\langle L\rangle_t\right] < \infty$,*[7] *then*

$$\mathbf{E}\left[\int_t^T\int_{\mathcal{Z}} L_{s-}\Phi(z,s)f(z,s)\mu(dz,ds)\;\middle|\;\mathcal{F}_t\right] = L_t\,\widetilde{\mathbf{E}}\left[\int_t^T\int_{\mathcal{Z}} f(z,s)\mu(dz,ds)\;\middle|\;\mathcal{F}_t\right].$$

Proof. Appendix. □

Intuitively, if $L_- + \Delta L$ is a factor of the integrand, then it can be factored out of the stochastic integral when expectations are taken.

Next we establish a similar connection for expectations of integrals with respect to time where we are allowed to take L_{s-} "out of the integral" and perform the change of measure.

Lemma 8 (Stochastic Integrals wrt. l). *Let L and $\widetilde{\mathbb{Q}}$ be as in theorem 5 and g be a predictable process with $\mathbf{E}\left[\int_0^T(\int_0^t |g_s|\,ds)^2\, d\langle L\rangle_t\right] < \infty$. Then*

$$\mathbf{E}\left[\int_t^T L_{s-}g_s\,ds\;\middle|\;\mathcal{F}_t\right] = L_t\,\widetilde{\mathbf{E}}\left[\int_t^T g_s ds\;\middle|\;\mathcal{F}_t\right]$$

Proof. Appendix. □

3.4 Classification and Properties of the AD Y

There is a large literature on AD in the context of pricing default free and defaultable bonds. The main feature of ADs is (see Duffie and Kan (1996)) that under technical conditions, for any affine function $f : \mathbb{R}^N \to \mathbb{R}$ there exist deterministic functions $\mathcal{A}, \mathcal{B}$ such that

$$\mathbf{E}\left[e^{\int_t^T f(Y_s)ds}\;\middle|\;\mathcal{F}_t\right] = e^{\mathcal{A}(T-t)+\mathcal{B}(T-t)'Y_t},$$

[6] I.e. $L_T\int_0^T\int_{\mathcal{Z}}|f(z,s)|\mu(dz,ds)$ is $\mathbb{Q}$-integrable.

[7] This condition can be replaced by any condition ensuring that $\int_0^t h_{s-}dL_s$ is a martingale.

where $\mathcal{A}, \mathcal{B}$ are determined by a set of ordinary differential equations (ODEs) involving $\gamma, \sigma\sigma'$ and f. (Note that $\mathbf{E}\left[e^{\int_t^T f(Y_s)ds} \,\middle|\, \mathcal{F}_t \right] = \mathbf{E}\left[e^{\int_t^T f(Y_s)ds} \,\middle|\, \mathcal{F}_t^W \right]$ by hypothesis $\mathbb{H}$.)

When one wants to parameterize the functions γ and $\sigma\sigma'$ in (2), it has to be taken into account that σ is a square root of an affine matrix function $(\sigma\sigma'(Y))$, hence it may not be well-defined for all values of Y. In the light of this problem Dai and Singleton (2000) introduced a parametrization under which the *admissibility* of the matrix $\sigma\sigma'$ can be checked easily. They call a pair $(\gamma, \sigma\sigma')$ *admissible* if (2) admits a unique strong solution.[8] Furthermore their parametrization allows for a simple classification of ADs. We present a (very slight) extension of their parameterization[9].

For $m \in \{0, 1, \ldots, N\}$ fixed, $\mathbb{A}_m(N)$ is the class of admissible N-dimensional ADs with σ depending on exactly m components of Y. Consider the following parameterized version of the SDE (2)

$$dY = (\Theta - \mathcal{K}Y)dt + \sqrt{S}\, dW, \qquad Y(0) = Y_0, \tag{7}$$

where S is a diagonal matrix with $S_{ii} = a_i + \sum_{j=1}^m b_{ij} Y_j$ for $a \in \mathbb{R}^N$ and $b \in \mathbb{R}^{N \times N}$, $Y_0, \Theta \in \mathbb{R}^N$ and $\mathcal{K} \in \mathbb{R}^{N \times N}$. Now, if the following conditions are satisfied then (γ, σ) is admissible and the solution Y of (7) belongs to $\mathbb{A}_m(N)$:

- $b \geq 0$, and for all $1 \leq i \neq j \leq m$ we have :

$$\begin{array}{ccc} Y_{0i} \geq 0, & \Theta_i \geq 0, & a_i = 0, \\ \mathcal{K}_{ij} \leq 0, & b_{ii} = 1, & b_{ij} = 0. \end{array}$$

- and for all $m < k \leq N$ and $1 \leq i \leq m$ we have

$$\begin{array}{cc} \Theta_k = 0, & a_k \in \{0, 1\} \\ b_{ik} = 0 & \mathcal{K}_{ik} = 0. \end{array}$$

Then the first m components $Y_{t1}, \ldots, Y_{tm}$ of Y_t are nonnegative and $\sigma(Y_t)$ depends only on these components. Y is called a *canonical representative* of the class $\mathbb{A}_m(N)$ which is formed by all regular affine transforms of Y (i.e. all processes $Z = \eta + \theta Y$, where $\eta \in \mathbb{R}^N$, $\theta \in \mathbb{R}^{N \times N}$ invertible).

If furthermore: (i) $\mathcal{K}_{ii} > 0$ for $1 \leq i \leq N$, (ii) the inequalities concerning Y_0 hold strictly, (iii) $\Theta_i > \frac{1}{2}$, $1 \leq i \leq m$ and $a_i + \sum_{j=1}^N b_{ij} > 0$ for all $1 \leq i \leq N$, then $(Y_{t1}, \ldots, Y_{tm})'$ remains strictly positive and Y is non-explosive $\mathbb{Q}$-a.s. on $[0, T^*]$, and $\mathbb{F}^W = \mathbb{F}^Y$. In the sequel of the paper we will always consider this case.

Proposition 1 (Quadratic Variation). *Let Y be a canonical representative of $\mathbb{A}_m(N)$ and $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}^N$. Then the quadratic variation of $\alpha + \beta' Y$ satisfies*

$$\frac{d}{dt}[\alpha + \beta' Y]_t = v(\beta) + w(\beta)' Y_t$$

where v and w in matrix notation are given by

$$v(\beta) = a' \mathrm{diag}(\beta)\beta = \beta' \mathrm{diag}(a)\beta \qquad \text{and} \qquad w(\beta) = b' \mathrm{diag}(\beta)\beta.$$

[8] In particular, a real matrix σ_t with $\sigma_t \sigma_t' = \sigma\sigma'(Y_t)$ is $\mathbb{Q}$-a.s. definable for all $t \in [0, T^*]$.

[9] A comment in Ait-Sahalia and Kimmel (2002) shows that the Dai and Singleton (2000) parametrization does not include all ADs.

Proof. We have

$$\frac{d}{dt}[\beta' Y] = \beta' S \beta = \sum_{i=1}^{N} \beta_i^2 \Big(a_i + \sum_{j=1}^{N} b_{ij} Y_j \Big) = \sum_{i=1}^{N} \beta_i^2 a_i + \sum_{i,j=1}^{N} \beta_i^2 b_{ij} Y_j.$$

□

We observe that $v(\cdot) \geq 0$ and $w(\cdot)$ is $\mathbb{R}_+^m \times \{0\}^{N-m}$-valued. Further note the rules

$$\begin{aligned} v(\beta_1 + \beta_2) &= v(\beta_1) + v(\beta_2) + 2\beta_1 \mathrm{diag}(a) \beta_2 \\ w(\beta_1 + \beta_2) &= w(\beta_1) + w(\beta_2) + 2b' \mathrm{diag}(\beta_1) \beta_2 \end{aligned}$$

and that the partial derivatives of v and w wrt. β in matrix notation are given by

$$\frac{\partial v(\beta)}{\partial \beta} = 2a' \mathrm{diag}(\beta) = 2\beta' \mathrm{diag}(a) \qquad \text{and} \qquad \frac{\partial w(\beta)}{\partial \beta} = 2b' \mathrm{diag}(\beta).$$

The following lemmata for the calculation of *extended transforms* are well-known (see e.g. Duffie and Kan (1996) or Dai and Singleton (2000)), in our parametrization they are:

Lemma 9. *Let Y be the canonical representative of $\mathbb{A}_m(N)$ satisfying (7), $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}^N$, and $\mathcal{A} : \mathbb{R} \to \mathbb{R}$ and $\mathcal{B} : \mathbb{R} \to \mathbb{R}^N$ solve the Riccati ODEs*

$$\begin{aligned} \frac{\partial}{\partial x} \mathcal{A}(x) &= \alpha + \Theta' \mathcal{B}(x) + \frac{1}{2} v\Big(\mathcal{B}(x)\Big), \\ \frac{\partial}{\partial x} \mathcal{B}(x) &= \beta - \mathcal{K}' \mathcal{B}(x) + \frac{1}{2} w\Big(\mathcal{B}(x)\Big) \end{aligned} \tag{8}$$

with initial conditions $\mathcal{A}(0) = 0$ and $\mathcal{B}(0) = 0$. If there exists $\mathcal{B}^* < \infty$ with $|\mathcal{B}| \leq \mathcal{B}^*$ on $[0, T]$ and $\mathbb{E}\big[e^{\frac{1}{2}\int_0^T w(\mathcal{B}^*)' Y_s ds}\big] < \infty$, then

$$\mathbb{E}\Big[e^{\int_t^T \alpha + \beta' Y_s ds} \Big| \mathcal{F}_t\Big] = e^{\mathcal{A}(T-t) + \mathcal{B}(T-t)' Y_t}$$

The following lemma generalizes the result of lemma 9.

Lemma 10. *Let the assumptions of lemma 9 be satisfied, $\zeta \geq 0$ and $\xi \in \mathbb{R}_+^m \times \{0\}^{N-m}$ and $A : \mathbb{R} \to \mathbb{R}$ and $B : \mathbb{R} \to \mathbb{R}^N$ solve the ODEs*

$$\begin{aligned} \frac{\partial A(x)}{\partial x} &= \Theta' B(x) + \mathcal{B}(x)' \mathrm{diag}(a) B(x) \\ \frac{\partial B(x)}{\partial x} &= -\mathcal{K}' B(x) + b' \mathrm{diag}\Big(\mathcal{B}(x)\Big) B(x). \end{aligned} \tag{9}$$

with initial conditions $A(0) = \zeta$ and $B(0) = \xi$. If $\mathbf{E}\left[e^{\frac{1}{2}\int_0^T w\left(\mathcal{B}^* + \frac{B^*}{A_* + B_*' Y_t}\right)' Y_t dt} \right] < \infty$, where $A_* = \min_{0 \leq t \leq T} A(t)$, $B_* = \min_{0 \leq t \leq T} B(t)$, then

$$\mathbf{E}\left[e^{\int_t^T \alpha + \beta' Y_s ds} (\zeta + \xi' Y_T) \;\middle|\; \mathcal{F}_t \right] = \big(A(T-t) + B(T-t)' Y_t\big) e^{\mathcal{A}(T-t) + \mathcal{B}(T-t)' Y_t}.$$

The proofs of lemma 9 and 10 can be found in the appendix. We say (α, β) *is* $\mathbb{Q}$*-regular* if α, β and the parameters $(\Theta, \mathcal{K}, a, b)$ governing the dynamics of Y under $\mathbb{Q}$ satisfy the conditions of lemma 9, and if the conditions of lemma 10 are satisfied, we say $(\alpha, \beta, \zeta, \xi)$ *is* $\mathbb{Q}$*-regular*.

Remark 11. (i) The following relationship is useful for applications and follows immediately from lemma 10. If $(\alpha, \beta, \zeta = \alpha, \xi = \beta)$ is $\mathbb{Q}$-regular, then

$$\mathbf{E}\left[\, e^{\int_t^T \alpha+\beta' Y_s ds}(\alpha+\beta' Y_T) \;\middle|\; \mathcal{F}_t \right] = \frac{\partial}{\partial T} e^{\mathcal{A}(T-t)+\mathcal{B}(T-t)' Y_t}$$

i.e. A and B are the derivatives of $\mathcal{A}$ and $\mathcal{B}$ with respect to time.

(ii) The result of of lemma 10 can be generalized to values of $\zeta \in \mathbb{R}$ and $\xi \in \mathbb{R}^N$, but then one has to impose a condition which is more difficult to check than $\mathbf{E}\left[\, \exp\{\frac{1}{2}\int_0^T w\big(\mathcal{B}^* + \frac{B^*}{A_* + B_*' Y_t}\big)' Y_t dt\} \,\right] < \infty$.

Duffie and Singleton (1999) directly calculate expectations of the form $\mathbf{E}\left[\, e^{\int_t^T \alpha+\beta' Y_s ds+\gamma' Y_T}(\zeta+\xi' Y_T) \,\right]$ using differentiation through the integral. We believe we have found a more natural way to look at this, namely we choose $\phi = \sqrt{S}\,\gamma$ and $\Phi = 1$ in Girsanov's theorem 5. Then

$$L_t = e^{\gamma'(Y_t - Y_0) - \int_0^t \Theta - \mathcal{K} Y_s ds - \frac{1}{2}\int_0^t v(\gamma) + w(\gamma)' Y_s ds}$$

and (see also proposition 3) Y is also an AD under $\widetilde{\mathbb{Q}}$. Thus we can always find a measure $\widetilde{\mathbb{Q}} \sim \mathbb{Q}$ under which Y remains an AD and $\widetilde{\alpha}, \widetilde{\beta}$ such that

$$\mathbf{E}\left[\, e^{\int_t^T \alpha+\beta' Y_s ds+\gamma' Y_T}(\zeta+\xi' Y_T) \;\middle|\; \mathcal{F}_t \right] = e^{\gamma' Y_t}\, \widetilde{\mathbf{E}}\left[\, e^{\int_t^T \widetilde{\alpha}+\widetilde{\beta}' Y_s ds}(\zeta+\xi' Y_T) \;\middle|\; \mathcal{F}_t \right],$$

hence lemma 9 or 10 apply again.

In the sequel of this paper the circuitous detour described above to find $\widetilde{\alpha}, \widetilde{\beta}$ in order to derive the respective expectations will never be necessary. The martingale L, rather than the factor $e^{\gamma' Y_T}$, will have an important financial interpretation.

4 Fixed Income Securities in Different Currencies

In this section we discuss how to determine the prices of default-sensitive foreign currency instruments such as bonds and CDSs denominated in different currencies. It is well-known that with the change of numeraire method the problem of pricing foreign currency claims *at fixed times* can be reduced to a related domestic currency pricing problem. The recovery of defaultable bonds or the protection payments of CDSs, however, are payoffs at *stopping times* which require a modification of these techniques, in particular when the value process of the numeraire (and thus the density of the change of measure) is discontinuous at the stopping time. The general form of this modification is given in lemma 7, here we apply this modification to the case of foreign currency claims payable at default and transform these pricing problems into an equivalent domestic currency pricing problem, which is then solved easily using the AJD-specification of the model. All notation from section 3 is carried over to this part of the paper.

4.1 Default Intensities under Domestic and Foreign Martingale Measures

The model for foreign exchange risk in the presence of default risk is set up as follows:

Assumption 12 (The Defaultable FX Model). *(i) The market is modelled by the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{Q})$ defined in 3.1, where $\mathbb{Q}$ is a domestic spot martingale measure (DSMM)*[10]*, and $\mathbb{F}$ is the information of the market to which all processes are adapted.*

(ii) The time of default of the obligor is the stopping time τ defined in (1)*. i.e. the time of the first jump of $N(t)$. Given default, the severity of default is characterized by the realization of the marker $z_\tau := \Delta J_\tau$ of the marked point process.*

(iii) The exchange (FX) rate between foreign currency c_f and domestic currency c_d is denoted with $X \geq 0$.[11] *r_i are the short-term interest rates and $b_i(t) := e^{\int_0^t r_i(s)ds}$ instantaneous bank accounts in the respective currencies c_i, $i = d, f$.*

We sometimes write $\mathbb{Q}_d$ instead of $\mathbb{Q}$ when we find it necessary to emphasize the fact that $\mathbb{Q}$ is the *domestic* SMM, or λ_d instead of λ for the default intensity under $\mathbb{Q}_d$.[12] $\mathbb{Q}_d$ does not need to be unique, we only assume it is the pricing measure chosen by the market. From assumption 12 *(i)* and *(iii)* it follows directly that X satisfies a SDE of the form

$$\frac{dX_t}{X_{t-}} = (r_d - r_f)(t)dt + \phi_X(t)' dW_t - \int_{\mathcal{Z}} \delta(z,t)(\mu - \nu)(dz, dt) \tag{10}$$

where ϕ_X is predictable process, and $\delta(\cdot, \cdot) \leq 1$ is a predictable function. To see this note that $X_t \frac{b_f(t)}{b_d(t)}$ is the discounted value in c_d of a foreign bank account and hence needs to be a $\mathbb{Q}$-local martingale. Furthermore our probability space has the predictable representation property and X must be a nonnegative process. Regarding the drift term in (10) we say *X satisfies the FX drift restriction under $\mathbb{Q}$.*

The dependency between defaults and the movements of the exchange rate X has important consequences for the dynamics of the model under the pricing measures that we will introduce in the following. Here, equation (10) has two implications. First, FX rate and *default intensity* may be conditionally *correlated*, if $[\lambda, X]$ is not identically zero. If for example we have positive local correlation, an increasing (decreasing) FX rate will indicate a rise (lowering) in the default intensity. Second, there is a direct jump-influence from *the default event itself* on the exchange rate which is captured in the function δ via $d[N, X]_t = -X_{t-} \int_{\mathcal{Z}} \delta(z,t)\mu(dz, dt)$ or

$$\Delta X_\tau = -X_{\tau-}\delta(z_\tau, t).$$

At default τ, the foreign currency c_f is *devaluated* (relative to c_d) in a jump of a fraction $\delta(z_\tau, \tau)$ of the pre-default value of X. As defaults are the only jumps in this model, there is no discontinuity in X before τ.

[10] I.e. every traded asset (denominated in domestic currency) is a $\mathbb{Q}$-local martingale.

[11] I.e. X_t is the value at time t of one unit of c_f, expressed in units of c_d.

[12] We also sometimes call λ itself *default intensity*, omitting the indicator function in the mathematically correct expression $\mathbf{1}_{\{t \leq \tau\}}\lambda(t)$.

Of course not only X and λ but also r_d, r_f, λ, X may show mutual correlation and r_d, r_f might also jump at τ. However, in this paper we mainly focus on the dependence of FX and credit risk, and we will not treat these additional features in detail. For technical reasons, we make the following assumption.

Assumption 13 (FX Martingale Property). *The discounted value in c_d of a foreign bank account*

$$L_t := \frac{X_t}{X_0}\frac{b_f(t)}{b_d(t)}$$

is uniformly integrable, i.e. a true martingale under $\mathbb{Q}$, and $L_{T^*} > 0$ $\mathbb{Q}$-a.s.

Starting from a DSMM $\mathbb{Q}$, the following pricing measure is commonly used to analyze FX-related instruments when assumption 13 is satisfied (see Musiela and Rutkowski (1997)).

Definition 1 (FSMM). The equivalent measure $\mathbb{Q}_f \sim \mathbb{Q}_d$ on $(\Omega, \mathcal{F})$ defined by

$$\left.\frac{\mathrm{d}\mathbb{Q}_f}{\mathrm{d}\mathbb{Q}_d}\right|_{\mathcal{F}_t} := L_t = \frac{X_t}{X_0}\frac{b_f(t)}{b_d(t)}$$

is called the *foreign spot martingale measure* (FSMM) induced by $\mathbb{Q}_d$ and X.[13] The default intensity under $\mathbb{Q}_f$ is denoted with λ_f.

The FSMM is useful for pricing foreign currency payoffs because the price in c_d at time t of an instrument that pays Z units of c_f at a *fixed time* $T \geq t$ (i.e. $X_T Z$ units of c_d) is

$$p(t) = \mathbf{E}^{\mathbb{Q}_d}\left[e^{-\int_t^T r_d(s)ds} X_T Z \;\middle|\; \mathcal{F}_t \right] = X_t\, \mathbf{E}^{\mathbb{Q}_f}\left[e^{-\int_t^T r_f(s)ds} Z \;\middle|\; \mathcal{F}_t \right], \tag{11}$$

i.e. $\mathbf{E}^{\mathbb{Q}_f}\left[e^{-\int_t^T r_f(s)ds} Z \;\middle|\; \mathcal{F}_t \right]$ is the price of this instrument in c_f. Thus, in order to price a foreign currency contingent claim at *fixed times*, only the distributional properties of foreign interest rates and the considered claim under the FSMM are needed.

If the FX rate X_t (and so the Radon-Nikodym density $\frac{\mathrm{d}\mathbb{Q}_f}{\mathrm{d}\mathbb{Q}_d}$) jumps at default, then, in general, the *domestic* and the *foreign* default intensity do not coincide, and thus, defaults occur with different probabilities under the FSMM and under the DSMM. As was already shown in corollary 6, $\lambda_f = \lambda_d$ if and only if $\langle N, L\rangle_{t\wedge\tau} \equiv 0$. Here, given the particular nature of our numeraire asset, more can be said about the link between the devaluation fraction of the currency and the two default intensities:

Proposition 2 (FSMM Default Intensity). *Let assumptions 12 and 13 hold. Define the* locally expected devaluation fraction $\widehat{\delta}(t)$ *at time t (under $\mathbb{Q}_d$, conditional on default occurring at $\tau = t$)*

$$\widehat{\delta}(t) := \int_{\mathcal{Z}} \mathbf{1}_{\{t\leq\tau\}}\delta(z,t)dF_t(z). \tag{12}$$

Then, the default intensity under the foreign *spot martingale measure (FSMM) equals*

$$\lambda_f(t) = \big(1 - \widehat{\delta}(t)\big)\lambda_d(t).$$

[13] We emphasize that every DSMM induces another associated FSMM. If $\mathbb{Q}_d$ is unique, then so is $\mathbb{Q}_f$.

Proof. Substitute (10) in theorem 5. ◻

Intuitively, the adjustment factor between local and foreign default intensity is equal to the locally expected devaluation of the FX rate X, if a default were to happen at time t:

$$\frac{\lambda_f(t)}{\lambda_d(t)} = 1 - \widehat{\delta}(t) = \frac{\mathbf{E}^{\mathbb{Q}}[\, X(t) \,|\, \tau = t,\ \mathcal{F}_{t-}\,]}{\mathbf{E}^{\mathbb{Q}}[\, X(t) \,|\, \tau > t,\ \mathcal{F}_{t-}\,]}.$$

To specify a tractable model that can be estimated statistically we return to an AJD framework and use the canonical representative $Y \in \mathbb{A}_m(N)$ defined in (7) as an observable background process driving the continuous dynamics of the other variables in the market.

Assumption 14 (AJD Economy). *Let Y be as in (7) with admissible parameters.*

(i) Default intensity and interest rates are affine in Y, i.e. there exist $\alpha_i \in \mathbb{R}$, $\beta_i \in \mathbb{R}^N$, $i = d, f$ and $\alpha_\lambda \geq 0$, $\beta_\lambda \in \mathbb{R}_+^m \times \{0\}^{N-m}$ [14] such that

$$\begin{aligned} r_i &= \alpha_i + \beta_i' Y, \qquad i = d, f \qquad \text{and} \\ \lambda &= \alpha_\lambda + \beta_\lambda' Y. \end{aligned}$$

(ii) There exist $\gamma \in \mathbb{R}^N$ and $x \in \mathbb{R}^K$ with $\mathcal{L}(t;x) < \infty$ $\mathbb{Q}$-a.s. for all t such that in the FX dynamics (10)

$$\phi_X(t) = \sqrt{S_t}\,\gamma, \qquad \text{and} \qquad \delta(z,t) = \delta(z) = 1 - (1-z)^x.$$

Occasionally, we prefer to write α_{λ_d} and β_{λ_d} instead of α_λ and β_λ. Under assumption 14 the covariation between intensity and exchange rate $[\lambda, X]$ is completely determined by the inner product $\gamma' S \beta_\lambda$ because $d[\lambda, X] = X_- \gamma' S \beta_\lambda dt$. As we did not make any assumptions on the conditional distribution $F_t(z)$ of the jump severity, the assumption regarding $\delta(\cdot,\cdot)$ is not restrictive and was only chosen in order to express $\widehat{\delta}(t) = 1 - \mathcal{L}(t;x)$ with the Laplace transform $\mathcal{L}(t;x)$.

The following proposition provides the relations between the domestic and the foreign spot martingale measure in an AJD framework.

Proposition 3 (AJD FX Rate). *Let assumption 14 be satisfied. Then*

(i) Y satisfies the SDE

$$dY = (\Theta_f - \mathcal{K}_f Y)dt + \sqrt{S}\,dW_f, \qquad Y(0) = Y_0,$$

where $\Theta_f = \Theta + \mathrm{diag}(\gamma)a$, $\mathcal{K}_f = \mathcal{K} - \mathrm{diag}(\gamma)b$ and W_f is a BM under $\mathbb{Q}_f$, i.e. Y is also an AD under $\mathbb{Q}_f$.

(ii) The default intensity under $\mathbb{Q}_f$ is given by

$$\lambda_f(t) = \mathcal{L}(t;x)\lambda(t),$$

(iii) and the stochastic Laplace transform of the default severity distribution under $\mathbb{Q}_f$ satisfies

$$\mathcal{L}_f(t;u) = \frac{\mathcal{L}(t;u+x)}{\mathcal{L}(t;x)}.$$

[14] This ensures nonnegativity of λ.

(iv) λ_f *is an AD under* $\mathbb{Q}_f$, *if and only if* $\mathcal{L}(t;x) = \mathcal{L}(x) < \infty$ *is time-invariant.*[15] *In this case we define constant coefficients* $\alpha_{\lambda_f} := \mathcal{L}(x)\alpha_\lambda$ *and* $\beta_{\lambda_f} := \mathcal{L}(x)\beta_\lambda$ *with* $\lambda_f = \alpha_{\lambda_f} + \beta'_{\lambda_f} Y$, *and we call* X *an* affine FX rate.

Proof. Apply Girsanov's theorem 5 to our AJD framework. ☐

Mean reversion speed and level of Y (and thus of λ, r_d, r_f) transform under the changes of measure considered in proposition 3, whereas the parameters a and b governing the volatility of Y are clearly invariant. Note that the parameter restrictions for canonical ADs given in section 3.4 are automatically satisfied for $\mathcal{K}_f$ and the m first components of Θ_f, but in general not for the $N-m$ last components of Θ_f. Of course $(\Theta_f, \mathcal{K}_f, a, b)$ is an admissible parameter set, but it does not necessarily belong to a canonical representative of $\mathbb{A}_m(N)$. Further the inequalities $\mathcal{K}_{fii} > 0$, $i = 1, \dots, N$ may be violated, i.e. we might lose mean reversion under the FSMM, which would not be economically meaningful. In applications one should check that these inequalities hold.

Remark 15.

(i) In general it is difficult to check the validity of assumption 13. In the AJD case however, the martingale property of L is ensured if the following condition holds (see e.g. Lepingle and Memin (1978)). Let X be as in proposition 3, and

$$\mathbf{E}\left[\exp\{\frac{1}{2}\int_0^{T^*} w(\gamma)' Y_t dt + \int_0^{T^*} \big(\int_{\mathcal{Z}} (x' \log(1-z) \cdot (1-z)^x - 1) dF_t(z)\big)\lambda(t) dt\}\right] < \infty$$

For $x \geq 0$, i.e. if the foreign currency can only be devaluated at default, it suffices to check the Novikov condition $\mathbf{E}\left[e^{\frac{1}{2}\int_0^{T^*} w(\gamma)' Y_t dt}\right] < \infty$.

(ii) Proposition 3 remains true for all sufficiently regular[16] interest rates.

4.2 Basic Default-Free and Default-Sensitive Instruments

We turn to the pricing of domestic and foreign securities which may be sensitive to both, time and severity of default. Basically one has to distinguish two types of default-sensitive instruments. First, payoffs *upon survival* until a fixed maturity T (e.g. a defaultable bond with zero recovery), and second, instruments with payments that become due *at default* (e.g. recovery payments or protection payments in CDS).

As hypothesis $\mathbb{H}$ underlies our probability space, we can always reduce the pricing problem of a defaultable claim to a related default-free pricing problem, following the tools provided in section 3.2. But first we address the well-known problem of pricing default-free zero coupon bonds (ZCBs).

4.2.1 Default-Free Zero Coupon Bonds

If the interest rate coefficients (α_i, β_i) are $\mathbb{Q}_i$-regular for $i = d, f$, then, according to (11) with $Z = 1$ and lemma 9, domestic and foreign default-free ZCB prices exist for all

[15] $F_t = const.$ is a sufficient but not a necessary condition for this.

[16] E.g. $b_i(t) < \infty$ $\mathbb{Q}$-a.s. for $i = d, f$.

$0 \leq t \leq T \leq T^*$ and are given (in their respective payoff currency) by

$$B_i(t,T) = \mathbf{E}^{\mathbb{Q}_i}\left[e^{-\int_t^T r_i(s)ds} \;\middle|\; \mathcal{F}_t \right] = e^{\mathcal{A}_i(T-t)+\mathcal{B}_i(T-t)'Y_t},$$

where $\mathcal{A}_i, \mathcal{B}_i$ solve (8) with $(-\alpha_i, -\beta_i, \Theta_i, \mathcal{K}_i, a, b)$ for $i = d, f$. In the sequel of the article, we will always assume that default free ZCB prices exist.

If we do not allow for negative interest rates, i.e. $\alpha_i \geq 0$ and $\beta_i \in \mathbb{R}^m \times \{0\}^{N-m}$ for $i = d, f$, then the existence of ZCB prices is immediate. However, in that case λ can only have nonnegative correlation with r_i (because then $\beta_\lambda' S \beta_i \geq 0$), which is not always a desirable property (see e.g. Duffee (1998) for empirical evidence of *negative* correlation).

Assumption 16 (Recoveries). *Generally, we model the loss given default (of a c_i-bond) with a predictable $[0,1]$-valued function $q_i(z,t)$* [17] *which captures the dependency of recovery on the default severity marker z.*
In the AJD framework we assume that the recoveries can be written as $1-q_i(z) := (1-z)^{u_i}$ for some fixed $u_i \in \mathbb{R}_+^K$. We also assume $\mathcal{L}_i(t;u) = \mathcal{L}_i(u)$ for all t.

In combination with assumption 14 (ii) on the devaluation fraction, the assumption on the functional form of $q(z,t)$ is restrictive, but it preserves the affine structure. Despite this, using a higher-dimensional marker space $\mathcal{Z}$ we still have a large degree of flexibility in the modelling of the correlation between loss given default q and FX devaluation δ.

4.2.2 Defaultable Zero Coupon Bonds

We present the prices of ZCB under a variety of recovery assumptions that have been proposed in the literature:

Zero Recovery. A c_i-ZCB with zero recovery (ZR) pays $\mathbf{1}_{\{T>\tau\}}$ units of c_i at its maturity T, $i = d, f$. By equation (11) with $Z = \mathbf{1}_{\{T>\tau\}}$, the price of domestic and foreign ZCB with ZR is

$$\overline{B}_i(t,T) := \mathbf{E}^{\mathbb{Q}_i}\left[e^{-\int_t^T r_i(s)ds}\,\mathbf{1}_{\{T>\tau\}} \;\middle|\; \mathcal{F}_t \right], \qquad i = d, f$$

in the respective payoff currency. Using hypothesis $\mathbb{H}$ and lemma 2, the survival function $\mathbf{1}_{\{T>\tau\}}$ can be replaced by $\mathbf{1}_{\{t>\tau\}} e^{-\int_t^T \lambda_i(s)ds}$ in the above expectation. Then lemma 9 yields for the AJD setup

$$\overline{B}_i(t,T) = \mathbf{1}_{\{t>\tau\}} \mathbf{E}^{\mathbb{Q}_i}\left[e^{-\int_t^T (r_i+\lambda_i)(s)ds} \;\middle|\; \mathcal{F}_t \right] = \mathbf{1}_{\{t>\tau\}} e^{\mathcal{A}_i(T-t)+\mathcal{B}_i(T-t)'Y_t}, \tag{13}$$

where $\mathcal{A}_i$ and $\mathcal{B}_i$ solve (8) with $(-\overline{\alpha}_i, -\overline{\beta}_i, \Theta_i, \mathcal{K}_i, a, b)$ where $\overline{\alpha}_i := \alpha_i + \alpha_{\lambda_i}$ and $\overline{\beta}_i = \beta_i + \beta_{\lambda_i}$, $i = d, f$. Note that not only are the payoffs in different currencies discounted with different interest rates, but also with different default intensities.

Any positive recovery is a payment at a stopping time, and thus, its value is equal to the expectation of a stochastic integral wrt. the jump measure μ. This renders the pricing problem a bit more complicated. We consider the following recovery assumptions.

[17]The recovery rate $1-q$ of a foreign-issued bond is in general not equal to that of a domestic bond because the respective bankruptcy courts use different legal rules.

Recovery of Par. (See e.g. Duffie (1998).) In addition to the survival payoff of $\mathbf{1}_{\{T>\tau\}}$units of c_i at maturity T, a defaultable ZCB pays $1-q_i(z_\tau,\tau)$ units of c_i at the default time τ if $\tau \le T$ in the recovery of par setting (RP). This (the recovery payment) can be written as

$$\int_0^T\!\!\int_{\mathcal{Z}}\big(1-q_i(z,t)\big)\mathbf{1}_{\{t\le\tau\}}\mu(dz,dt).$$

The no-arbitrage price of the domestic bond is thus

$$\overline{B}_d^{RP}(t,T) = \overline{B}_d(t,T) + \mathbf{E}^{\mathbb{Q}_d}\left[\int_t^T\!\!\int_{\mathcal{Z}} e^{-\int_t^s r_d(v)dv}\big(1-q_d(z,s)\big)\mathbf{1}_{\{s\le\tau\}}\mu(dz,ds)\;\middle|\;\mathcal{F}_t\right]$$

on the set $\{t<\tau\}$. The valuation of the foreign recovery payment is slightly more involved. In units of c_d, the foreign recovery payment is

$$\int_0^T\!\!\int_{\mathcal{Z}} X_{t-}\big(1-\delta(z,t)\big)\big(1-q_f(z,t)\big)\mathbf{1}_{\{t\le\tau\}}\mu(dz,dt).$$

The standard formula (11) to eliminate X by changing to the FSMM is not directly applicable here because the payment takes place at a stopping time. We use lemma 7 instead, again on $\{t<\tau\}$. Assume that L and $f(z,t) := e^{-\int_0^t r_f(v)dv}(1-q_f(z,t))\mathbf{1}_{\{t\le\tau\}}$ satisfy the conditions of lemma 7. [18] Then

$$\begin{aligned}
&\mathbf{E}^{\mathbb{Q}_d}\left[\int_t^T\!\!\int_{\mathcal{Z}} e^{-\int_t^s r_d(v)dv}\,X_{s-}\big(1-\delta(z,s)\big)\big(1-q_f(z,s)\big)\mathbf{1}_{\{s\le\tau\}}\mu(dz,ds)\;\middle|\;\mathcal{F}_t\right]\\
=\;& X_t\,\mathbf{E}^{\mathbb{Q}_d}\left[\int_t^T\!\!\int_{\mathcal{Z}} \frac{L_{s-}}{L_t}\big(1-\delta(z,s)\big)e^{-\int_t^s r_f(v)dv}\big(1-q_f(z,s)\big)\mathbf{1}_{\{s\le\tau\}}\mu(dz,ds)\;\middle|\;\mathcal{F}_t\right]\\
=\;& X_t\,\mathbf{E}^{\mathbb{Q}_f}\left[\int_t^T\!\!\int_{\mathcal{Z}} e^{-\int_t^s r_f(v)dv}\big(1-q_f(z,s)\big)\mathbf{1}_{\{s\le\tau\}}\mu(dz,ds)\;\middle|\;\mathcal{F}_t\right]. \qquad (14)
\end{aligned}$$

The important difference to a "naive" application of (11) is, that not just the predictable part of the exchange rate X_{t-}, but also the jump term $1-\delta(z,s)$ is removed in the change of measure. Further note that pricing equation (14) remains true for all regular[19] payoff functions $p(z,t)$ instead of $1-q_f$. Hence (14) is an equivalent to (11) for the valuation of foreign currency payments *at default*.

Then, by corollary 4 and lemma 10 on $\{t<\tau\}$

$$\begin{aligned}
\overline{B}_i^{RP}(t,T) &= \overline{B}_i(t,T) + \mathbf{E}^{\mathbb{Q}_i}\left[\int_t^T\!\!\int_{\mathcal{Z}} e^{-\int_t^s r_i(v)dv}\,(1-z)^{u_i}\mathbf{1}_{\{s\le\tau\}}\mu(dz,ds)\;\middle|\;\mathcal{F}_t\right]\\
&= \overline{B}_i(t,T) + \mathcal{L}_i(u_i)\int_t^T \mathbf{E}^{\mathbb{Q}_i}\left[e^{-\int_t^s (r_i+\lambda_i)(v)dv}\lambda_i(s)\;\middle|\;\mathcal{F}_t\right]ds\\
&= \overline{B}_i(t,T) + \mathcal{L}_i(u_i)\int_t^T \big(A_i(s-t)+B_i(s-t)'Y_t\big)e^{\mathcal{A}_i(s-t)+\mathcal{B}_i(s-t)'Y_t}ds
\end{aligned}$$

where A_i, B_i solve (9) with $(-\overline{\alpha}_i, -\overline{\beta}_i, \zeta_i := \alpha_{\lambda_i}, \xi_i := \beta_{\lambda_i}, \Theta_i, \mathcal{K}_i, a, b)$. Note that by remark 11 the existence of $\mathbf{E}^{\mathbb{Q}_i}\left[e^{-\int_t^s (r_i+\lambda_i)(v)dv}\lambda_i(s)\;\middle|\;\mathcal{F}_t\right]$ is immediate if $r_i\ge 0$.

[18] If $r_f\ge 0$, then the process $h_t=\int_0^t\int_{\mathcal{Z}} f(z,s)\mu(dz,ds)$ is bounded by 1. In this case the conditions of lemma 7 are already satisfied, when L is a square-integrable martingale.

[19] Again, the conditions of lemma 7 must be satisfied.

Remark 17. A special case in the RP setting is obtained when $u_i = 0$. Such a ZCB pays 1 unit of c_i at $\tau \wedge T$ [20] and we denote its price by $\overline{B}_i^*(t,T)$.

Multiple Defaults. (See e.g. Schönbucher (1998) or Duffie et al. (2003).) A multiple default (MD) ZCB with maturity T has the payoff $p_i^{MD}(T)$ units of c_i at T which solves the SDE

$$\frac{dp_i^{MD}(t)}{p_i^{MD}(t-)} = -\int_{\mathcal{Z}} q_i(z,t)\mu(dz,dt), \qquad p_i^{MD}(0) = 1.$$

An MD bond can default more than once and q_i is the loss fraction at a default. The value $p_i^{MD}(t)$ can be seen as the remaining promised payoff at time t.

In order to derive closed form solutions of ZCB prices in the MD setting, knowledge of the compensator ν *after* τ is needed. That is the distributional properties of the stopping times $\tau_j := \inf\{t > \tau_{j-1};\ N_t > N_{\tau_{j-1}}\}$ (i.e. the time of the jth default) and the conditional distribution function of the jth jump size $F_{\tau_j}(z)$ (given the jth default occurs). Hence additional assumptions have to made.

Assumption 18 (MD). *The predictable compensator of μ under $\mathbb{Q}_d$ is of the form*

$$\nu(dz,dt) = dF(z)\lambda(t)dt,$$

where F is a time-invariant distribution function on $\mathcal{Z}$ and the loss fractions at each default are

$$q_i(z) = 1 - (1-z)^{u_i}.$$

Under assumption 18, N_t is a Cox process and thus

$$\overline{B}_i^{MD}(t,T) = \mathbf{E}^{\mathbb{Q}_i}\left[e^{-\int_t^T r_i(s)ds}\, p_i^{MD}(T) \;\middle|\; \mathcal{F}_t \right] = p_i^{MD}(t)\, e^{\mathcal{A}_i^{MD}(T-t)+\mathcal{B}_i^{MD}(T-t)'Y_t}, \tag{15}$$

where $\mathcal{A}_i^{MD}$ and $\mathcal{B}_i^{MD}$ solve (8) with $(-\overline{\alpha}_i^{MD}, -\overline{\beta}_i^{MD}, \Theta_i, \mathcal{K}_i, a, b)$, where $\overline{\alpha}_i^{MD} := \alpha_i + \mathcal{L}_i(u_i)\alpha_{\lambda_i}$ and $\overline{\beta}_i^{MD} := \beta_i + \mathcal{L}_i(u_i)\beta_{\lambda_i}$, $i = d, f$. (A proof can be found in the appendix.)

Recovery of Treasury. (See e.g. Jarrow and Turnbull (1995).) If a default occurs before maturity ($\tau \leq T$), a ZCB holder receives under recovery of treasury (RT)

$$\int_0^T\int_{\mathcal{Z}} \big(1-q_i(z,t)\big)\mathbf{1}_{\{t\leq\tau\}}\mu(dz,dt) \tag{16}$$

default-free ZCBs with the same maturity T and par value $1c_i$. By a simple hedging argument, the value of the recovery is equal to that of receiving $(1-q_i(z_\tau,\tau))\mathbf{1}_{\{\tau<T\}}$ units of c_i at time T. Using the domestic and foreign T-forward measures

$$\left.\frac{\mathrm{d}\mathbb{P}_i^T}{\mathrm{d}\mathbb{Q}_i}\right|_{\mathcal{F}_t} := L_i^T(t) := \frac{B_i(t,T)}{B_i(0,T)b_i(t)} \qquad \text{for } i = d, f,$$

[20] This is also true for arbitrary u_i if $F_t(0) = 1$ $\mathbb{Q}$-a.s. for all $t \leq T$

we can always reduce the RT case to a related RP case[21], namely on $\{t < \tau\}$

$$\begin{aligned} \overline{B}_i^{RT}(t,T) &= \overline{B}_i(t,T) + \mathbf{E}^{\mathbb{Q}_i}\left[\int_t^T \int_{\mathcal{Z}} e^{-\int_t^s r_i(v)dv} \big(1-q_i(z,s)\big) B_i(s,T) \mathbf{1}_{\{s\le\tau\}} \mu(dz,dt) \,\middle|\, \mathcal{F}_t \right] \\ &= \overline{B}_i(t,T) + B_i(t,T)\, \mathbf{E}^{\mathbb{P}_i^T}\left[\int_t^T \int_{\mathcal{Z}} \big(1-q_i(z,s)\big) \mathbf{1}_{\{s\le\tau\}} \mu(dz,dt) \,\middle|\, \mathcal{F}_t \right]. \qquad (17) \end{aligned}$$

If we postulate again $1-q_i = (1-z)^{u_i}$, then this leads to a AJD pricing problem with deterministic but time-dependent coefficients $(\Theta_i(t), \mathcal{K}_i(t), a, b)$. We will not further treat the time-heterogeneous case in this paper.

4.2.3 Credit Default Swaps

The cash flows involved in a CDS contract were already described in section 2. After default a CDS contract is unwound, thus we always assume that default has not yet occurred ($t < \tau$) in order to avoid trivialities. We also assume the amount of notional insured is always 1 unit of c_i, $i = d, f$. For a contract with maturity T entered at t, the fee leg then consists of payments $\overline{s}_i(t,T) \times (T_j - T_{j-1})$ in c_i at quarterly dates T_j. We approximate this payment stream by an integral.

The **Value of the Fee Stream** (in its payoff currency) is thus given by

$$V_i^{fee}(t,T) = \overline{s}_i(t,T) \int_t^T \overline{B}_i(t,s) ds = \overline{s}_i(t,T) \int_t^T \mathbf{E}^{\mathbb{Q}_i}\left[e^{-\int_t^s (r_i+\lambda_i)(v)dv} \,\middle|\, \mathcal{F}_t \right] ds.$$

The fee stream of a CDS can be interpreted as a defaultable coupon bond with continuously paid coupon $\overline{s}_i(t,T)ds$ and par value zero.

Value of Protection Leg. The protection payment of a CDS takes place if and only if $\tau \le T$. In this case it is made at the time of default τ, and its size is $1 - \overline{B}_{ctd}(\tau)$ units of c_i, where $\overline{B}_{ctd}$ is the price of the cheapest-to-deliver bond. As argued in section 2, the cheapest-to-deliver bond will usually be a coupon-bearing bond and the only relevant quantity is its relative discount to par value in the currency in which it was issued. We model this using the RP setup which is the most appropriate choice for CDS recovery modelling (see e.g. Houweling and Vorst (2005)). In this case, we can write $q(z_\tau, \tau) = 1 - \overline{B}_{ctd}(\tau) = 1 - (1 - z_\tau)^{u_{ctd}}$. If e.g. $\mathcal{L}_i(t; u_{ctd}) = \mathcal{L}_i(u_{ctd})$ is deterministic and time-independent, then the value of the protection leg satisfies

$$V_i^{prot}(t,T) = \big(1-\mathcal{L}_i(u_{ctd})\big) \int_t^T \mathbf{E}^{\mathbb{Q}_i}\left[e^{-\int_t^s (r_i+\lambda_i)(v)dv}\, \lambda_i(s) \,\middle|\, \mathcal{F}_t \right] ds. \qquad (18)$$

The fair CDS rate $\overline{s}_i(t,T)$ is obtained when the value of fee and protection leg are equal:

$$\overline{s}_i(t,T) = \frac{V_i^{prot}(t,T)}{\int_t^T \overline{B}_i(t,s)ds}. \qquad (19)$$

[21] First (14) shows that (17) is also valid for $i = f$. Then lemma 7 with L_i^T applies again because L_i^T is continuous at τ. Moreover theorem 5 yields that Y has the coefficients $\Theta_i^T(t) := \Theta_i + \mathrm{diag}\big(\mathcal{B}_i(T-t)\big)a$ and $\mathcal{K}_i^T(t) := \mathcal{K}_i - \mathrm{diag}\big(\mathcal{B}_i(T-t)\big)b$ under the respective measure $\mathbb{P}_i^T$.

It is not hard to see that

$$\lim_{T \downarrow t} \overline{s}_i(t,T) = \big(1 - \mathcal{L}_i(t; u_{ctd})\big)\lambda_i(t) \qquad \mathbb{Q}_i\text{-a.s.} \tag{20}$$

This limiting property remains true when $\mathcal{L}_i(t; u_{ctd})$ is a right-continuous stochastic process.

4.3 The relationship between domestic and foreign CDS rates

In this subsection we want to achieve some intuition regarding the relationships between domestic and foreign CDS rates $\overline{s}_d$ and $\overline{s}_f$. We will identify the three components of credit risk (default intensity risk, default event risk and default severity risk) and the default-free term structures of interest rates as the driving factors of this relation.

For simplicity we assume that the expected recovery rates remain constant over time as in (18) for both currencies and write $\widehat{q}_i := 1 - \mathcal{L}_i(u_{ctd})$, $i = d, f$. Note that if default occurs, there will be *one unique* recovery rate to both CDS due to the protection buyer's delivery option discussed on section 2. The quantities $\mathcal{L}_i(u_{ctd})$, however, may differ from each other because they are the expectations of this recovery rate under different measures. Mathematically this becomes clear from the definition of $\widetilde{F}$ in *(ii)* of Girsanov's theorem 5 or from *(iii)* in proposition 3. From an economic point of view this phenomenon is explained by the possible dependence of recovery and devaluation fraction of the foreign currency at default. E.g. if recovery and devaluation show negative correlation, then the protection buyer of a foreign CDS is doubly punished. In a light default scenario, i.e. when recovery is large, the LGD payed to the protection buyer will be small. In the case of a severe default, i.e. when recovery is small, then the LGD amount in foreign currency will be large, but its value in domestic currency will typically be reduced by a high devaluation fraction. Clearly, when the recovery rate (or/and the devaluation fraction) is constant, then $\widehat{q}_d = \widehat{q}_f := q$.[22] In the sequel we assume such a q exists but one could easily include the considerations concerning recovery risk (=default severity risk) in the subsequent discussion.

The value of the protection leg in (18) was expressed as an integral over securities with a payoff in units of defaultable ZCBs. The following probability measures are often used in this situation (see Schönbucher [1999, 2004]). $\overline{\mathbb{P}}_d^T$ and $\overline{\mathbb{P}}_f^T$

$$\left.\frac{\mathrm{d}\overline{\mathbb{P}}_i^T}{\mathrm{d}\mathbb{Q}_i}\right|_{\mathcal{F}_t} := \overline{L}_i^T(t) := \frac{\overline{B}_i(t,T)}{\overline{B}_i(0,T)b_i(t)}, \qquad i = d, f$$

are called the domestic and the foreign T-survival measure.[23] Further we define the *term structures of default intensities* $\lambda_i(t,s) := \mathbf{E}^{\overline{\mathbb{P}}_i^s}[\,\lambda_i(s) \mid \mathcal{F}_t\,]$ and defaultable weights $\overline{w}_i(s;t,T) := \frac{\overline{B}_i(t,s)}{\int_t^T \overline{B}_i(t,s)ds}$. Then (19) simplifies to

$$\overline{s}_i(t,T) = q \int_t^T \overline{w}_i(s;t,T)\,\lambda_i(t,s)ds. \tag{21}$$

[22] This is also true when recovery and devaluation are independent, given default occurs.

[23] We assume $\overline{L}_i^T$ is a $\mathbb{Q}_i$-martingale.

The weights $\overline{w}_i$ are proportional to the defaultable ZCB price for the corresponding maturities. Overall, the price curve of defaultable ZCBs will be downward-sloping. The currency with the higher level of defaultable interest rates will have a stronger downward slope in the defaultable ZCB price $\overline{B}_i(t,s)$ and thus it will have a higher weight on *early* (small s) values of $\lambda_i(t,s)$, compared to the currency with a lower level of defaultable interest rates.

In many cases the fact that we have different weights $\overline{w}_d$ and $\overline{w}_f$ will only have a small influence on the differences in CDS rates, because the weights $\overline{w}_i$ will not differ by much, and the term-structure of default intensities will be quite *flat* (i.e. $\lambda_i(t,s)$ is close to a constant function of s). For flat term structures of $\lambda_i(t,s)$ the weighting has no influence at all; and if the slope is small, the influence of slightly varying weights will also be small. In these cases we may argue that $\overline{s}_i(t) \approx q\lambda_i(t)$.

It remains to analyze the difference between the domestic and foreign *term structure of default intensities*. Assume that X is an affine FX rate. Then $\widehat{\delta} := 1-\mathcal{L}(x)$, the expected devaluation fraction of X at default (under $\mathbb{Q}_d$ and given default occurs), is a constant and $\lambda_f(t) = (1-\widehat{\delta})\lambda_d(t)$. Further, chaining densities we can define the $\overline{\mathbb{P}}_d^T$-martingale

$$\overline{L}_{df}^T(t) := \left.\frac{d\overline{\mathbb{P}}_f^T}{d\overline{\mathbb{P}}_d^T}\right|_{\mathcal{F}_t} = L^X(t)\frac{\overline{L}_f^T(t)}{\overline{L}_d^T(t)} = \frac{X_t}{X_0}\,\frac{\overline{B}_f(t,T)/\overline{B}_d(t,T)}{\overline{B}_f(0,T)/\overline{B}_d(0,T)}.$$

It should be noted that when domestic and foreign ZCB price do not differ by much, then $\overline{L}_{df}^T$ is almost proportional to the FX rate X. In any case

$$\lambda_f(t,s) = (1-\widehat{\delta})\,\mathbf{E}^{\overline{\mathbb{P}}_d^s}\left[\left.\frac{\overline{L}_{df}^s(s)}{\overline{L}_{df}^s(t)}\lambda_d(s)\,\right|\,\mathcal{F}_t\right].$$

Thus, different term structures of default intensities may arise for two reasons. First, when domestic and foreign default intensities are not equal, i.e. when there is a non-zero expected devaluation fraction $\widehat{\delta}$. Second, when $\overline{L}_{df}^s$ and the domestic default intensity $\lambda_d(s)$ are correlated under the domestic s-survival measure $\overline{\mathbb{P}}_d^s$.

Overall we have identified four drivers of the difference between CDS rates in domestic and foreign currency. First, foreign and domestic default intensities are not equal, when the FX rate is subject to *default event risk*, i.e. when X jumps at default.[24] Second, foreign and domestic term structure of default intensities do not coincide (nor are they proportional) when there is covariance between λ_d and the $\overline{\mathbb{P}}_d^s$-martingale $\overline{L}_{df}^s$ under the domestic survival measure $\overline{P}_d^s$ at any time s. It can be shown that these term structures are equal, up to the multiplicative constant $(1-\widehat{\delta})$, when the default-free interest rates r_d, r_f are independent of λ and $[\lambda, X] = 0$, i.e. when X is not subject to *default intensity risk*.[25]

Third, the expected LGD $\widehat{q}_i$ may differ between the pricing measures $\mathbb{Q}_d$ and $\mathbb{Q}_f$ when devaluation is not independent of *default severity risk*. Fourth, the slopes of domestic and foreign term structure of default free interest rates determine (via their impact on the weights $\overline{w}_i$) how the respective term structures of default intensities must be weighted in order to derive the CDS rate in each currency.

[24] Unless the expected devaluation fraction is equal to zero.

[25] Precisely, we mean they have orthogonal volatilities ($[\lambda, X] = 0$).

5 Empirical Results

Here we focus on the first two drivers. There is theoretical and empirical evidence that the correlation between default-free interest-rates and default intensities has only a very small effect on CDS rates (see e.g. Houweling and Vorst (2005) or Schönbucher (2002)). Regarding the *relative* prices of CDS, this effect is likely to be even smaller here, so we feel justified in ignoring this effect. Furthermore, we will also assume that the correlation between recovery rate and devaluation fraction is not significant.

Assumption 19 (Empirical Estimation Setup).

(i) We assume that domestic and foreign interest-rates have zero covariation with exchange rate and default intensity:

$$[r_d, X] = 0 = [r_d, \lambda] \qquad \textit{and} \qquad [r_f, X] = 0 = [r_f, \lambda].$$

(ii) The cheapest-to-deliver bond of a CDS has a constant LGD rate q and, at default, the FX rate is devaluated by a constant fraction δ.

By virtue of assumption 19 *(i)* we do not have to model an affine model for the default-free interest-rates any more and we directly use the current 1M Libor rates as approximation for the short rate in the FX drift (23) and the current term-structure of interest-rates to discount future cash-flows.

As data source for CDS quotes we use the *ValuSpread* CDS database by *Lombard Risk Systems Ltd.*, a provider of a data pooling service for the CDS market. [26] The data starts in the third quarter of 1999. At the beginning of the period the set of CDSs on the same reference entity which are available in more than one currency is relatively sparse and the data frequency is only monthly but from 2002 onwards the data quality improves significantly both in frequency (weekly, then daily from 2003) and in the number of obligors with CDS rates in both JPY *and* USD. From the available set of Japanese reference names we selected the 25 obligors with the largest number of simultaneous data points in both JPY and USD. For default-free interest-rates we used JPY and USD term-structures of interest-rates based upon Bloomberg swap and money-market data. The JPY/USD exchange rate data was also taken from Bloomberg.

The snapshots in figure 5 act as a good example for the magnitude of the spread between domestic and foreign CDS rates in our dataset. JPY CDS rates on many large Japanese reference entities trade typically around 20 percent lower than their US$ equivalent.

5.1 A Simple Estimator for the Devaluation Fraction

As a first step, we tested for the presence of significant differences between JPY and USD CDS rates used a simple statistic based upon the limiting property (20), which states that

[26] At regular (daily) intervals, Lombard Risk Systems collects CDS quotes on a large number of reference names from a set of major market makers and dealers. The submitted data is cleaned and averaged and then added to the ValuSpread CDS data base and distributed back to the original data providers and to other subscribers of the service who use this data to mark their books and for various risk management purposes.

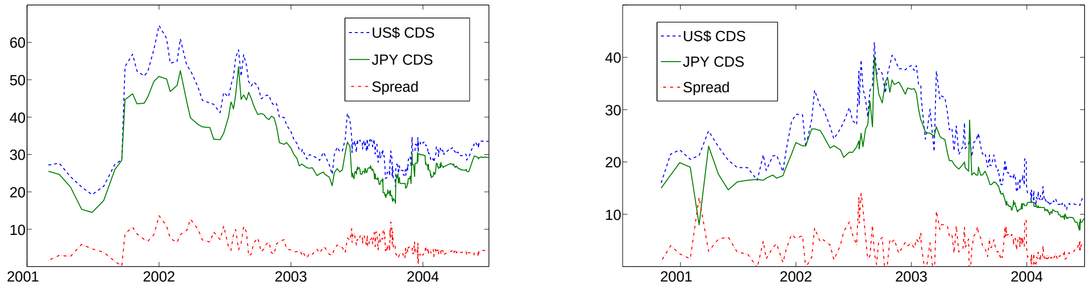

Figure 1: 5y CDS rates on Sony (left) and Tokyo Electric Power US$ and JPY. The mean spread amounts 5.4 bp for Sony and 3.6 bp for Tokyo Electric Power.

CDS rates for short maturities are approximately equal to LGD times default intensity. Remembering $\lambda_f = (1-\delta)\lambda_d$ (proposition 2), the following (approximate) estimator for δ is straightforward.

$$\widehat{\delta}_k := \frac{1}{n}\sum_{i=1}^{n}\left\{1 - \frac{\overline{s}_Y(t_i, t_i+k)}{\overline{s}_\$(t_i, t_i+k)}\right\}, \qquad k > 0. \tag{22}$$

Apart from the limit argument (20), we will see another argument why this estimator can be interpreted as an implied devaluation fraction in equation (28). We give the estimates

Ticker *(Company)*	1 year CDS $\widehat{\delta}_1$ [%]	1 year CDS p-Value [%]	5 year CDS $\widehat{\delta}_5$ [%]	5 year CDS p-Value [%]
ASAGLA *(Asahi Glass Company, Limited)*	15.50	0.000	11.19	0.000
BOT1 *(Bank of Tokyo-Mitsubishi, Ltd.)*	5.86	3.438	18.98	0.000
EJRAIL *(East Japan Railway Company)*	22.34	0.000	18.12	0.000
FUJITS1 *(Fujitsu Ltd)*	15.78	0.000	13.91	0.000
HITACH *(Hitachi, Ltd.)*	26.28	0.000	17.55	0.000
HONDA *(Honda Motor Co., Ltd.)*	16.77	0.000	20.86	0.000
MATSEL *(Matsushita Electric Industrial Co., Ltd.)*	25.79	0.000	17.42	0.000
MITCO1 *(Mitsubishi Corp)*	17.99	0.000	11.74	0.000
MITSCO *(Mitsui & Co., Ltd.)*	16.56	0.000	12.88	0.000
NECORP *(NEC Corpoartion)*	20.24	0.000	13.58	0.000
NIPOIL/NIPOIL1 *(Nippon Oil Corporation)*	14.59	0.000	14.55	0.000
NIPSTL *(Nippon Steel Corporation)*	14.23	0.000	12.51	0.000
NTT *(Nippon Telegraph & Telephone Corporation)*	28.50	0.000	19.05	0.000
NTTDCM *(NTT DoCoMo Inc.)*	35.75	0.000	15.27	0.000
ORIX *(Orix Corporation)*	18.86	0.000	12.93	0.000
SHARP *(Sharp Corporation)*	31.80	0.000	21.76	0.000
SNE *(Sony Corporation)*	28.21	0.000	17.17	0.000
SUMIBK2 *(Sumitomo Mitsui Banking Corporation)*	23.54	0.000	21.94	0.000
SUMT *(Sumitomo Corporation)*	14.32	0.000	9.61	0.000
TAKFUJ *(Takefuji Corporation)*	8.38	0.000	6.56	0.000
TOKELP *(Tokyo Electric Power Co., Inc.)*	29.22	0.000	18.73	0.000
TOKIO *(Tokio Marine and Fire Insurance Company Limited)*	30.11	0.000	22.37	0.000
TOSH *(Toshiba Corporation)*	22.62	0.000	14.01	0.000
TOYOTA1 *(Toyota Motor Corporation)*	31.26	0.000	27.41	0.000
YAMAHA *(Yamaha Motor Co., Ltd.)*	19.83	0.000	17.68	0.000

Table 1: Estimates for the devaluation fraction implied by the data.

$\widehat{\delta}_k$ for the maturities $k = 1$ years and $k = 5$ years in table 1. The 1Y maturity is the shortest maturity available to us, it is reported in the first pair of columns. We also added the 5Y maturity because market liquidity is usually concentrated at this point of the CDS term structure. Using a t-test on $1 - \overline{s}_Y/\overline{s}_\$$ we tested the hypothesis whether the deviations of the CDS ratio $\overline{s}_Y/\overline{s}_\$$ from 1 is just noise in the data. In all cases this hypothesis could be clearly rejected: We are facing a systematic feature of the data. Interestingly the devaluation fractions implied by the data is typically higher for the 5 CDS rates than for the 1 year CDS rates.

5.2 A Correlation Model

The differences between JPY and USD CDS rates reported in table 1 need not necessarily be caused by an implied devaluation of the JPY at default (i.e. a $\widehat{\delta} > 0$). As seen above, a difference between CDS rates in different currencies can also arise when default intensities and FX rate are correlated via their diffusion components. Thus, we now want to investigate whether it is possible to reproduce the observed differences in CDS rates *without* assuming a *devaluation* at default, but only using the dependency between defaults and exchange rate X that is generated in a purely diffusion-based setup.

We build up a concrete model for the observable background driving process Y using the classification of ADs into the families $\mathbb{A}_m(N)$. Because assumption 19 relieves us from modelling default-free interest-rates, we are left with the task of modelling the correlation structure of two variables: λ and the diffusion part of X, hence we need a dimension of at least $N \geq 2$. Second we have to choose the number $m \leq N$ of components of Y that we want to remain positive. In our case $\lambda > 0$ is desirable, hence we choose $m \geq 1$. Thus, an $\mathbb{A}_1(2)$ or an $\mathbb{A}_2(2)$ model is appropriate. We chose $Y \in \mathbb{A}_1(2)$ because for fixed N, the number of parameter restrictions is increasing in m (see 3.4), and if $Y \in \mathbb{A}_2(2)$, then the two components driving the stochastic volatility of Y, and thus of λ and X, can only have nonnegative correlation. (See Dai and Singleton (2000))

Then λ is a CIR process up to a positive additive constant. We restrict this constant and the matrix b defined in (7) to zero. This yields the following model under the DSMM $\mathbb{Q}_d$.

$$\begin{aligned} d\lambda &= \kappa(\theta - \lambda)dt + \sigma\sqrt{\lambda}\,dW^1 \\ \frac{dX}{X_-} &= (r_d - r_f)dt + \gamma_1\sqrt{\lambda}\,dW^1 + \gamma_2\,dW^2 - \delta(dN - \lambda dt) \end{aligned} \tag{23}$$

with $\kappa, \theta, \sigma, \gamma_2 > 0$, $\gamma_1 \in \mathbb{R}$ and $\delta < 1$ and $(W^1, W^2)'$ a standard BM under $\mathbb{Q}_d$. The instantaneous correlation of default intensity and log FX rate,

$$\rho(\log X^c, \lambda) := \frac{\frac{d}{dt}[\log X^c, \lambda]}{\sqrt{\frac{d}{dt}[\log X^c]\frac{d}{dt}[\lambda]}} = \operatorname{sgn}(\gamma_1)\left(1 + \frac{\gamma_2^2}{\gamma_1^2\lambda}\right)^{-\frac{1}{2}},$$

is essentially controlled by the ratio γ_1/γ_2, but also depends on the current level of the stochastic process λ. Importantly its sign depends only on the sign of γ_1. Further we have to link the DSMM to the physical measure $\mathbb{P}$, under which the data was generated. For tractability we assume that the state price density is of the form

$$\left.\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{Q}}\right|_{\mathcal{F}_T} := L_T \quad \text{with} \quad \frac{dL}{L_-} = \phi_1\sqrt{\lambda}\,dW^1 + \phi_2\,dW^2 - \Phi(dN - \lambda dt)$$

for $\phi_1, \phi_2 \in \mathbb{R}$ and $\Phi < 1$. Then Y is also an AD $\in \mathbb{A}_1(2)$ under $\mathbb{P}$.

In order to estimate the parameters of an $\mathbb{A}_m(N)$ model a large number of relatively demanding estimation methodologies have been proposed for non-Gaussian AD models (Singleton (2001), Ait-Sahalia (2002), Gallant and Tauchen (1996)). Given the simplicity of our model and the high frequency of our data (daily for most of the dataset) we used a simple quasi maximum likelihood (QML) estimator by approximating (23) with its Euler discretisation. Then the parameter estimation reduces to a linear regression problem in transformed variables. Note that for every volatility estimator of the above model the value of δ plays no role as long as default has not yet occurred.

As the intensity λ is not directly quoted in the markets, we need to find a proxy for it. Here, the limiting property (20) suggests that CDS rates with a short maturity are approximately proportional to λ, and equation (21) tells us that also for longer times to maturity, CDS rates are proportional to a weighted average of forward hazard rates. Therefore, we decided to use CDS rates as approximation for $q\lambda_d(t)$. Ideally, we would have liked to use $1Y$ CDS rates, but the data for the $5Y$ maturity turned out to be cleaner and more liquid, so we used $5Y$ USD CDS rates, $\overline{s}_d(t, t+5)$.

If q is unknown, then γ_1 can only be estimated up to a positive constant. Therefore, we give QML estimates and 95%-confidence intervals for $\frac{\gamma_1}{q\sigma}$ in table 2. However, the

Ticker *(Company)*	$\widehat{(\frac{\gamma_1}{q\sigma})}$	95%-CI	$\rho(\log X^c, \lambda)$ *(mean, [%])*
ASAGLA *(Asahi Glass Company Ltd.)*	−2.26	[−10.18, 5.67]	−3.97
BOT1 *(Bank of Tokyo-Mitsubishi Ltd.)*	−2.80	[5.62, 0.01]	−16.58
EJRAIL *(East Japan Railway Co.)*	−2.34	[−10.57, 5.90]	−3.38
FUJITS1 *(Fujitsu Ltd)*	−0.15	[−1.58, 1.27]	−1.09
HITACH *(Hitachi Ltd.)*	−1.14	[−4.78, 2.50]	−3.58
HONDA *(Honda Motor Co. Ltd.)*	−2.52	[−7.80, 2.75]	−5.48
MATSEL *(Matsushita Electric Industrial Co. Ltd.)*	1.35	[−4.43, 1.73]	−5.03
MITCO1 *(Mitsubishi Corp.)*	−4.10	[−8.55, 0.35]	−10.69
MITSCO *(Mitsui & Co. Ltd.)*	−4.43	[−8.72, −0.13]*	−10.25
NECORP *(NEC Corp.)*	−1.51	[−3.60, 0.58]	−7.23
NIPOIL1/NIPOIL *(Nippon Oil Corp.)*	−3.92	[−10.28, 2.44]	−7.98
NIPSTL *(Nippon Steel Corp.)*	−5.67	[−9.44, −1.89]*	−15.04
NTT *(Nippon Telegraph & Telephone Corp.)*	−3.16	[−11.59, 5.28]	−4.80
NTTDCM *(NTT DoCoMo Inc.)*	−3.90	[−9.76, 1.95]	−8.61
ORIX *(Orix Corp.)*	−0.38	[−1.84, 1.07]	−2.94
SHARP *(Sharp Corp.)*	−3.96	[−7.28, −0.65]*	−15.47
SNE *(Sony Corp.)*	−2.84	[−7.43, 1.76]	−7.06
SUMIBK2 *(Sumitomo Mitsui Banking Corp.)*	−1.42	[−5.34, 2.50]	−4.63
SUMT *(Sumitomo Corp.)*	−3.88	[−7.65, −0.12]*	−11.72
TAKFUJ *(Takefuji Corp.)*	0.07	[−0.48, 0.62]	1.32
TOKELP *(Tokyo Electric Power Co. Inc.)*	−3.15	[−8.02, 1.72]	−6.98
TOKIO *(Tokyo Marine & Fire Insurance Co. Ltd.)*	−1.51	[−4.39, 1.37]	−6.95
TOSH *(Toshiba Corp.)*	−0.70	[−2.78, 1.37]	−3.58
TOYOTA1 *(Toyota Motor Corp.)*	−3.68	[−13.93, 6.58]	−4.22
YAMAHA *(Yamaha Motor Co., Ltd.)*	0.38	[−1.63, 2.40]	1.98

Table 2: QLM estimates of the Correlation Parameter and Averaged Correlation

level of correlation seems to be rather low: For only four companies, MITSCO, NIPSTL, SHARP and SUMT, the null hypothesis $H_0: \ \gamma_1 = 0$ can be rejected on the 95% level. To provide a quantity which is more intuitively understandable, we also computed the average instantaneous correlation function $\rho(\log X^c, \lambda)$ by combining the estimates for $\frac{\gamma_1}{q\sigma}$ with the estimates of γ_2 and σ. This quantity can be viewed as the average correlation between default intensity and exchange rate over our sample period.

5.3 Evidence for Devaluation at Default

As we outlined in 4.3, $\gamma_1 \neq 0$ if and only if $[X, \lambda] \neq 0$, which implies different term structures of default intensities in USD and JPY. We now attempt to answer the question whether this purely correlation based dependence is able to explain the whole spread between USD and JPY CDS rates, i.e. the question whether we can we can make our life much easier and set the jump interaction parameter δ to zero? Formally, our null hypothesis is: "The difference between JPY and USD CDS spreads is caused by a model like (23) with no devaluation at default, i.e. $\delta = 0$."

In order to evaluate the Null, we must first completely specify the model (23). Given $\delta = 0$, and for a given value of the loss given default q (which is also provided in our CDS database), we are able to compute QML-estimates for all parameters in (23) including the market price of risk ϕ_1, ϕ_2. In the estimates of these parameters only USD CDS rates (and *not* JPY CDS rates) are used besides the exchange rate and interest-rate data.

Next, we try to find out whether there are *any* combinations of λ and the default-free term structures for which the resulting ratio of JPY CDS spreads to USD CDS spreads is of a similar order of magnitude as the one observed in table 1. We do not need to consider the *current* value of X as it only enters the CDS spreads through its dependency with λ under the different measures (see e.g. equation (21)). For this, we took the mean term structures of default-free interest rates over the sample period in US$ and JPY. We also considered the term structure which turned out to be most favorable (disfavorable) to our null hypothesis: the steepest (flattest) default-free USD term structure of interest-rates of this period, together with the JPY term structure of the same day. We first used $\lambda(t_i) \approx \frac{1}{q}\overline{s}_{\$}(t_i, t_i{+}5)$ as an approximation for the default intensity at t_i and then calculated at each t_i the corresponding 5Y USD and JPY CDS rates (21).

The resulting spread between USD and JPY CDS rates was in no case even close to the spread empirically observed in the data: Even with the steepest term structure of USD-interest-rates we typically found a theoretical relative spread of less than 5%, (with the mean term structure one of less than 3% and with the flattest less than 2%): We applied the estimator (22) to the resulting theoretical 5Y CDS rates and also calculated $\min_i \left\{1 - \frac{\overline{s}_{Y}(t_i, t_i+5)}{\overline{s}_{\$}(t_i, t_i+5)}\right\}$. With the mean and the steepest term structures of interest rates in this period we reached the values reported in table 3. (We considered only the tickers

	mean term structure			steepest term structure		
Ticker *(Company)*	$\widehat{\delta}_5$	95%-CI	$\delta_{5,max}$	$\widehat{\delta}_5$	95%-CI	$\delta_{5,max}$
BOT1 *(Bank of Tokyo-Mitsubishi Ltd.)*	0.9	[−1.8, 3.4]	2.1	−0.2	[−2.9, 2.4]	1.7
EJRAIL *(East Japan Railway Co.)*	0.0	[−0.7, 0.7]	1.1	−0.1	[−0.8, 0.6]	1.7
HONDA *(Honda Motor Co. Ltd.)*	0.3	[0.3, 0.4]	0.8	0.6	[0.5, 0.6]	1.5
MITCO1 *(Mitsubishi Corp.)*	1.4	[1.2, 1.5]	3.8	2.2	[2.1, 2.4]	6.4
MITSCO *(Mitsui & Co. Ltd.)*	0.9	[0.5, 1.3]	3.2	1.2	[0.9, 1.6]	5.1
NTT *(Nippon Telegraph & Telephone Corp.)*	1.9	[1.8, 2.0]	3.9	3.3	[3.2, 3.3]	6.8
NTTDCM *(NTT DoCoMo Inc.)*	2.1	[2.0, 2.3]	4.4	3.6	[3.5, 3.8]	7.5
SHARP *(Sharp Corp.)*	0.7	[0.4, 0.9]	1.5	0.9	[0.7, 1.2]	2.3
SUMT *(Sumitomo Corp.)*	1.3	[0.4, 2.1]	3.7	1.5	[0.7, 2.3]	5.4
TAKFUJ *(Takefuji Corp.)*	1.5	[1.5, 1.5]	3.3	2.7	[2.7, 2.7]	5.8
TOKIO *(Tokyo Marine & Fire Insurance Co. Ltd.)*	0.9	[0.8, 1.0]	2.4	1.5	[1.4, 1.7]	4.2
TOYOTA1 *(Toyota Motor Corp.)*	1.2	[1.1, 1.2]	2.1	2.1	[2.0, 2.1]	3.8
YAMAHA *(Yamaha Motor Co., Ltd.)*	2.7	[2.7, 2.8]	6.4	4.6	[4.7, 4.8]	10.9

Table 3: Implied devaluation fractions [%] in the pure-diffusion setup.

which showed positive mean reversion κ under $\mathbb{Q}$.) Comparing these results to the relative spreads reported in table 1 we reject the null hypothesis of $\delta = 0$.

On the other hand, for any given relative spread of USD and JPY CDS rates (and for any given term structures of default-free interest-rates) we can find a *nonzero* value of $\delta \neq 0$ such that this relative spread is reproduced. So we can*not* reject the hypothesis that the devaluation fraction δ is *not* zero.

6 Applications and Extensions

6.1 Discussion of the Empirical Results

The spread between domestic and foreign CDS rates implies a currency devaluation of between 15 and 25% at a default of one of the larger obligors in our sample. This may seem rather large, and it may be (at least partially) caused by market imperfections. Nevertheless, the size and persistence of the effect (it did not change significantly although the market liquidity has multiplied in our sample) indicates that there is – at least to some extent – a real foundation to it. For example, a default of a large Japanese firm will most likely happen in a serious recession scenario, or it will be an indicator of severe fundamental problems in the economy (e.g. a bank default would be a strong indicator that the Japanese bad-loans problem is more serious than expected).

Furthermore, we would like to point out that we excluded some sources of dependency in our setup, in particular the possibility of joint jumps of FX rate and default intensity *before* default, and the dependency *across* obligors. In particular the latter effect should be investigated further: A macro-economic effect of a default (e.g. FX devaluation) should be particularly large if the default itself was caused by a systematic, macro-economic reason. But then the default is also more likely to affect other obligors. Thus, the implied FX devaluation at default may even give us information on the dependency between the obligors and the macro economy if all variables (defaults and FX rate) are driven by the same macro-variables.

We would have expected to see significantly different results for firms that are active in foreign trade as exporters, as importers, and firms that mostly service the domestic market. Making this distinction for large Japanese corporations is rather difficult and from a qualitative inspection of our results we did not see any such systematic connection.

To the methodology of our study it was irrelevant that the reference obligors were Japanese companies. What we needed was that X was the exchange rate between the CDS-currencies, but not that X had any connection to the reference credit. Thus, a similar study can also be performed using reference credits that are not incorporated in the "foreign" country. A particularly interesting case arises if the reference credit of the CDS is a sovereign itself. Usually, this sovereign will not be a G7 country, so (unless this is explicitly stated in the term sheet) the CDS will not be denominated in its currency. Nevertheless, many developing countries have issued debt in multiple currencies, e.g. USD, EUR, and JPY, and CDS on these sovereigns can also be traded in these currencies. The relative spreads of these CDS will then allow us to make statements about the implied effect of a sovereign default of e.g. Brazil on the EUR/USD exchange rate in much the same way as we made statements about the JPY/USD exchange rate upon default of a corporate reference credit. In this case, it will not be clear which currency we would expect to devalue, i.e. $\hat{\delta}$ may also be negative.

6.2 Other Default-Sensitive FX Derivatives

We used CDS with denomination in different currencies to illustrate the key ideas and methods which must be used to analyse credit risk in multiple currencies, but there are also a number of other situations in which the methods of this paper can be used, for example some exotic credit derivatives and the analysis of counterparty credit risk in OTC derivatives transactions. In this section we present some of these connections.

As a first application, we define default-sensitive equivalents for the most common FX derivatives such as FX swaps and FX forwards. These derivatives behave like their default-free equivalents except that they are *cancelled at default.* It will turn out that some of these instruments have a natural connection to the problem of pricing CDSs in different currencies.

We define the **defaultable FX forward** rate $\overline{X}$ as

$$\overline{X}(t,T) = X(t)\frac{\overline{B}_f(t,T)}{\overline{B}_d(t,T)}. \tag{24}$$

This is the forward exchange rate to be used in a FX forward contract which is *cancelled at default* (i.e. if a default occurs before the settlement date T). Using the defaultable zero coupon bond prices given in section 4.2.2, the defaultable FX forward rates can be given in closed-form.

In a **defaultable currency swap** one side of the swap pays a stream of the *defaultable currency swap rate* $\overline{x}(t,T)$ in c_d, and the other side of the swap pays a stream of 1 in c_f, and both payment streams stop at default, or — if no default occurs before T — at the maturity date T. (In contrast to a classical currency swap we assume no exchange of principal at maturity.) As both legs of the currency swap must have the same value, we reach the following representation for $\overline{x}(t,T)$:

$$\overline{x}(t,T)\int_t^T \overline{B}_d(t,s)ds = X(t)\int_t^T \overline{B}_f(t,s)ds = \int_t^T \overline{X}(t,s)\overline{B}_d(t,s)ds \tag{25}$$

and thus

$$\overline{x}(t,T) = X(t)\frac{\int_t^T \overline{B}_f(t,s)ds}{\int_t^T \overline{B}_d(t,s)ds} = \int_t^T \overline{w}_d(s;t,T)\overline{X}(t,s)ds.$$

Thus, we can view the rate $\overline{x}$ as a weighted average of the defaultable FX-forward rates $\overline{X}(t,s)$ over the life of the swap, or as a *survival-contingent exchange rate* for payment streams. Alternatively, it can be viewed as the relative price of a unit payoff stream (an annuity) in foreign-currency $\int_t^T \overline{B}_f(t,s)ds$, measured in units of the *domestic defaultable annuity* $\int_t^T \overline{B}_d(t,s)ds$.

The defaultable currency swap in (25) allows us to transform any CDS fee stream in c_f into a corresponding fee stream in c_d. This leads naturally to the following instrument:

A **Quanto CDS** is a credit-default swap, which has a protection payment in one currency (e.g. c_f), but the fee payment is made in another currency (e.g. c_d) at the rate $\overline{s}^{\text{quanto}}$. Using the defaultable currency swap to transform the fee streams we reach:

$$\overline{s}^{\text{quanto}}(t,T) = \overline{s}_f(t,T)\,\overline{x}(t,T). \tag{26}$$

A **Default-Contingent FX Forward** is a contract to exchange 1 unit of foreign currency for $\overline{X}^{\tau}(t,T)$ units of domestic currency *at the time of default* τ, if and only if default occurs before its maturity T.

This instrument may seem a bit artificial but we believe there should be interest in it. Often, investors in foreign companies have secured strong collateralisation of their loans. Upon a credit event, these investors will have to liquidate the collateral and convert a significant amount of cash back into their domestic currency. Such investors might be interested in a default-contingent FX forward, i.e. a FX forward contract which allows the investor to exchange if (and only if) a credit event has occurred.

Assuming constant recovery rates $1-q$, we can model this contract as a portfolio of two CDS contracts, one of which is short protection in foreign currency with notional $1/q$ (thus paying one unit of c_f at default), and one that is long protection in domestic currency with notional $\overline{X}^{\tau}/q$ (which pays one unit of c_d at default). We can convert the fees of the c_f CDS into domestic currency using the quanto CDS introduced above to reach a net value in c_d for the default-contingent FX forward of

$$\frac{1}{q}\Big(\overline{s}_f(t,T)\,\overline{x}(t,T) - \overline{X}^{\tau}(t,T)\,\overline{s}_d(t,T)\Big)\int_t^T \overline{B}_d(t,s)ds,$$

i.e. the fee in domestic currency that has to be paid for this contract is $\frac{1}{q}(\overline{s}_f\,\overline{x} - \overline{X}^{\tau}\overline{s}_d)$. Finally, for the contract to be fair (i.e. for the fee to be zero), we need to set the *default-contingent FX forward rate* to

$$\overline{X}^{\tau}(t,T) = \overline{x}(t,T)\,\frac{\overline{s}_f(t,T)}{\overline{s}_d(t,T)}. \tag{27}$$

The Ratio of Domestic to Foreign CDS Rates: From (27) we reach immediately

$$\frac{\overline{s}_f(t,T)}{\overline{s}_d(t,T)} = \frac{\overline{X}^{\tau}(t,T)}{\overline{x}(t,T)}. \tag{28}$$

The ratio of the foreign to the domestic CDS rate is the ratio of two exchange rates: The exchange rate $\overline{X}^{\tau}(t,T)$ that applies at the time of default *only if a default occurs over* $[t,T]$ to the exchange rate $\overline{x}(t,T)$ that applies over $[t,T]$ *only for the time the obligor survives.*

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A Appendix

Proof of Lemma 2. $M_t := \mathbf{E}\left[e^{-\int_0^T \lambda(s)ds} g_T \,\middle|\, \mathcal{F}_t \right]$, $t \in [0, T^*]$ is a uniformly integrable martingale. Under hypothesis $\mathbb{H}$ (see definition $\mathbb{H}_3$ in Jeanblanc and Rutkowski (2000)) we have that $\mathbf{E}\left[e^{-\int_0^T \lambda(s)ds} g_T \,\middle|\, \mathcal{F}_t \right] = \mathbf{E}\left[e^{-\int_0^T \lambda(s)ds} g_T \,\middle|\, \mathcal{F}_t^W \right]$ because $e^{-\int_0^T \lambda(s)ds} g_T$ is

$\mathcal{F}_T^W$-measurable and thus the strong predictable representation property (see e.g Karatzas and Shreve (1991)) implies that there exists a $\mathbb{F}^W$-adapted process ϕ_M such that

$$M_t = M_0 + \int_0^t \phi_M(s) dW_s. \tag{29}$$

$Y_t := 1_{\{t<\tau\}} e^{\int_0^t \lambda(s)ds}$ is a local martingale ($dY_t = Y_{t-}(-dN_t + \lambda(t)dt)$) with $Y_T = 1_{\{T<\tau\}} e^{\int_0^T \lambda(s)ds}$. MY is also a local martingale because $d[M,Y] = d[M^c, Y^c] + \Delta M \Delta Y = 0$. Note that this follows only from (29), i.e. from hypothesis $\mathbb{H}$. Further $\mathbf{E}\left[\, |g_T| \mid \mathcal{F}_t \,\right]$ is uniformly integrable and $|M_t Y_t| \leq \mathbf{E}\left[\, |g_T| \mid \mathcal{F}_t \,\right]$ (Jensen's inequality for cond. expectations). Thus MY is uniformly integrable, i.e. it is a martingale and we have

$$1_{\{t<\tau\}} \mathbf{E}\left[\, e^{-\int_t^T \lambda(s)ds} g_T \;\middle|\; \mathcal{F}_t \,\right] = M_t Y_t = \mathbf{E}\left[\, M_T Y_T \mid \mathcal{F}_t \,\right] = \mathbf{E}\left[\, 1_{\{T<\tau\}} g_T \;\middle|\; \mathcal{F}_t \,\right]. \qquad \square$$

Proof of corollary 3. $\mathbf{E}\left[\, \int_0^T 1_{\{s\leq\tau\}} |g_s| ds \,\right] < \infty$ guarantees that $1_{\{t\leq\tau\}} g_t$ is $\mathbb{Q}\times l$-integrable. Hence Fubini's theorem applies and $\mathbf{E}\left[\, 1_{\{t\leq\tau\}} |g_t| \,\right] < \infty$ l-a.e. For every $F \in \mathcal{F}_t$

$$\begin{aligned}
\mathbf{E}\left[\, 1_F \int_t^T 1_{\{s\leq\tau\}} g_s ds \,\right] &= \int_t^T \mathbf{E}\left[\, 1_{\{s\leq\tau\}} g_s \, 1_F \,\right] ds = \int_t^T \mathbf{E}\left[\, 1_F \, \mathbf{E}\left[\, 1_{\{s\leq\tau\}} g_s \;\middle|\; \mathcal{F}_t \,\right] \,\right] ds \\
&= \mathbf{E}\left[\, \int_t^T 1_F \, \mathbf{E}\left[\, 1_{\{s\leq\tau\}} g_s \;\middle|\; \mathcal{F}_t \,\right] ds \,\right] \\
&= \mathbf{E}\left[\, 1_F \int_t^T \mathbf{E}\left[\, 1_{\{s<\tau\}} g_s \;\middle|\; \mathcal{F}_t \,\right] ds \,\right].
\end{aligned}$$

In the last equality we used that $\mathbf{E}\left[\, 1_{\{t\leq\tau\}} g_t \,\right] = \mathbf{E}\left[\, 1_{\{t<\tau\}} g_t \,\right]$ $\mathbb{Q}\times l$-a.e. because $1_{\{t\leq\tau\}} = 1_{\{t<\tau\}}$ l-a.e. Together with lemma 2 this proves the claim. $\square$

Proof of corollary 4. By a variation of theorem 1.8 of chapter II in Jacod and Shiryaev (1988) (4) ensures $\int_0^t \int_{\mathcal{Z}} 1_{\{s\leq\tau\}} f(z,s)(\mu-\nu)(dz,ds)$ is a martingale. Thus, for all $F \in \mathcal{F}_t$[27]

$$\mathbf{E}\left[\, 1_F \int_t^T \int_{\mathcal{Z}} 1_{\{s\leq\tau\}} f(z,s) \mu(dz,ds) \,\right] = \mathbf{E}\left[\, 1_F \int_t^T 1_{\{s\leq\tau\}} \Big(\int_{\mathcal{Z}} f(z,s) dF_s(z) \Big) \lambda_s ds \,\right] < \infty$$

By Fubini's theorem $\mathbf{E}\left[\, \int_{\mathcal{Z}} |f(z,t)| dF_t(z) \, 1_{\{t\leq\tau\}} \lambda_t \,\right] < \infty$ l-a.e. and we can proceed as in the proof of lemma 3. $\square$

Proof of Lemma 7. We omit the argument of f and Φ. Recall $\Delta L = L_- \int_{\mathcal{Z}} (\Phi - 1) \mu(dz, dt)$. By Itô's lemma

$$\begin{aligned}
d\Big(L_s \int_0^s \int_{\mathcal{Z}} f \mu(dz,du) \Big) &= L_{s-} \int_{\mathcal{Z}} f \mu(dz,ds) + \Big(\int_0^{s-} \int_{\mathcal{Z}} f \mu(dz,du) \Big) dL_s \\
&\quad + \Delta L_s \int_{\mathcal{Z}} f \mu(dz,ds) \\
&= L_{s-} \int_{\mathcal{Z}} f \Phi \mu(dz,ds) + \Big(\int_0^{s-} \int_{\mathcal{Z}} f \mu(dz,du) \Big) dL_s.
\end{aligned}$$

[27] $\widetilde{f}(z,s) := 1_{\{s>t\}} 1_F f(z,s)$ is predictable for fixed t and $F \in \mathcal{F}_t$.

Condition $\mathbf{E}\left[\,\int_0^T h_{t-}^2\, d\langle L\rangle_t\,\right] < \infty$ ensures that the stochastic integral wrt. L, $\int_0^t h_{s-} dL_s$, is a true martingale, hence has zero expectation. The arguments hold true when f is replaced by $\widetilde{f} := 1_{(t,T]} 1_A f$ for arbitrary $A \in \mathcal{F}_t$, this proves the claim. $\square$

Proof of Lemma 8. From Itô's lemma it follows

$$d\Big(L_s \int_0^s g_u\, du\Big) = L_{s-} g_s\, ds + \Big(\int_0^s g_u\, du\Big) dL_s$$

Condition $\mathbf{E}\left[\,\int_0^T (\int_0^t |g_s|\, ds)^2\, d\langle L\rangle_t\,\right] < \infty$ guarantees that $\int_0^t (\int_0^s g_u du)\, dL_s$ is a true martingale. This remains valid for $\widetilde{g} := 1_{(t,T]} 1_A g$ for arbitrary $A \in \mathcal{F}_t$ instead of g and the claim follows. $\square$

Proof of lemma 9. The boundedness of $\mathcal{B}$ (and hence $\mathcal{A}$) guarantees that $\mathcal{A}$ and $\mathcal{B}$ are C^1-functions. Hence $M(t) = e^{\int_0^t \alpha + \beta' Y_s ds + \mathcal{A}(T-t) + \mathcal{B}(T-t)' Y_t}$, $t \in [0,T]$, is differentiable in t and it can be checked easily via Itô's lemma that M solves the SDE

$$\frac{dM_t}{M_t} = \mathcal{B}(T-t)' \sqrt{S_t}\, dW_t$$

if $\mathcal{A}$ and $\mathcal{B}$ solve the ODE (8), hence M is a local martingale. By $\mathbf{E}\left[\, e^{\frac{1}{2}\int_0^T w(\mathcal{B}^*)' Y_s ds}\,\right] < \infty$ (Novikov's condition) M is a martingale and the boundary conditions $\mathcal{A}(0) = 0$ and $\mathcal{B}(0) = 0$ imply the terminal value $M_T = e^{\int_0^T \alpha + \beta' Y_s ds}$. $\square$

Proof of lemma 10. If $\xi = 0$, then lemma 9 applies. Hence we consider only the case where ξ has at least one positive component. Note that (9) is linear in B with bounded coefficients as long as $\mathcal{B}$ is bounded, i.e. B is also bounded. Moreover $B \geq 0$, for at least one component $\min_{0 \leq t \leq T} B_i(t) > 0$ and $A \geq 0$. Thus

$$\mathcal{M}(t) := e^{\int_0^t \alpha + \beta' Y_s ds + \mathcal{A}(T-t) + \mathcal{B}(T-t)' Y_t} \big(A(T-t) + B(T-t)' Y_t\big),$$

is differentiable in t and $\mathbb{Q}$-a.s. positive for all $t \in [0,T]$. By Itô's lemma

$$\frac{d\mathcal{M}(t)}{\mathcal{M}(t)} = \left(\mathcal{B}(T-t) + \frac{B(T-t)}{A(T-t) + B(T-t)' Y_t}\right)' \sqrt{S_t}\, dW_t$$

i.e. $\mathcal{M}$ is a local martingale. Novikov's criterion $\mathbf{E}\left[\, e^{\frac{1}{2}\int_0^T w\left(\mathcal{B}^* + \frac{B^*}{A_* + B_*' Y_t}\right)' Y_t dt}\,\right] < \infty$ guarantees that $\mathcal{M}$ is also a martingale and by the initial condition $A(0) = \zeta$ and $B(0) = \xi$ its terminal value is $\mathcal{M}_T = e^{\int_0^T \alpha + \beta' Y_s ds}\big(\zeta + \xi' Y_T\big)$. $\square$

Proof of formula (15). By virtue of (11) it suffices to proof the domestic formula. We first define $\widehat{q}_d := 1 - \mathcal{L}_d(u_d)$ and choose an $\mathcal{F}_T^W$-measurable g_T with $\mathbf{E}^{\mathbb{Q}_d}[\,|g_T|\,] < \infty$. Then the proof goes much in line with that of lemma 2. Define the $\mathbb{Q}_d$-martingale $M_t := \mathbf{E}^{\mathbb{Q}_d}\left[\, e^{-\widehat{q}_d \int_0^T \lambda_d(s) ds} g_T \,\middle|\, \mathcal{F}_t\right]$. It can be checked easily that $Y_t := p_d^{MD}(t) e^{\widehat{q}_d \int_0^t \lambda_d(s) ds}$ is a $\mathbb{Q}_d$-local martingale. Due to assumption of hypothesis $\mathbb{H}$ we have $M_t = \mathbf{E}\left[\, e^{-\int_0^T \lambda(s) ds} g_T \,\middle|\, \mathcal{F}_t^W\right]$

and thus MY is also a $\mathbb{Q}_d$-local martingale. Further $|M_t Y_t| \leq \mathbf{E}^{\mathbb{Q}_d}\left[\, |g_T| \mid \mathcal{F}_t \,\right]$, hence MY is a true $\mathbb{Q}_d$-martingale and

$$p_d^{MD}(t)\, \mathbf{E}^{\mathbb{Q}_d}\left[\, e^{-\widehat{q}_d \int_t^T \lambda_d(s) ds} g_T \;\middle|\; \mathcal{F}_t \right] = M_t Y_t = \mathbf{E}^{\mathbb{Q}_d}\left[\, M_T Y_T \mid \mathcal{F}_t \,\right] = \mathbf{E}^{\mathbb{Q}_d}\left[\, p_d^{MD}(T)\, g_T \;\middle|\; \mathcal{F}_t \right].$$

We set $g_T := e^{-\int_t^T r_d(s) ds}$, then lemma 9 yields the claim. $\square$