A stochastic model for assets and liabilities of a pension institution

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Motivation

- We develop a stochastic model for assets and liabilities of a pension institution.
- The asset side of our model applies equally well to other financial institutions.
- The model can be used in
  - dynamic financial analysis,
  - dynamic portfolio optimization,
  - calculation of solvency capital requirements,
  - ...

Assets and liabilities

We model the total returns of

1. cash,
2. bonds,
3. Finnish equities,
4. European equities,
5. North American equities,
6. Asian equities,
7. European real estate

together with the cash-flows and technical reserves of a pension institution.

Assets and liabilities

- For cash investments, the return over a holding period of length Δt is determined by the short rate F^1,

$$R_t^1 = e^{\Delta t F_{t-1}^1}.$$

- The short rate will be modeled as a strictly positive stochastic process which will imply that $R^1 > 1$.

Assets and liabilities

- Bond prices can be expressed in terms of their payouts $(c_t)_{t=1}^T$ as

$$P(Y) = \sum_{t=1}^{T} \frac{c_t}{(1+Y)^t},$$

where Y is the yield to maturity.

- The duration

$$D = -\frac{dP(Y)}{dY}$$

is often used in the approximation

$$P(Y_t) \approx P(Y_{t-1}) - D(Y_t - Y_{t-1}).$$

- However, this may predict negative prices.

Assets and liabilities

- We will use the approximation

$$\ln P(Y_t) \approx \ln P(Y_{t-1}) - \tilde{D}(Y_t - Y_{t-1}),$$

where we use the logarithmic duration

$$\tilde{D} = -\frac{d \ln P(Y)}{dY}.$$

- This gives the approximation

$$\begin{aligned} R_t^2 &\approx \Delta t Y_{t-1} + \frac{P_t(Y_t)}{P_{t-1}(Y_{t-1})} \\ &\approx \Delta t Y_{t-1} + \exp(-\tilde{D}(Y_t - Y_{t-1})). \end{aligned}$$

Assets and liabilities

- The total returns R^j on equity as well as real estate investments are given simply in terms of the total return indices (in euros) S^j,

$$R_t^j = \frac{S_t^j}{S_{t-1}^j} \quad j = 3, \ldots, 7.$$

- The total return indices will be modeled as positive stochastic processes.

Assets and liabilities

- On the liability side, our goal is to model a pension institution's cash-flows and technical reserves associated with major insurance classes.
- In pension insurance, the cash-flows and technical reserves typically depend on wages and the distribution of insured into different states J within the cohorts I.
- For example, when modeling a pension institution's aggregate cash-flows and technical reserves associated with old age and disability pensions, the relevant states are

$$J = \{\text{working, disabled, old age, other, dead}\}.$$

Assets and liabilities

- The number of people in cohort i in state j in year t will be denoted by $K_t^{i,j}$. The development of the vector $K_t^i = (K_t^{i,j})_{j \in J}$ will be modeled by a Markov chain

$$K_{t+1}^{i+1} = \Pi_t^i K_t^i,$$

  where $i+1$ refers to the cohort one year older than i and Π_t^i is the matrix of transition probabilities.
- The elements of Π_t^i depend on the development of cohorts and the employment rate E.
- The employment rate will be modeled as a stochastic process with values in the interval $(0, 1)$.

Assets and liabilities

- The payroll of cohort i can be written as

$$P_t^i = K_t^{i,working} \bar{P}_t^i,$$

where $\bar{P}^i$ is the average wage in cohort i. Average wages will be approximated by

$$\bar{P}_t^i \approx \bar{P}_0^i \frac{W_t}{W_0},$$

where W is the general wage index and $\bar{P}_0^i$ is the average wage in cohort i in the initial year 0.

- The wage index will be modeled as a strictly positive stochastic process.

Data

- The asset returns have been expressed in terms of the short rate F^1, bond yield Y, the total return indices $S^3, \ldots, S^7$.
- The cash-flows and technical reserves have been described in terms of the wage index W and the employment rate E.
- The cohort sizes will be assumed deterministic.
- We will also include the consumer price index I in our model. This allows modeling relationships between inflation and interest rates as well as asset prices.

Data

Monthly values from January 1992 to December 2006 of

- F^1 German 3 month interest rate,
- Y German government 5 year bond,
- S^3 OMX HELSINKI CAP total return index,
- S^4 DJ EURO STOXX 50 total return index,
- S^5 DJ Americas 600 total return index,
- S^6 DJ Asia/Pacific 600 total return index,
- S^7 EPRA/NAREIT Euro Zone-index,
- E seasonally adjusted Finnish employment rate,
- W seasonally adjusted Finnish wage-level index,
- I seasonally adjusted Europe eurozone harmonized consumer price index.

Data

- The forward rate defined as

$$F^2 = \frac{t_2 Y - t_1 F^1}{t_2 - t_1}$$

  is always strictly positive, which corresponds to the fact that the "zero curve" is a decreasing function of maturity.

- The bond yield is in turn given by

$$Y_t = \frac{t_1 F^1 + (t_2 - t_1) F^2}{t_2}.$$

- The forward rate will be modeled as a strictly positive stochastic process.

Data

- To guarantee the positivity of the stochastic processes F^1, F^2, S^3, S^4, S^5, S^6, S^7, W and I, we will model their natural logarithms f^1, f^2, s^3, s^4, s^5, s^6, s^7, w and i as real-valued processes.
- To guarantee that the employment rate E takes values in the interval $(0, 1)$ we will model the transformed employment rate $e = \varphi(E)$ as a real-valued stochastic process. Here φ is the inverse of the cumulative normal distribution function.

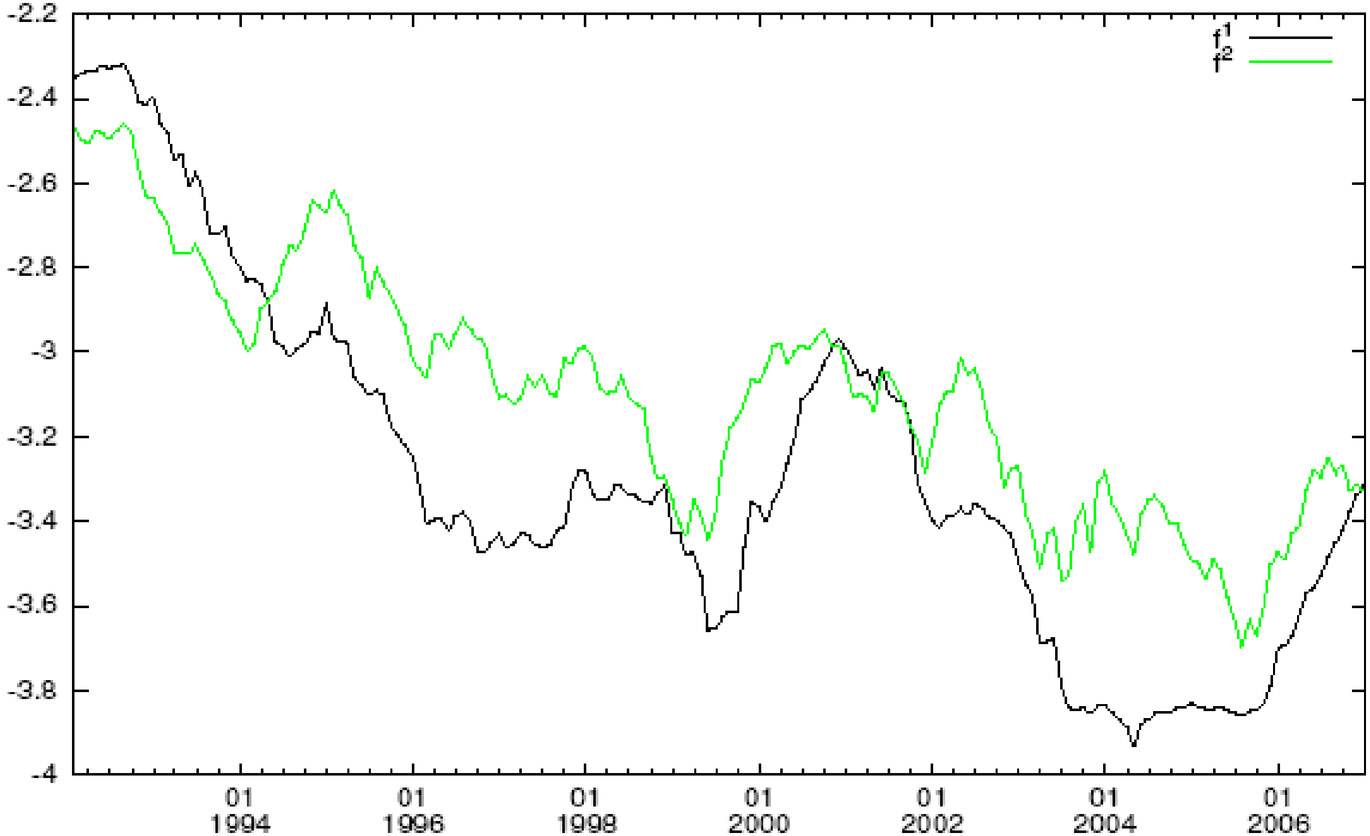
Data

Instead of the nominal logarithmic interest rates and price indices, we will model the “real” rates and indices defined as

$$\tilde{f}_t^1 = f_t^1 - p\Delta i_t$$
$$\tilde{f}_t^2 = f_t^2 - p\Delta i_t$$
$$\tilde{s}_t^3 = s_t^3 - i_t$$
$$\tilde{s}_t^4 = s_t^4 - i_t$$
$$\tilde{s}_t^5 = s_t^5 - i_t$$
$$\tilde{s}_t^6 = s_t^6 - i_t$$
$$\tilde{s}_t^7 = s_t^7 - i_t$$
$$\tilde{w}_t = w_t - i_t.$$

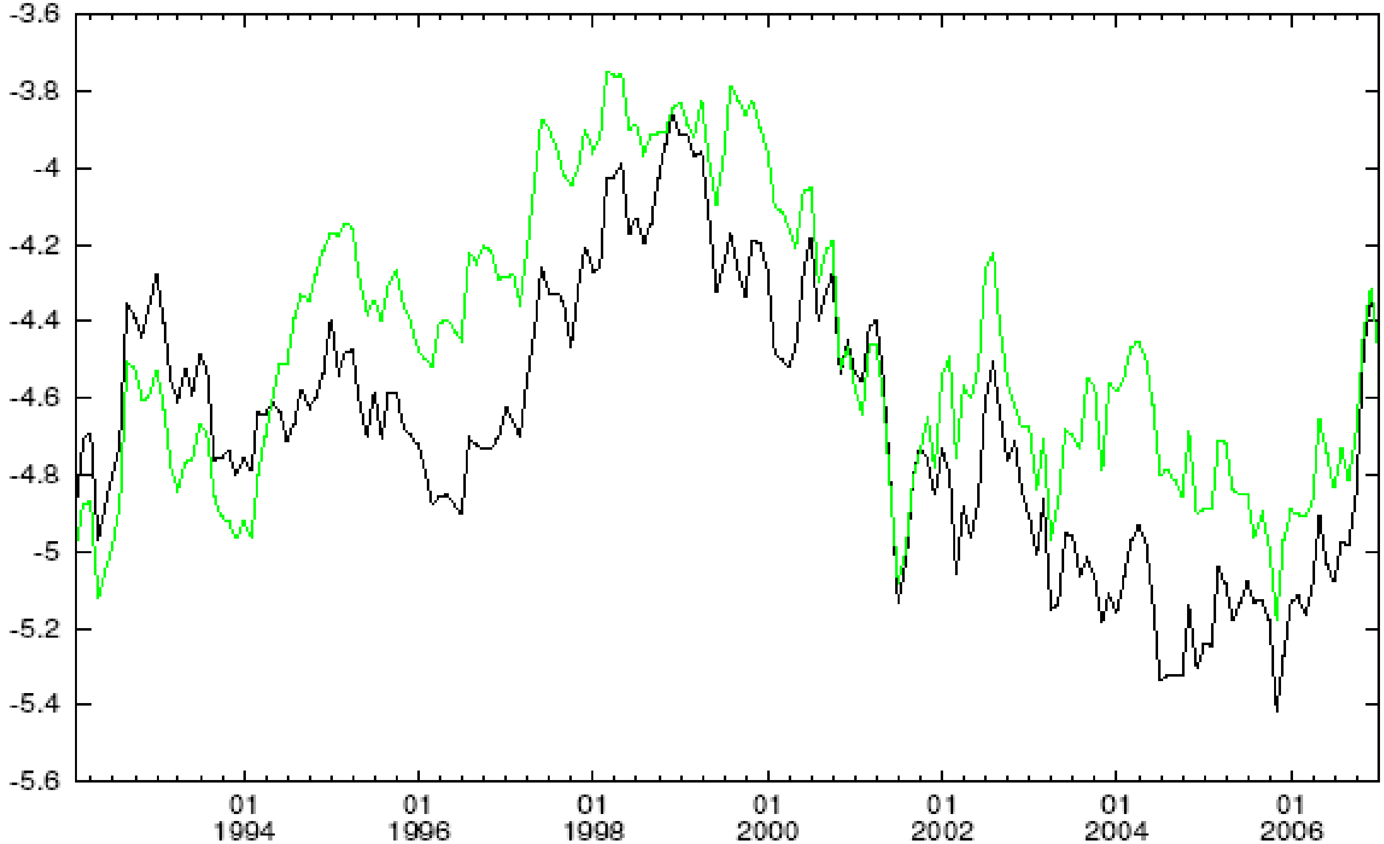
Data

Logarithmic interest rates



Data

Log-real interest rates



Time series model

- We will assume that the monthly values of

$$x = (\tilde{f}^1, \tilde{f}^2, \tilde{s}^3, \tilde{s}^4, \tilde{s}^5, \tilde{s}^6, \tilde{s}^7, e, \tilde{w}, i)^T$$

satisfy

$$\Delta_\delta x_t = A\Delta_\delta x_{t-1} + \alpha(\beta^T x_{t-1} - \gamma) + \varepsilon_t,$$

where

$$\Delta_\delta x_t := x_t - x_{t-1} - \delta$$

and ε_t is a 10-dimensional random vector with zero mean.

- The matrices $A \in \mathbb{R}^{10\times 10}$, $\beta \in \mathbb{R}^{10\times 3}$ and $\alpha \in \mathbb{R}^{10\times 3}$, the vectors $\gamma \in \mathbb{R}^3$ and $\delta \in \mathbb{R}^{10}$ as well as the distribution of ε_t are parameters of the model.

Time series model

- The vector δ specifies the average drift of x whereas γ gives the average value of the vector $\beta^T x$ in the long-run.
- We fix the values of δ, γ and β to set
  - average returns,
  - interest rates,
  - employment rate,
  - ...

  according to the views of the user.
- Once δ, β and γ are fixed, we estimate the values of A and α from historical data using ordinary least squares.

Time series model

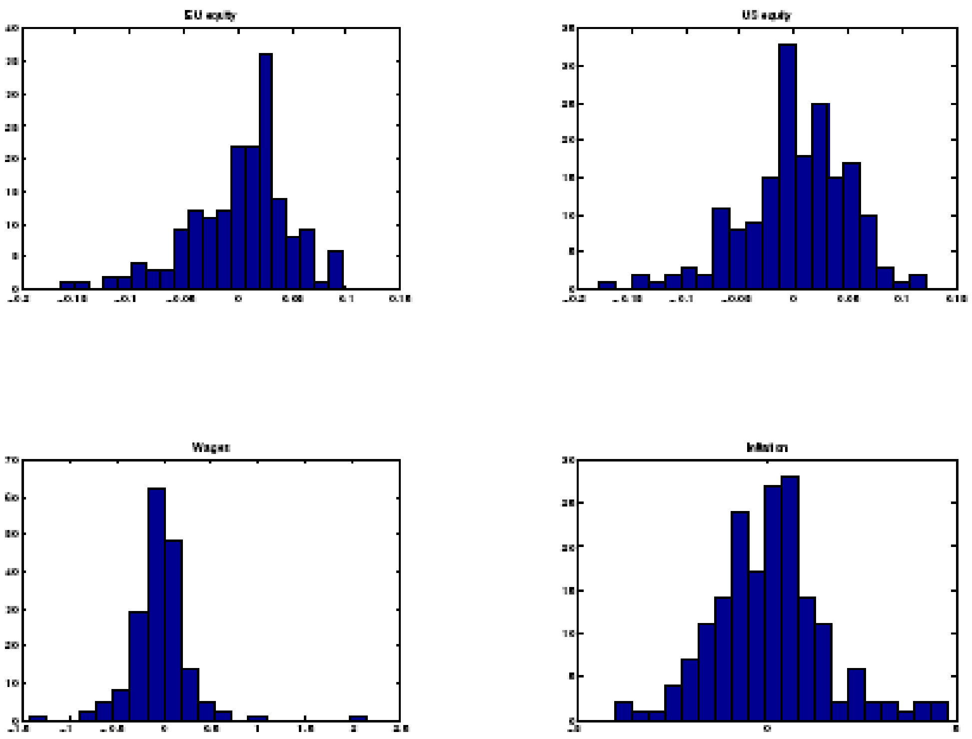
- It remains to model the residuals

$$\varepsilon_t := \Delta_\delta x_t - A\Delta_\delta x_{t-1} + \alpha(\beta^T x_{t-1} - \gamma).$$

- With our specification of δ, β and γ and with the OLS estimates of A and α, the historical values of ε_t
  - have zero mean,
  - are (essentially) serially uncorrelated and homoscedastic,
  - violate normality.

Time series model

Marginal histograms of ε.



Time series model

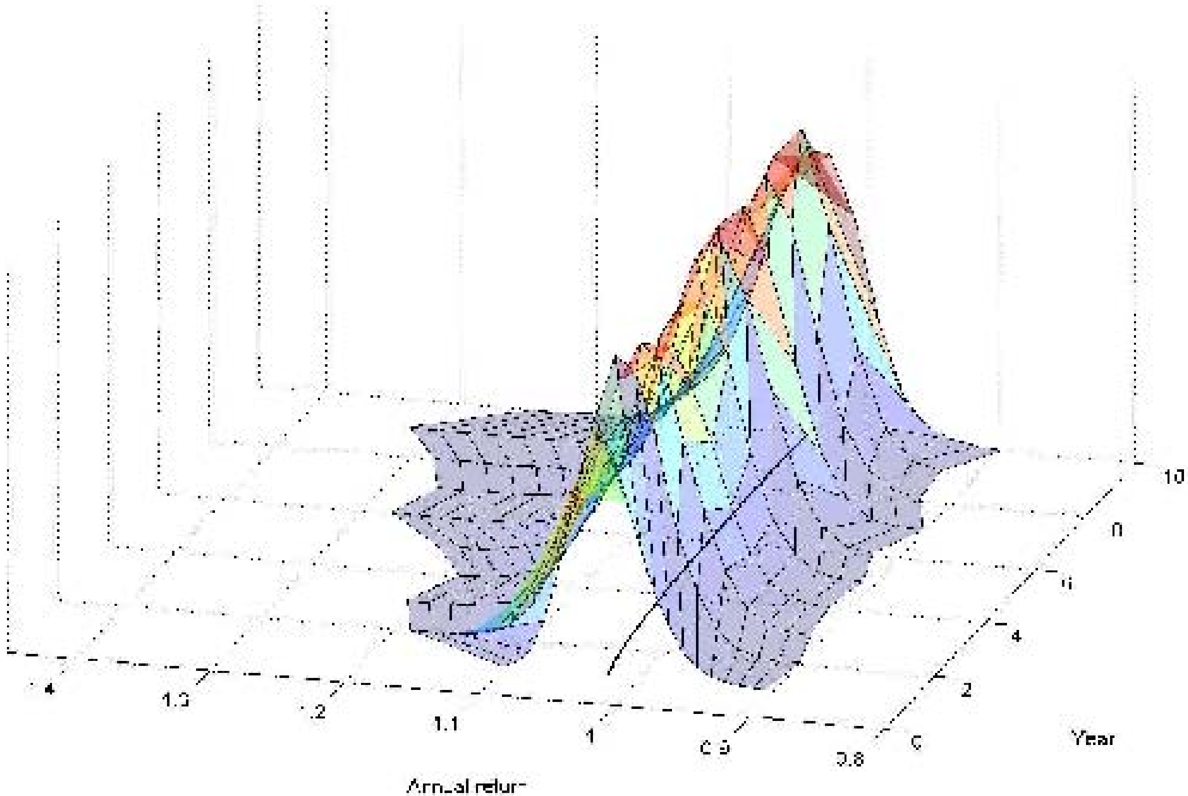
- We model ε with the kernel distribution

$$P = \sum_{t=1}^{T} \frac{1}{T} N(\varepsilon_t, \sigma^2 I),$$

  where $N(\varepsilon_t, \sigma^2 I)$ is the multivariate normal distribution with mean ε_t and covariance matrix $\sigma^2 I$.
- When $\sigma \searrow 0$, we have $P \to \sum_{t=1}^{T} \frac{1}{T} \delta_{\varepsilon_t}$, the “bootstrap distribution”.
- Larger values of σ correspond to greater uncertainty about the future.

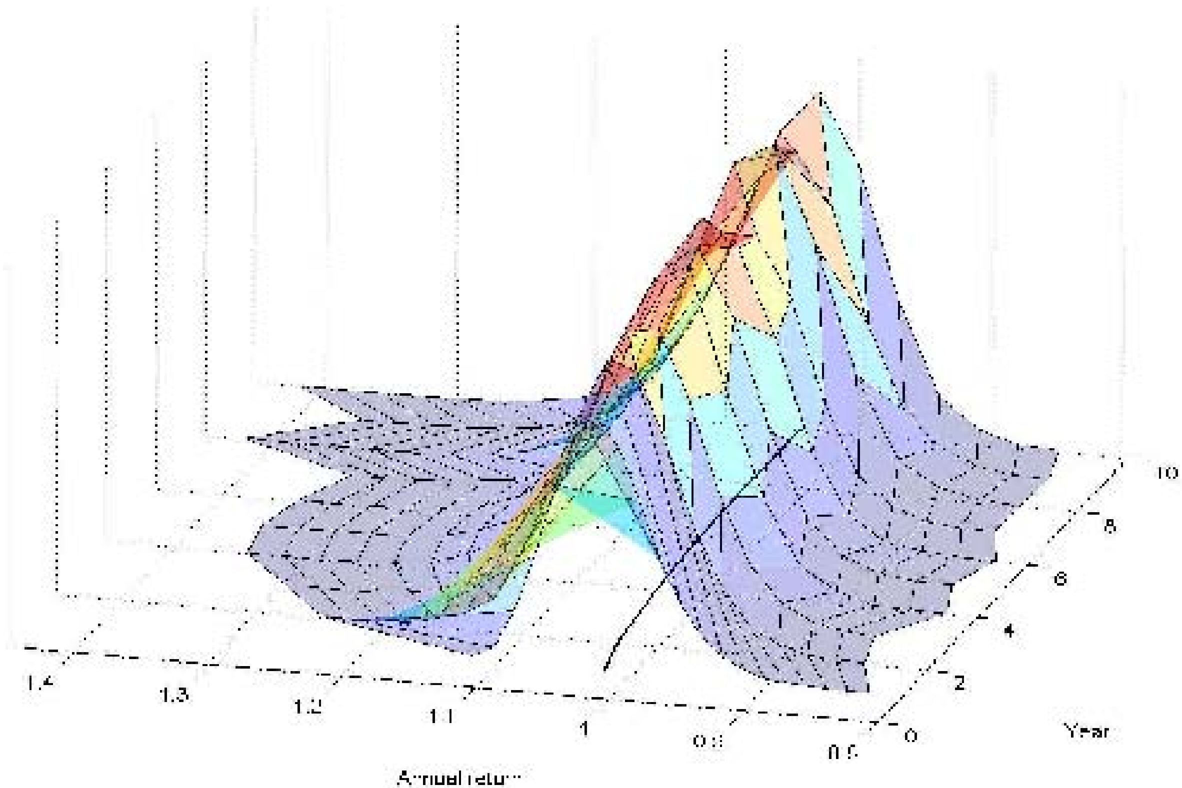
Numerical illustrations

Annual bond returns with $\sigma = 0$.



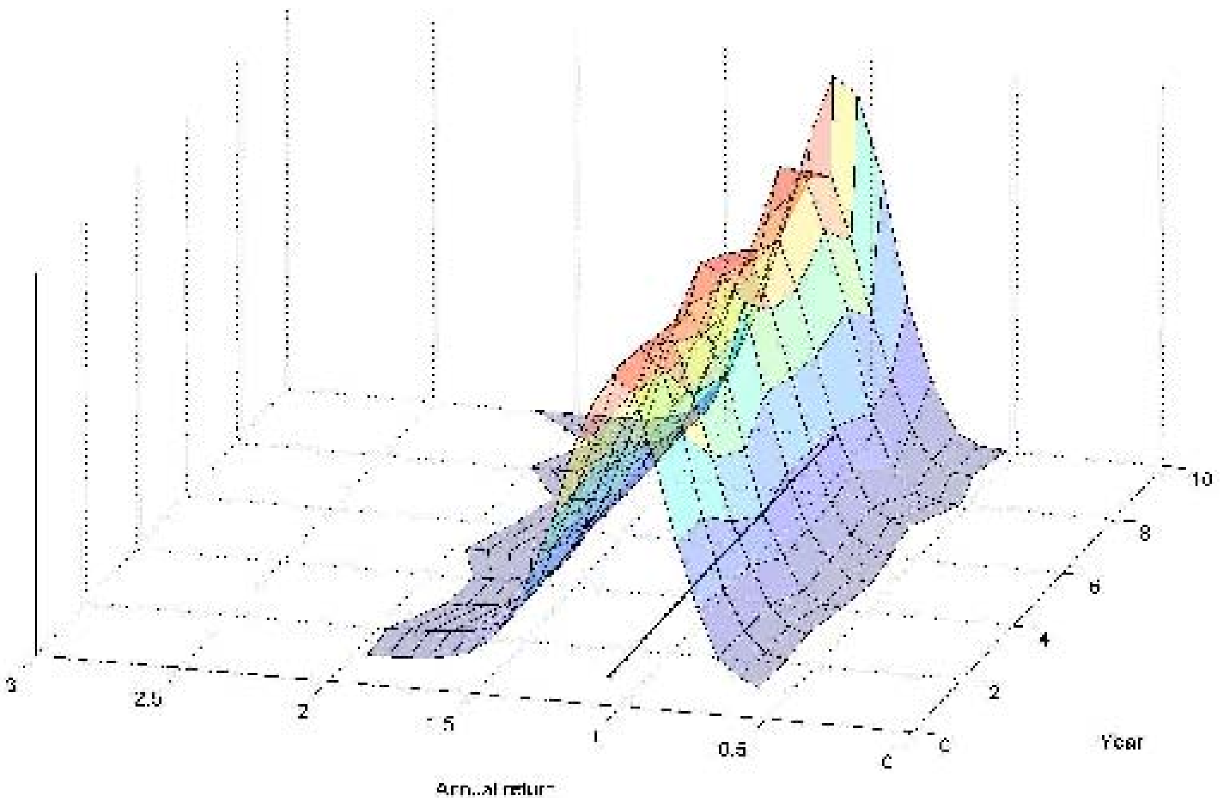
Numerical illustrations

Annual bond returns with $\sigma > 0$.



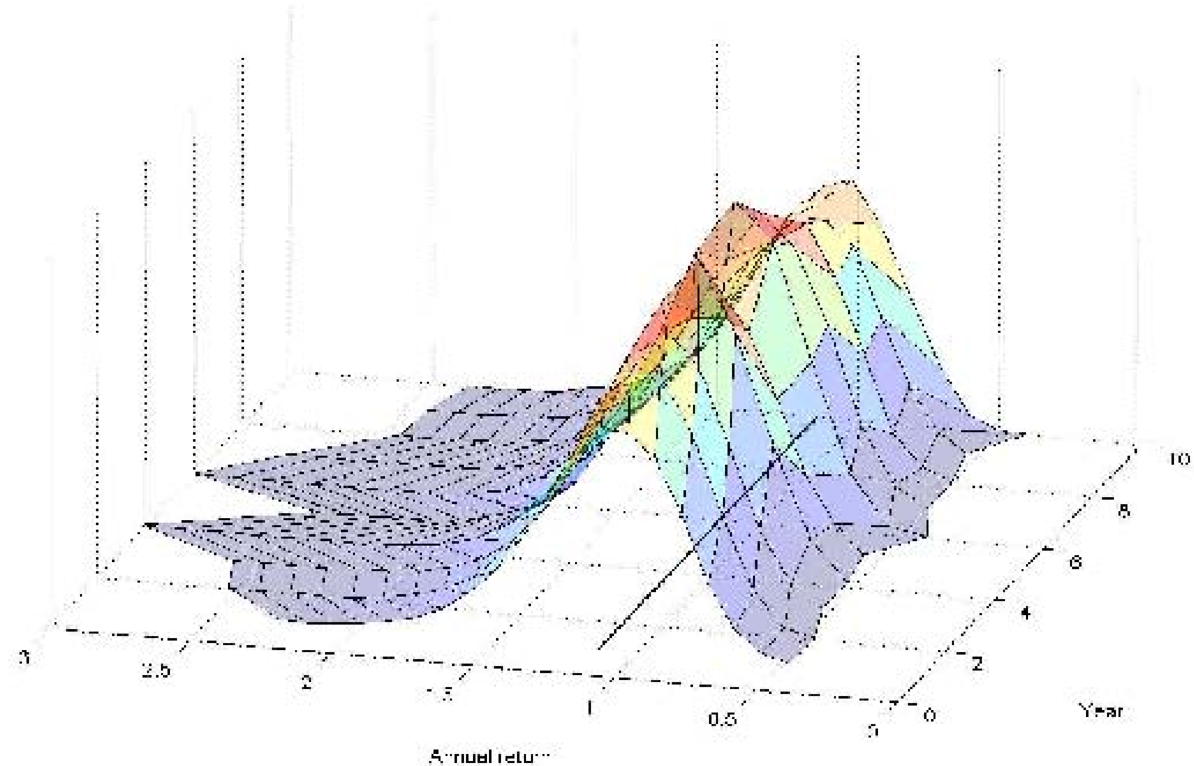
Numerical illustrations

Annual EU-stock returns with $\sigma = 0$.



Numerical illustrations

Annual EU-stock returns with $\sigma > 0$.



Numerical illustrations

- The model can be used e.g. to evaluate the performance of given investment strategies.
- An investment strategy together with the stochastic asset returns and cash-flows determine the wealth of an institution.
- The wealth and the technical reserves determine the

$$\text{solvency ratio} = \frac{\text{wealth} - \text{technical reserves}}{\text{technical reserves}}.$$

- In the following example, we examined the behavior of the solvency ratio when using simple fixed proportion strategies.

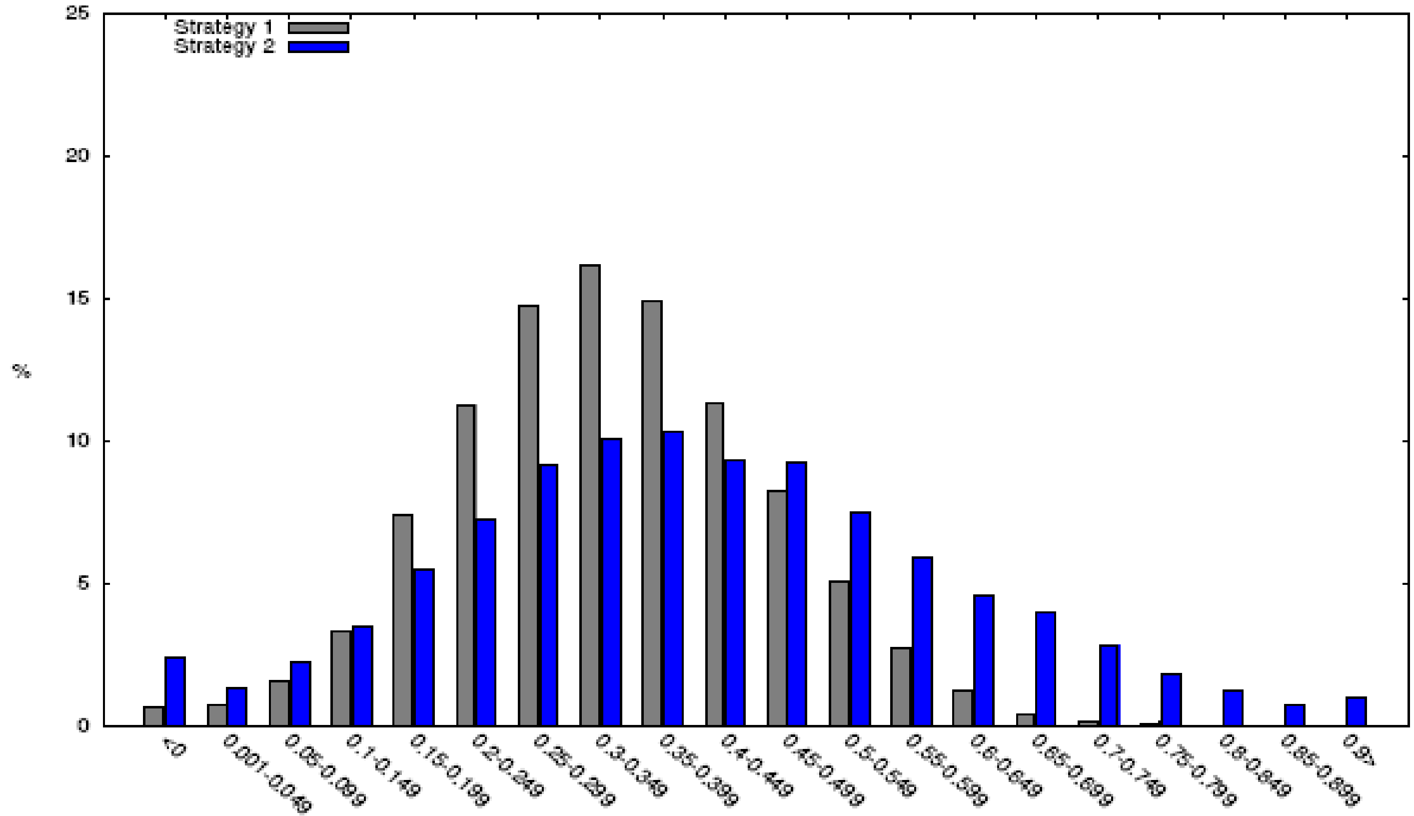
Numerical illustrations

Descriptive statistics of the terminal financial position.

Statistic	solvency ratio %	annualized return %
Mean	33.4	5.7
Median	33.2	5.6
St Dev	12.7	5.9
$V@R_1$	2.7	3.2
$V@R_5$	13.6	3.9
$CV@R_1$	-4.4	2.8
$CV@R_5$	6.7	3.5

Numerical illustrations

Distribution of terminal solvency ratio.



Numerical illustrations

The development of the median and the 90% confidence interval of the solvency ratio over time.