A stochastic model for assets and liabilities of a pension institution

Petri Hilli* Matti Koivu† Teemu Pennanen‡

February 28, 2007

Abstract

This paper develops a stochastic model for a pension institution that faces uncertainties both in investment returns and liabilities. The model is driven by a moderate number of risk factors that are modeled by a time series model incorporating statistical information with user specified expected growth rates and long term equilibria.

1 Introduction

When considering the financial risks of a pension institution, it is important to have a reasonable description of the investments as well as the liabilities extending far into the future. A reasonable model should take into account the inherent uncertainties as well as dynamics and relationships that exist between different risk factors. The purpose of this paper is to present a stochastic model that captures many such characteristics. The model allows for efficient computer implementations and, more importantly, it is easily adapted to user's views about the future development of certain key factors.

Many stochastic models for actuarial use have been reported in the literature. Most famous is probably the one presented by Wilkie [18, 19], that has been the prototype for many more recent models; see or example Yakoubov et al. [21] or Mulvey [15, 16] whose model was the basis for the simulation model of Towers Perrin Tillinghast. Ranne [17] and Heikkilä [11] presented stochastic models for Finnish pension insurance companies. More recent models are

*Department of Business Technology, Helsinki School of Economics

†Risk Management Division, European Central Bank

‡Department of Business Technology, Helsinki School of Economics and Institute of Mathematics, Helsinki University of Technology

those of Hibbert et al. [12] and Ahlgrim et al [1, 2]. All the above mentioned models are restricted by their hierarchic, so called cascade structure, where the dependencies between the risk factors go only in one direction.

The model presented in this paper is based on the Vector Equilibrium Correction (VEqC)-model of Engle and Granger [9]. A VEqC-model generalize the Vector Auto-Regressive (VAR) model for first differences by adding an equilibrium term to account for stochastic relationships between levels of the modeled time series. A definite advantage of VAR-models over Wilkie-type models is that they do not pose apriori restrictions on the causalities between the time series. VAR-models for investment returns have been presented by Dert [7], Wright [20], Harris [10] and Campbell et al. [5, 6]. VEqC-models for assets and liabilities have been presented by Boender et al. [3] and Koivu et al. [13].

An important feature of the model of Koivu et al. is that the user can incorporate his own views about the future development of e.g. asset returns and interest rates. The model reported here modifies the model of Koivu et al. by incorporating inflation into the model and dividing equity investments into four separate investment classes. On the other hand, the model has been simplified by modeling the returns on each asset in the form of total returns instead of distinguishing between price appreciation and dividends. Moreover, price indices and interest rates are modeled in real terms, which allows capturing the often reported relationships between inflation, interest rates and asset returns in the long run; see e.g. [14] and its references. We model interest rates through positive forward rates which guarantees that the corresponding zero curve is decreasing. All the time series are modeled on monthly basis.

The rest of this paper is organized as follows. Section 2 outlines the asset and liability classes treated by our model. Section 3 describes the initial data as well as the transformed data that will be modeled by our linear time series model. Section 4 presents the time series model, describes our approach for parameter specification and gives interpretations of some of the key parameters. Section 5 gives some numerical illustrations.

2 Assets and liabilities

Our aim is to express the asset returns as well as the cash-flows and technical reserves of a pension institution in terms of 10 economic time series which will be modeled with a time series model with iid Gaussian innovations.

2.1 Assets

On the asset side, we will model the "total returns" in euros of the major investment classes. The *total return* is the sum of value appreciation and cash return per unit investment. In this study we will consider the following investment classes

1. cash,
2. bonds,
3. Finnish equities,
4. European equities,
5. North American equities,
6. Asian equities,
7. European real estate.

For cash investments, the return over a holding period of Δt is determined by the short rate F^1,

$$R_t^1 = e^{\Delta t F_{t-1}^1}.$$

The short rate will be modeled as a strictly positive stochastic process which will imply that $R^1 > 0$. The total return on bond investments R^2 will be approximated by the formula

$$R_t^2 = \Delta t Y_{t-1} + \left(\frac{1 + Y_t}{1 + Y_{t-1}}\right)^{-D},$$

where Y is the bond yield. The first term approximates the payouts by the current yield which corresponds to the fact that newly issued bonds usually sell at par. The second term is an approximation of the relative price change of the bond investments; see Appendix A.1. The total returns R^j on equity as well as real estate investments are given simply in terms of the *total return indices* (in euros) S^j,

$$R_t^j = \frac{S_t^j}{S_{t-1}^j} \quad j = 3, \ldots, 7.$$

In practice, total return indices are calculated from the total returns, not vice versa as above, but it is the indices that are available from most sources of financial data. For real estate investment, the index is an industrial index of real estate investment companies.

2.2 Liabilities

On the liability side, the goal is to model a pension institution's cash-flows and technical reserves associated with major insurance classes. In pension insurance, the cash-flows and technical reserves typically depend on wages and the distribution of insured into different states within the cohorts. A *cohort* is the group of all citizens of given age and sex. The population in each cohort is distributed into *states* of distinct characteristics. For example, when modeling a pension institution's aggregate cash-flows and technical reserves associated with old age and disability pensions, the relevant states may include working, disabled, unemployed and old age.

We will denote the set of cohorts by I and the set of states by J. The number of people in cohort i in state j in year t will be denoted by $K_t^{i,j}$. In particular, the size of cohort i in year t equals the sum $\sum_{j\in J} K_t^{i,j}$. In this study, we will consider the following set of states

$$J = \{\text{working, disabled, old age, other, dead}\},$$

which are relevant when modeling, for example, old age and disability pensions in Finland. The payroll of cohort i can be written as

$$P_t^i = K_t^{i,working}\bar{P}_t^i,$$

where $\bar{P}^i$ is the average wage in cohort i. Average wages will be approximated according to

$$\bar{P}_t^i \approx \bar{P}_0^i \frac{W_t}{W_0},$$

where W is the general wage index and $\bar{P}_0^i$ is the average wage in cohort i at the initial year 0.

The development of the population will be modeled by a Markov chain

$$K_{t+1}^{i+1} = \Pi_t^i K_t^i,$$

where $i+1$ refers to the cohort one year older than i, $K_t^i = (K_t^{i,j})_{j\in J}$ and Π_t^i is the matrix of transition probabilities from the states of cohort i in year t to the states of cohort $i+1$ in year $t+1$. The elements of Π_t^i can be estimated from historical data of yearly transitions. Additionally, Π_t^i can incorporate forecasts concerning, for example, sizes of cohorts and the employment rate E. Instead of deterministic forecasts, one can model such factors as stochastic processes thus introducing uncertainty into the $K^{i,j}$s.

3 Data

As described above, the total asset returns can be expressed in terms of the short rate F^1, bond yield Y, the total return indices $S^3, \dots, S^7$. The cash-flows and technical reserves associated with major insurance classes can be described in terms of the wage index W and the $K^{i,j}$s where the latter are determined by the transition probability matrices Π_t^i. In this study, we assume that Π_t^is are determined not only by historical data but also by cohort sizes and the employment rate E. The cohort sizes will be assumed deterministic and given and E will be modeled stochastically. We will also include the consumer price index I in our model. This allows modeling the often reported relationships between inflation and interest rates as well as asset prices. Moreover, in some pension schemes the cash-flows and liabilities depend directly on I.

Our data consists of monthly observations of the following series from January 1992 to December 2006:

- F^1 German 3 month interest rate (Datastream),
- Y German government 5 year bond (Datastream),
- S^3 OMX HELSINKI CAP total return index (Datastream),
- S^4 DJ EURO STOXX 50 total return index (`http://www.stoxx.com`),
- S^5 DJ Americas 600 total return index (`http://www.stoxx.com`),
- S^6 DJ Asia/Pacific 600 total return index (`http://www.stoxx.com`),
- S^7 EPRA/NAREIT Euro Zone-index (`http://www.epra.com`),
- E seasonally adjusted Finnish employment rate (Statistics Finland),
- W seasonally adjusted Finnish wage-level index (Statistics Finland),
- I seasonally adjusted Europe eurozone harmonized consumer price index (`http://epp.eurostat.cec.eu.int`).

Since monthly observations of wage index were not available, monthly observations were interpolated from quarterly observations. All the data is presented graphically in Appendix A.4. For wage and price indices, we present log-differences (percentual changes) instead of levels.

3.1 Data transformations

The *forward rate* defined as

$$F^2 = \frac{t_2 Y - t_1 F^1}{t_2 - t_1}$$

is always strictly positive, which corresponds to the fact that the "zero curve" is a decreasing function of the maturity; see e.g. Cairns [4, s. 5]. Given the values of the short rate F^1 and the forward rate F^2, the bond yield is in turn given by

$$Y_t = \frac{t_1 F^1 + (t_2 - t_1) F^2}{t_2}.$$

To guarantee the positivity of the processes F^1, F^2, S^3, S^4, S^5, S^6, S^7, W and I, we will model their natural logarithms f^1, f^2, s^3, s^4, s^5, s^6, s^7, w and i as real-valued processes. The employment rate E, on the other hand, takes values in the interval $(0, 1)$. To guarantee this feature in our model, we will model the transformed employment rate $e = \varphi(E)$ as a real-valued stochastic process. Here φ is an invertible function from the interval $(0, 1)$ onto the real line $\mathbb{R}$. In this study, we take φ to be the inverse of the cumulative normal distribution function.

Instead of the nominal logarithmic interest rates and price indices, we will model the "real" rates and indices defined as

$$\begin{aligned}
\tilde{f}_t^1 &= f_t^1 - p\Delta i_t \\
\tilde{f}_t^2 &= f_t^2 - p\Delta i_t \\
\tilde{s}_t^3 &= s_t^3 - i_t \\
\tilde{s}_t^4 &= s_t^4 - i_t \\
\tilde{s}_t^5 &= s_t^5 - i_t \\
\tilde{s}_t^6 &= s_t^6 - i_t \\
\tilde{s}_t^7 &= s_t^7 - i_t \\
\tilde{w}_t &= w_t - i_t.
\end{aligned}$$

We do this in order to model the long term relationship between inflation and the interest rates and price indices, which has been reported in many empirical studies; see e.g. [14] and its references. The transformed data is presented graphically in Appendix A.4. The parameter p is chosen so as to optimize the statistical fit of the above time series into the chosen time series model. In this study, the value $p = 750$ was chosen.

3.2 Unit root tests

This section studies the stationarity properties of the time series derived in Section 3 by performing augmented Dickey-Fuller tests [8] for all the considered series. The unit root test results are displayed in Table 1. In the tests, the deterministic terms (constant, trend) are selected based on economic intuition and historical behavior of the time series under analysis. The number of lags in the tests are selected using the Schwartz information criterion with a maximum of 12 lags. The results indicate that the transformed series constructed above are $I(1)$-processes whereas their first differences are stationary. It seems thus reasonable to model the first differences of the time series as stationary processes.

Table 1: Unit root test results.

Time-series	Deterministic terms	Number of lags	ADF test t-statistic
$\tilde{f}^1$	constant	0	-2.288
$\tilde{f}^2$	constant	0	-2.419
$\tilde{s}^3$	constant, trend	0	-1.813
$\tilde{s}^4$	constant, trend	0	-1.019
$\tilde{s}^5$	constant, trend	0	-0.924
$\tilde{s}^6$	constant, trend	1	-2.208
$\tilde{s}^7$	constant, trend	1	-0.862
e	constant	3	-2.350
$\tilde{w}$	constant, trend	1	-2,245
$\tilde{i}$	constant, trend	1	-2.545
$\Delta\tilde{f}^1$	constant	1	-11.242***
$\Delta\tilde{f}^2$	constant	0	-13.269***
$\Delta\tilde{s}^3$	constant	0	-11.428***
$\Delta\tilde{s}^4$	constant	0	-11.847***
$\Delta\tilde{s}^5$	constant	0	-11.939***
$\Delta\tilde{s}^6$	constant	0	-10.909***
$\Delta\tilde{s}^7$	constant	0	-9.777***
Δe	constant	2	-3.453**
$\Delta\tilde{w}$	constant	0	-3,130***
$\Delta\tilde{i}$	constant	0	-2.718*

***, **, * denotes the rejection of the unit root null hypothesis at 1%, 5%, 10% confidence level, respectively.

4 Time series model

It is customary to model $I(1)$-processes with ARIMA-type models. In this study, we will use Vector Equilibrium Correction-models, which augment ARIMA-models by the so called Equilibrium Correction-term; see e.g. Engle and Granger [9]. More precisely, we will assume that the monthly values of the 10-dimensional process

$$x = \begin{bmatrix} \tilde{f}^1 \\ \tilde{f}^2 \\ \tilde{s}^3 \\ \tilde{s}^4 \\ \tilde{s}^5 \\ \tilde{s}^6 \\ \tilde{s}^7 \\ e \\ \tilde{w} \\ i \end{bmatrix}$$

satisfies

$$\Delta_\delta x_t = A\Delta_\delta x_{t-1} + \alpha(\beta^T x_{t-1} - \gamma) + \varepsilon_t,$$

where

$$\Delta_\delta x_t := x_t - x_{t-1} - \delta$$

and ε_t is a 10-dimensional Gaussian vector with zero mean and variance matrix $\Omega \in \mathbb{R}^{10\times 10}$. The matrix Ω as well as the matrices $A \in \mathbb{R}^{10\times 10}$, $\beta \in \mathbb{R}^{10\times 3}$ and $\alpha \in \mathbb{R}^{10\times 3}$ and the vectors $\gamma \in \mathbb{R}^3$ and $\delta \in \mathbb{R}^{10}$ are parameters of the model.

If the eigenvalues of the matrix

$$\begin{bmatrix} A & \alpha \\ \beta^T A & \beta^T\alpha + I \end{bmatrix}$$

are all inside the open unit disc and if $\beta^T\delta = 0$, then the expectations of the vectors $\Delta_\delta x_t$ and $\beta^T x_t - \gamma$ converge to zero; see Appendix A.2. In other words, in the long run,

$$E\Delta x_t = \delta \quad \text{and} \quad E\beta^T x_t = \gamma.$$

We fix the values of the vectors δ and γ and the matrix β before estimating the remaining parameters A, α and Ω from the data. This will not only simplify the specification of the model but it gives a simple way to incorporate expert views into the model; see Koivu, Pennanen and Ranne [13]. Writing

$$\Delta x_t = \begin{bmatrix} \Delta \tilde{f}_t^1 \\ \Delta \tilde{f}_t^2 \\ \Delta \tilde{s}_t^3 \\ \Delta \tilde{s}_t^4 \\ \Delta \tilde{s}_t^5 \\ \Delta \tilde{s}_t^6 \\ \Delta \tilde{s}_t^7 \\ \Delta e \\ \Delta \tilde{w}_t \\ \Delta i_t \end{bmatrix} = \begin{bmatrix} \Delta \tilde{f}_t^1 \\ \Delta \tilde{f}_t^2 \\ \Delta s_t^3 - \Delta i_t \\ \Delta s_t^4 - \Delta i_t \\ \Delta s_t^5 - \Delta i_t \\ \Delta s_t^6 - \Delta i_t \\ \Delta s_t^7 - \Delta i_t \\ \Delta e \\ \Delta w_t - \Delta i_t \\ \Delta i_t, \end{bmatrix}$$

we see that δ^1 and δ^2 give the average drifts of the log-real interest rates, $\delta^3, \delta^4, \delta^5, \delta^6, \delta^7$ give the average real total returns on the equity and real estate investments, δ^8 gives the average drift of the transformed employment rate, δ^9 gives the average growth rate of the real wages and δ^{10} gives the average inflation. Since there is no reason to assume that the log-real interest rates or the transformed employment rate would have nonzero drift, we set $\delta^1 = \delta^2 = \delta^8 = 0$. The remaining values of the vector δ are set according to the views of the user. In this study, we use the values

$$\delta = \begin{bmatrix} 0 \\ 0 \\ 0.0060 \\ 0.0052 \\ 0.0052 \\ 0.0060 \\ 0.0060 \\ 0 \\ 0.0014 \\ 0.0015 \end{bmatrix}$$

The last component corresponds to the average inflation of 1.8% and the second last component to the average of 1.7% yearly real growth of the wages. The remaining components of δ correspond to an average of 9% yearly nominal total returns in Finnish and Asian equity markets as well as real estates and 8% yearly nominal total returns in Europe and North America.

The matrix β is chosen as

$$\beta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

which simply gives the levels of the log-real interest rates and the transformed employment rate, the underlying assumption being that these series are stationary. The vector γ then determines the long term averages of these series. In this study, we use

$$\gamma = \begin{bmatrix} -4.552 \\ -4.212 \\ 0.500 \end{bmatrix},$$

which corresponds to approximately 3.25% and 4.5% nominal rates and to employment rate of 0.7[1].

The values of the remaining parameters A, α and Ω were estimated from historical data. These are given in Appendix A.3. Most of the elements of the matrices A and α were set to zero in order to avoid forecasting based solely on historical data and, correspondingly, to increase the variance of the innovation term ε.

The auto-regression matrix A has diagonal terms for real estate, employment rate, wages and inflation. Inflation also affects the employment rate. Rows 1, 2 and 8 of the matrix α correspond to mean reversion terms for the log-real interest rates and the transformed employment rate. The 9th row of α shows that the logarithmic interest rate spread has a positive effect on the growth of wages. As for the correlation matrix ρ, the total return on real estate has a notably low correlation with the total returns on equities.

5 Numerical illustrations

Figures 19 – 26 in Appendix A.4 present graphically a simulation from the estimated model. The simulated series seem to have similar characteristics as

[1] This was set as a target by the Finnish government parties in 2007

Table 2: Averages and standard deviations for logarithmic total returns and log-differences of wage and price indices.

	Historical		Simulations	
	Average	St. dev.	Average	St. dev.
$\ln R^1$	0.041	0.006	0.033	0.001
$\ln R^2$	0.063	0.038	0.041	0.039
$\ln R^3$	0.158	0.236	0.091	0.236
$\ln R^4$	0.113	0.160	0.081	0.161
$\ln R^5$	0.100	0.173	0.081	0.174
$\ln R^6$	0.038	0.203	0.090	0.205
$\ln R^7$	0.114	0.137	0.093	0.114
$\Delta \ln W$	0.032	0.003	0.032	0.003
$\Delta \ln I$	0.022	0.002	0.018	0.001

Table 3: Averages of simulated interest rates and employment rate.

	Historical	Simulations
F^1	0.041	0.033
Y	0.047	0.044
E	0.650	0.692

the historical ones in Figures 1 – 8 in Appendix A.4. Short rate, bond yield, inflation and wage growth rate tend to move together. A similar phenomenon can be observed in total return indices except real estate index which moves largely independently from stock indices.

To examine the average behavior of the model, we performed 10000 random simulations spanning over 15 years starting from January 2007. Tables 2 and 3 present some statistics calculated from the simulations as well as historical data.

Acknowledgments Petri Hilli and Teemu Pennanen are grateful to the Finnish Ministry of Social Affairs and Health for funding the research. Teemu Pennanen's research was partially funded by Academy of Finland under contract number 3385.

References

[1] Kevin C. Ahlgrim, Stephen P. D'Arcy, and Richard W. Gorvett. Report on Modeling of Economic Series Coordinated with Interest Rate Scenarios. Technical report, Casual Actuarial Society, 2004.

[2] Kevin C. Ahlgrim, Stephen P. D'Arcy, and Richard W. Gorvett. A Comparison of Actuarial Financial Scenario Generators. In *Proceedings of the American Risk and Insurance Association annual meeting*, 2006.

[3] C. G. E. Boender, P.C. van Aalst, and F. Heemskerk. Modelling and management of assets and liabilities of pension plans in the Netherlands. In William T. Ziemba and John M. Mulvey, editors, *Worldwide asset and liability management*, pages 561–580. Cambridge University Press, 1998.

[4] A. J. G. Cairns. *Interest rate models: An introduction.* Princeton University Press, 2004.

[5] John Y. Campbell, Yeung Lewis Chan, and Luis M. Viceira. A multivariate model of strategic asset allocation. *Journal of Financial Economics*, 67(1):41–80, 2003.

[6] John Y. Campbell and Luis M. Viceira. Strategic Asset Allocation in a Continuous-Time VAR Model. *Journal of Economic Dynamics and Control*, 28(11):2195–2214, 2004.

[7] C. L. Dert. A dynamic model for asset liability management for defined benefit pension funds. In William T. Ziemba and John M. Mulvey, editors, *Worldwide asset and liability management*, pages 501–536. Cambridge University Press, 1998.

[8] D. A. Dickey and W. A. Fuller. Likelihood ratio statistics for autoregressive time series with a unit root. *Econometrica*, 49:1057–1072, 1981.

[9] Robert F. Engle and C. W. J. Granger. Co-integration and error correction: representation, estimation, and testing. *Econometrica*, 55(2):251–276, 1987.

[10] G. Harris. Markov chain Monte Carlo estimation of regime switching vector autoregressions. *ASTIN Bulletin*, 29:47–79, 1999.

[11] M. Heikkilä. Tel-järjestelmän yhteinen tuottovelvoite ja yksittäisen yhtiön sijoitustuottojen jäännösvarianssi. Working paper, Suomen Aktuaariyhdistys, 2004.

[12] John Hibbert, Philip Mowbray, and Craig Turnbull. A Stochastic Asset Model & Calibration for Long-Term Financial Planning Purposes. Technical report, Barrie & Hibbert Limited, 2001.

[13] M. Koivu, T. Pennanen, and A. Ranne. Modeling assets and liabilities of a Finnish pension insurance company: a VEqC approach. *Scandinavian Actuarial Journal*, 2005(1):46–76, 2005.

[14] James R. Lothian and Cornelia McCarthy. Equity returns and inflation: the puzzlingly long lags. *Research in Banking and Finance*, 2:149–166, 2001.

[15] John M. Mulvey. Generating scenarios for the Towers Perrin investment system. *INTERFACES*, 26(March-April):1–15, 1996.

[16] John M. Mulvey and A. Eric Thorlacius. The Towers Perrin global capital market scenario generation system. In William T. Ziemba and John M. Mulvey, editors, *Worldwide asset and liability management*, pages 286–312. Cambridge University Press, 1998.

[17] A. Ranne. The Finnish stochastic investment model. In *Transactions of the 26th ICA*, pages 213–238, 1998.

[18] A. D. Wilkie. A stochastic investment model for actuarial use. *Transactions of the Faculty of Actuaries*, 39:341–403, 1986.

[19] A. D. Wilkie. More on a stochastic asset model for actuarial use. *British Actuarial Journal*, 1:777–964, 1995.

[20] I. D. Wright. A stochastic asset model using vector auto-regression. Actuarial research report no. 108, City University, 1998.

[21] Y.H. Yakoubov, M. H. Teeger, and D. B. Duval. Stochastic investment model for asset and liability management. AFIR Colloquium, 1999. *www.actuaries.org/members/en/AFIR/colloquia/Tokyo/papers.cfm.*

A Appendices

A.1 Fixed income investments

The market price of a fixed income instrument (such as a government bond) described by its payouts $(c_t)_{t=1}^T$ can be approximately expressed as

$$P(Y) = \sum_{t=1}^{T} \frac{c_t}{(1+Y)^t},$$

where Y is the *yield to maturity;* see e.g. Cairns [4]. The price is thus a positive function of both the payouts c_t and the yield Y. The number

$$D = -\frac{dP(Y)}{dY}$$

is known as the *duration* of the fixed income instrument. The duration is often used in the first order Taylor approximation

$$P(Y_t) \approx P(Y_{t-1}) - D(Y_t - Y_{t-1})$$

to describe the effect of the yield on the price. An obvious handicap of this is that when $(Y_t - Y_{t-1})$ is large it may predict negative prices even though $c_t \geq 0$. In this study, we have avoided this problem by using the first order Taylor approximation to the logarithmic prices,

$$\ln P(Y_t) \approx \ln P(Y_{t-1}) - D(Y_t - Y_{t-1}),$$

where D is the *logarithmic duration,* given by

$$D = -\frac{d \ln P(Y)}{dY}.$$

The relative price change is then approximated by

$$\frac{P_t(Y_t)}{P_{t-1}(Y_{t-1})} \approx \exp(-D(Y_t - Y_{t-1})) = \left(\frac{e^{Y_t}}{e^{Y_{t-1}}}\right)^{-D} \approx \left(\frac{1+Y_t}{1+Y_{t-1}}\right)^{-D}.$$

A.2 Stationarity of a VEqC-model

The VEqC-model of Section 4 can be written in the autoregressive form

$$z_t = Bz_{t-1} + b + \eta_t,$$

where

$$z_t = \begin{bmatrix} \Delta_\delta x_t \\ \beta^T x_t - \gamma \end{bmatrix}, \qquad B = \begin{bmatrix} A & \alpha \\ \beta^T A & \beta^T \alpha + I \end{bmatrix},$$
$$b = \begin{bmatrix} 0 \\ -\beta^T \delta \end{bmatrix}, \qquad \eta_t = \begin{bmatrix} \varepsilon_t \\ 0 \end{bmatrix}.$$

If $\beta\delta = 0$ we get

$$Ez_t = BEz_{t-1} = B^t z_0,$$

so if the eigenvalues of the matrix B are inside the unit disc, we have $Ez_t \to 0$ as $t \to \infty$.

A.3 Parameter estimates

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.295 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.818 & 0 & -0.970 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.932 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.954 \end{bmatrix}$$

$$\alpha = 10^{-3} \begin{bmatrix} -53.575 & 0 & 0 \\ 0 & -41.913 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2.933 \\ -0.287 & 0.287 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{\Omega} = \mathrm{diag}(\sigma)\rho\,\mathrm{diag}(\sigma),$$

where correlation matrix

$$\rho = \begin{bmatrix} 1 \\ 0.89 & 1 \\ -0.07 & -0.01 & 1 \\ -0.02 & 0.03 & 0.73 & 1 \\ -0.03 & 0.01 & 0.65 & 0.81 & 1 \\ -0.11 & -0.04 & 0.54 & 0.53 & 0.54 & 1 \\ -0.07 & -0.1 & 0.03 & 0.03 & 0.05 & -0.05 & 1 \\ -0.08 & -0.03 & -0.03 & -0.05 & 0.06 & -0.04 & -0.09 & 1 \\ 0.56 & 0.5 & -0.05 & -0.01 & -0.05 & 0.02 & -0.08 & -0.03 & 1 \\ -0.93 & -0.89 & 0.07 & 0.02 & 0.04 & 0.06 & 0.03 & 0.08 & -0.61 & 1 \end{bmatrix}$$

and deviation vector

$$\sigma = 10^{-3} \begin{pmatrix} 117.505 & 114.234 & 67.609 & 46.262 & 50.006 & 58.637 & 31.720 & 1.612 & 0.321 & 0.146 \end{pmatrix}.$$

A.4 Figures

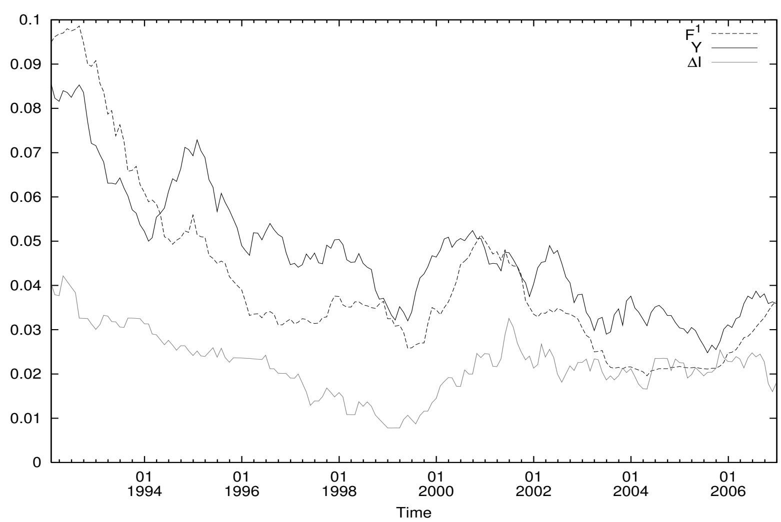

Figure 1: Historical $F^1, Y, \Delta i$

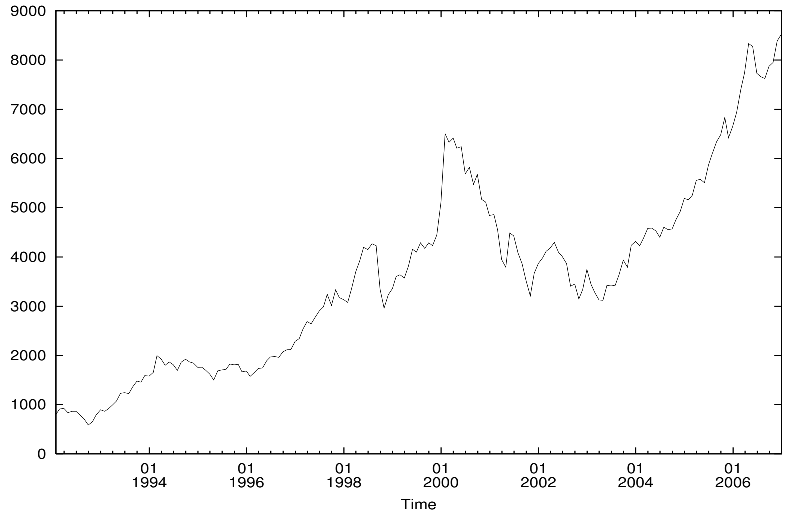

Figure 2: Historical S^3

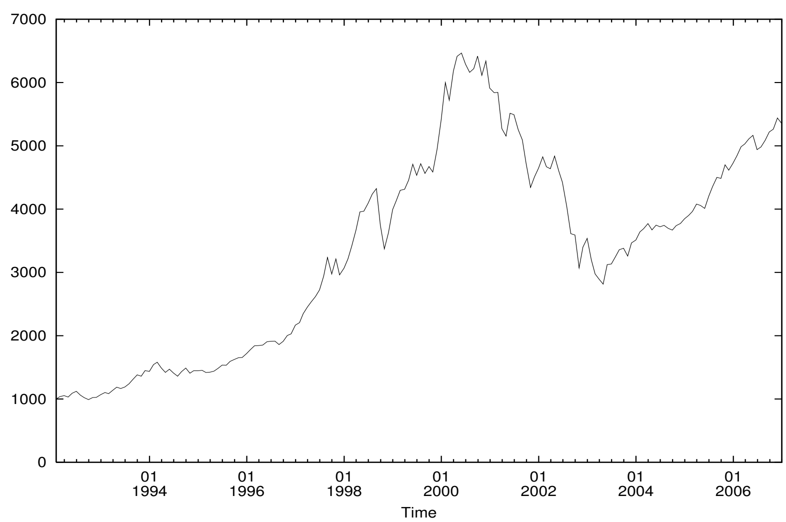

Figure 3: Historical S^4

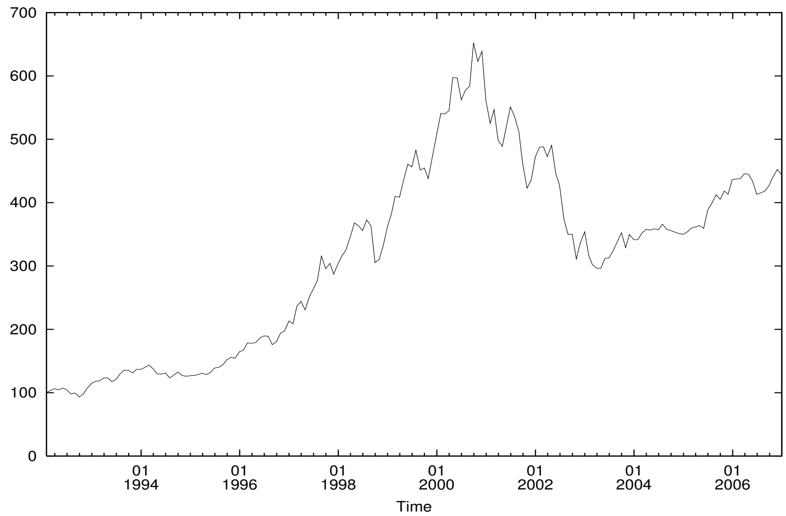

Figure 4: Historical S^5

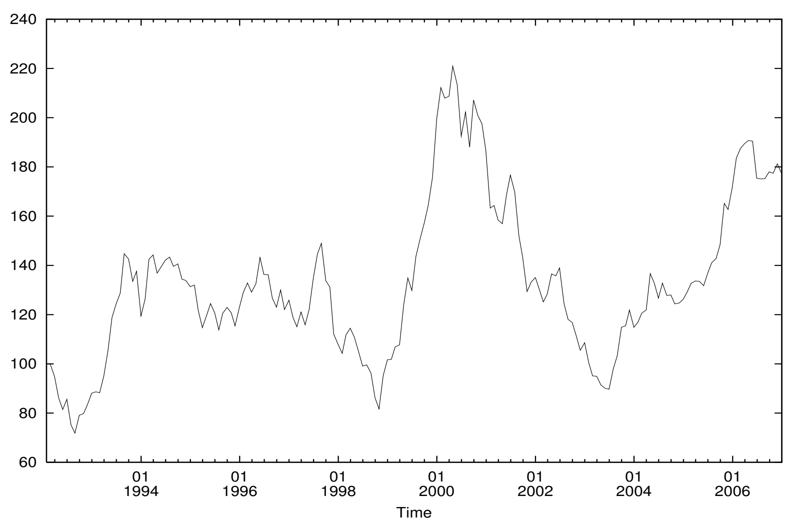

Figure 5: Historical S^6

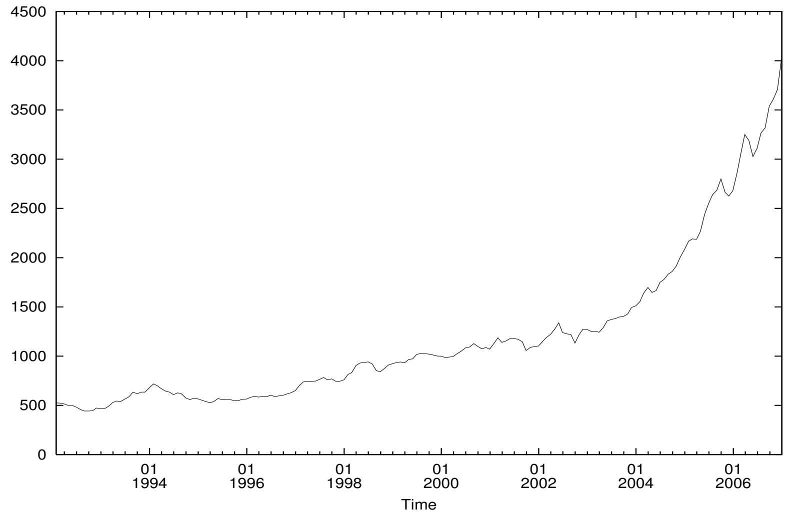

Figure 6: Historical S^7

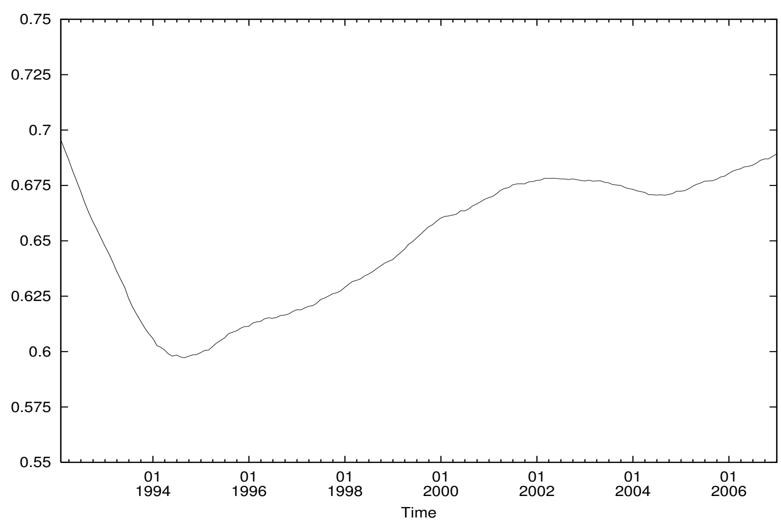

Figure 7: Historical E

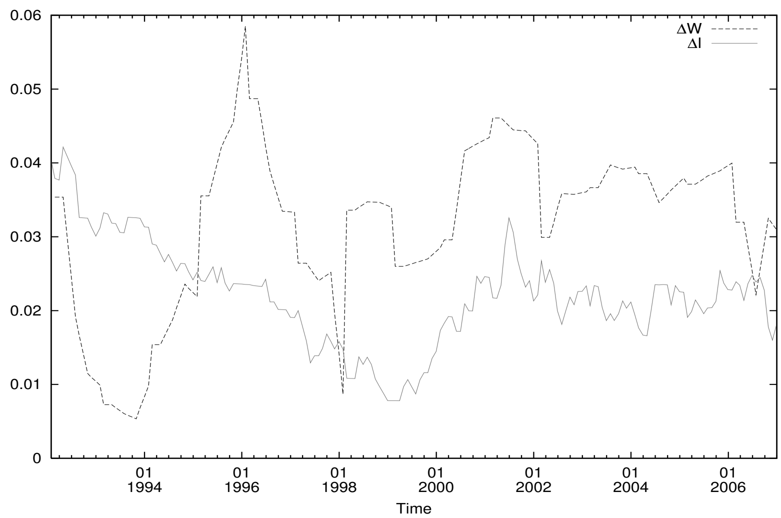

Figure 8: Historical $\Delta W, \Delta i$

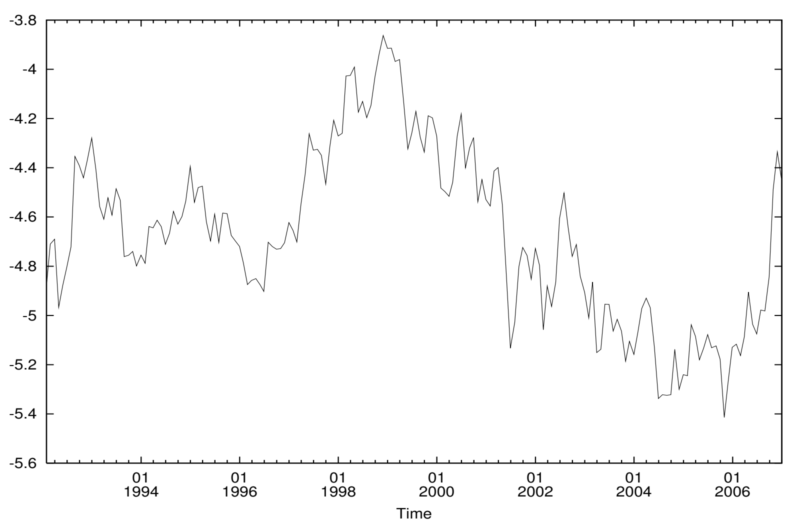

Figure 9: Historical $\tilde{f}^1$

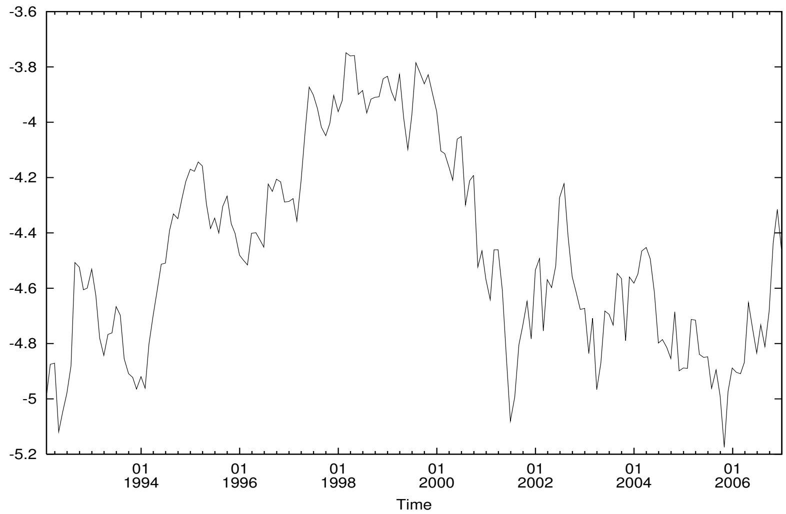

Figure 10: Historical $\tilde{f}^2$

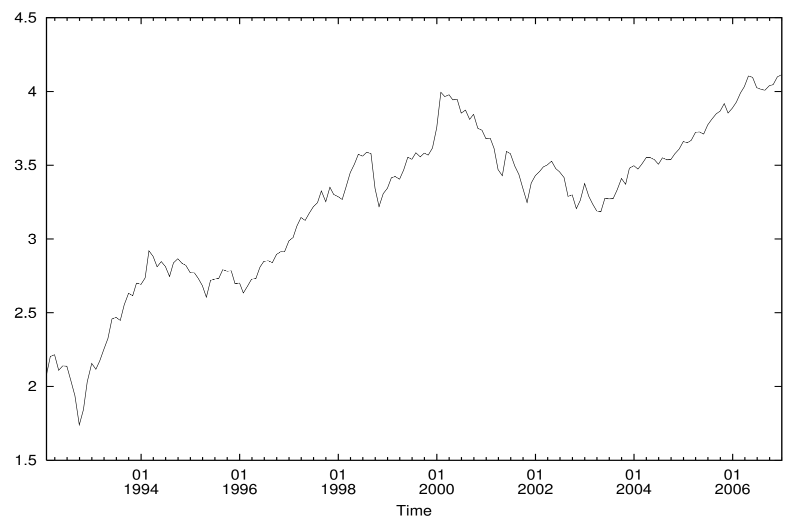

Figure 11: Historical $\tilde{s}^3$

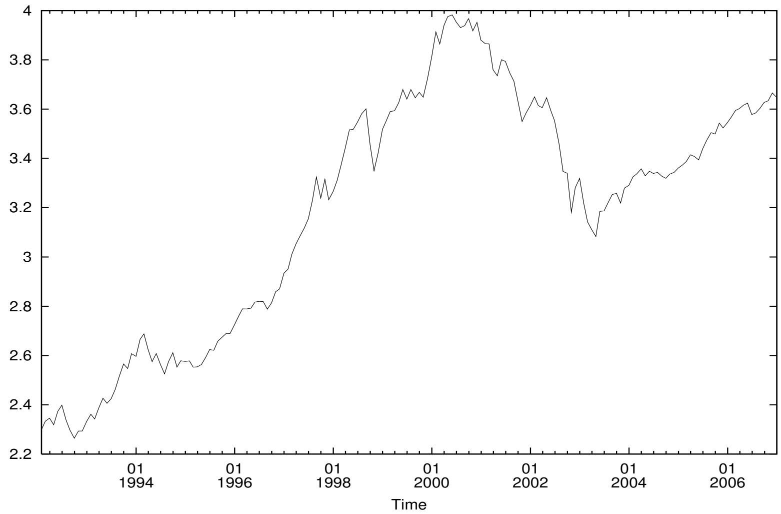

Figure 12: Historical $\tilde{s}^4$

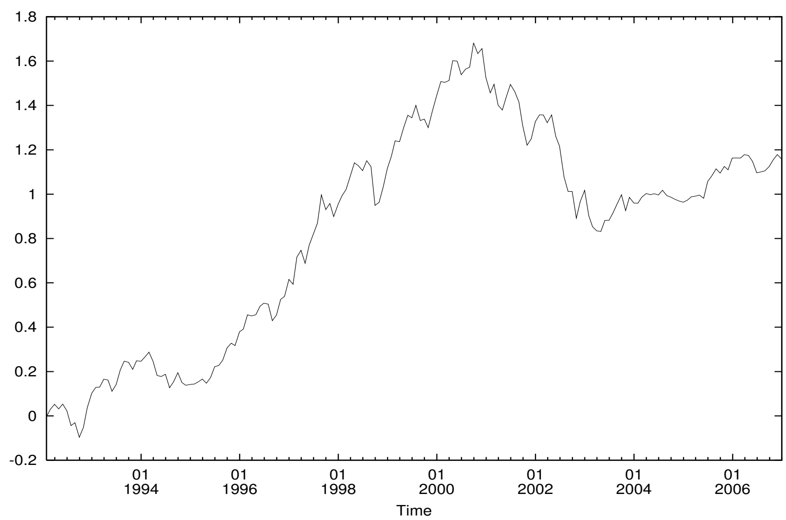

Figure 13: Historical $\tilde{s}^5$

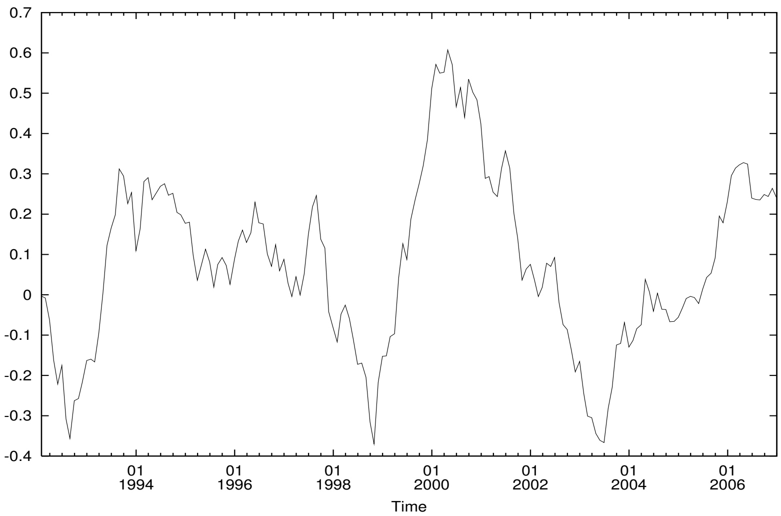

Figure 14: Historical $\tilde{s}^6$

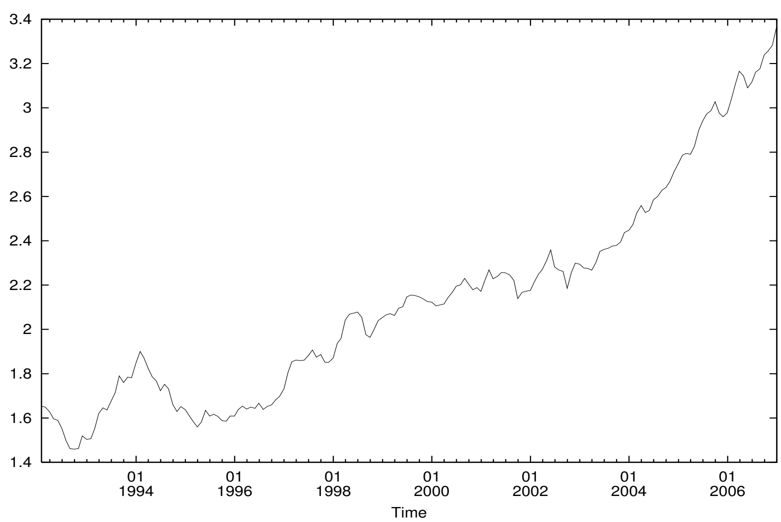

Figure 15: Historical $\tilde{s}^7$

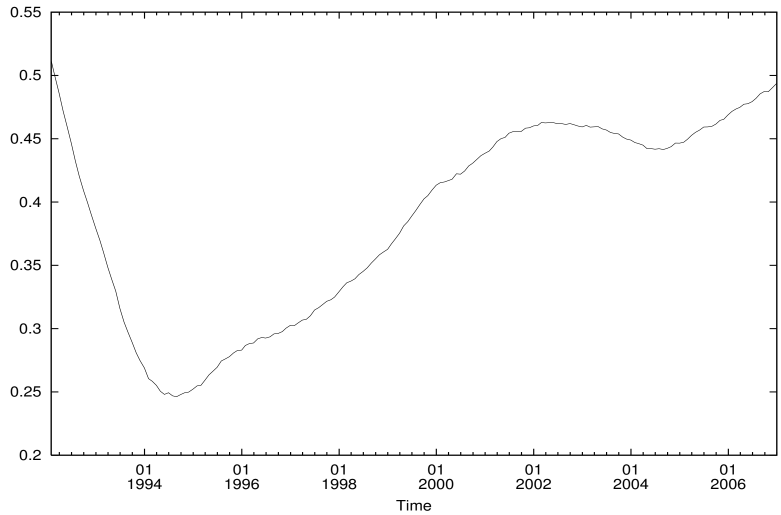

Figure 16: Historical e

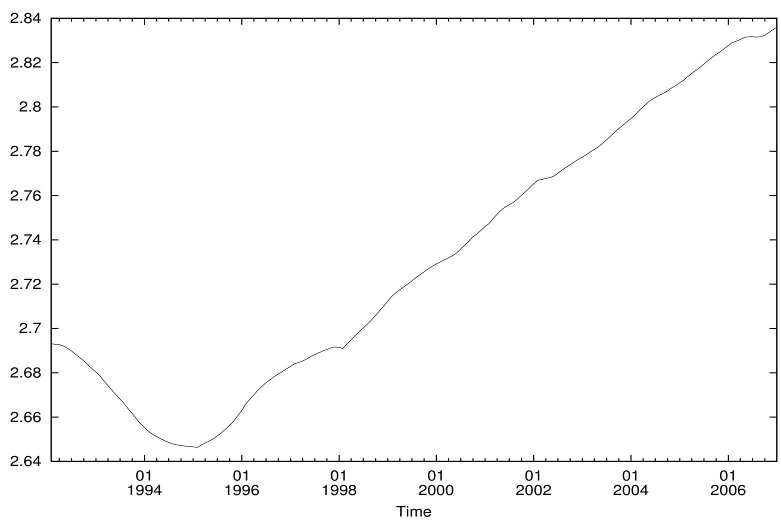

Figure 17: Historical $\tilde{w}$

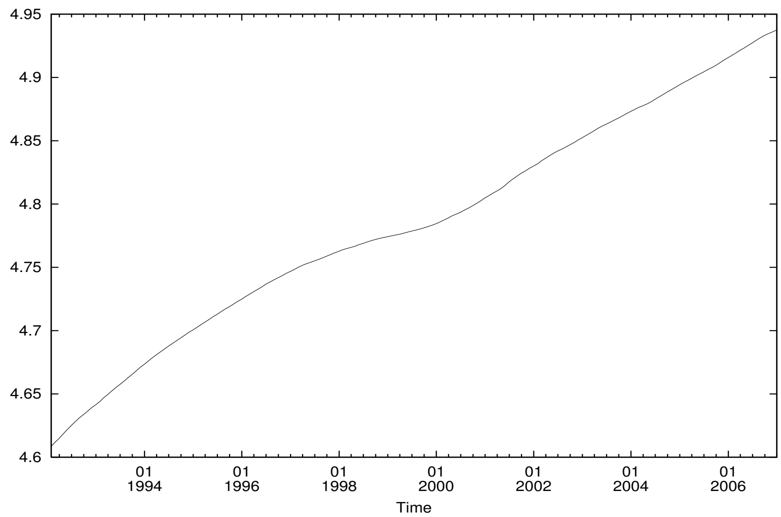

Figure 18: Historical i

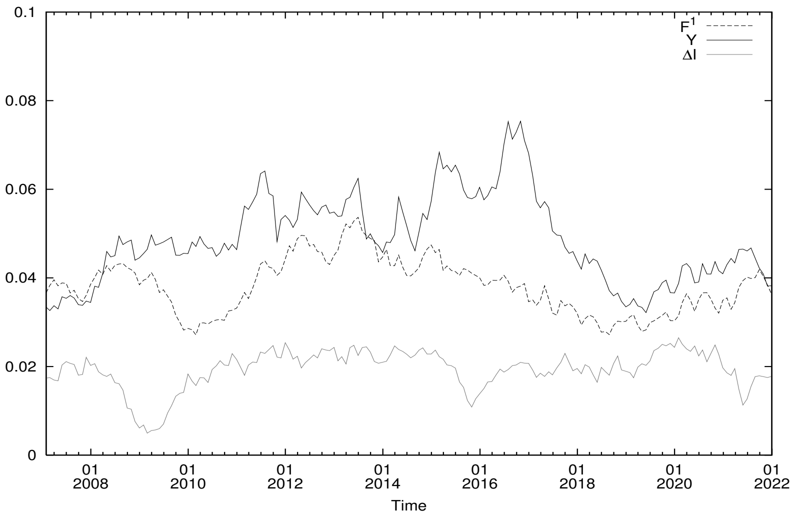

Figure 19: Simulated $F^1, Y, \Delta i$

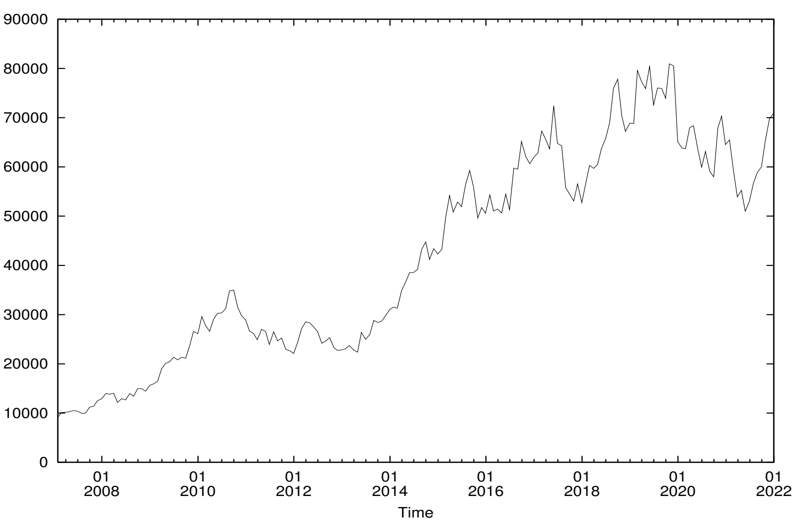

Figure 20: Simulated S^3

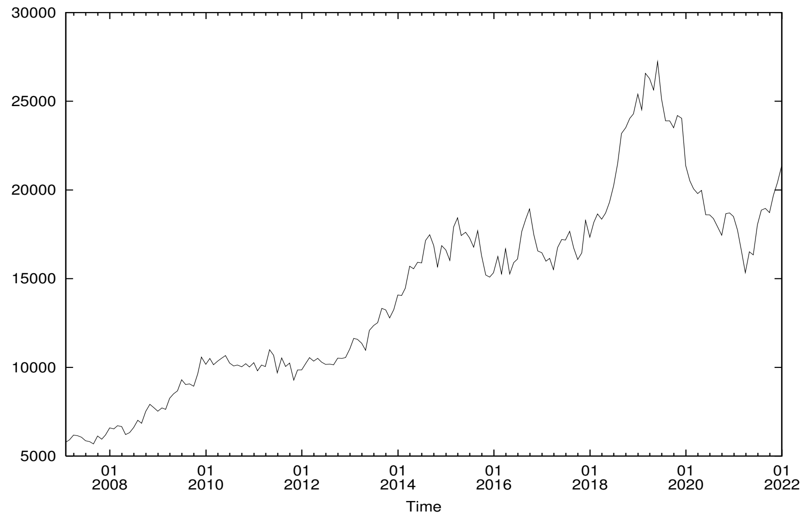

Figure 21: Simulated S^4

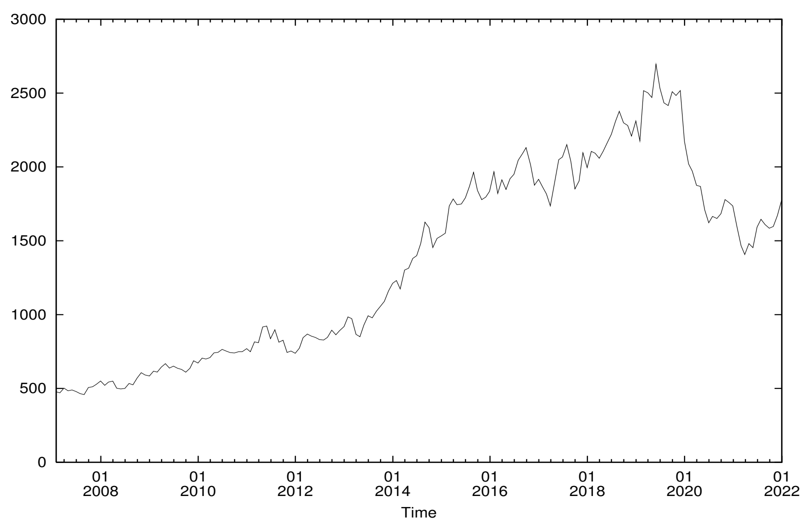

Figure 22: Simulated S^5

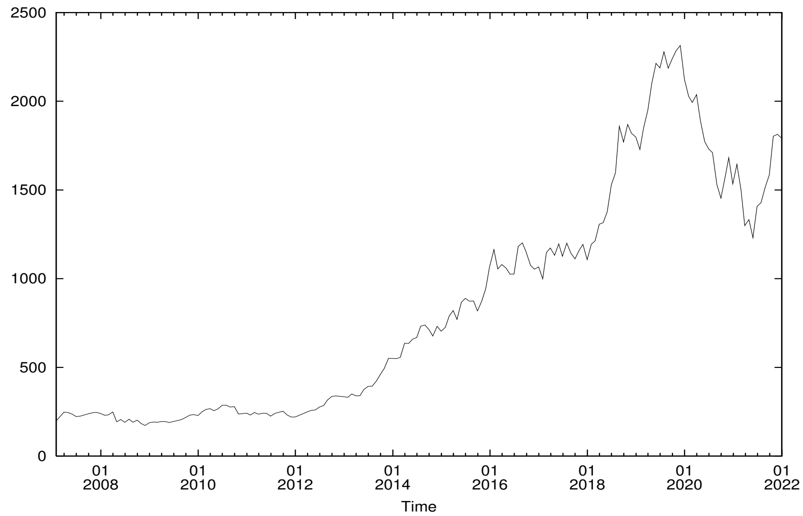

Figure 23: Simulated S^6

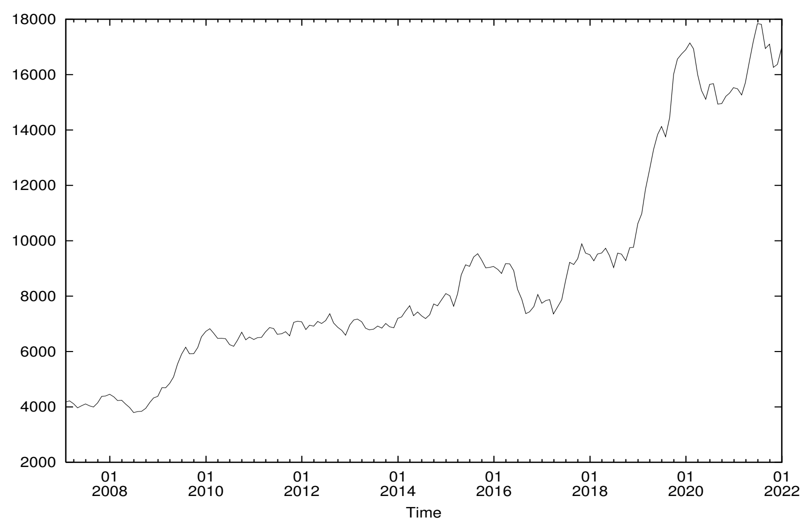

Figure 24: Simulated S^7

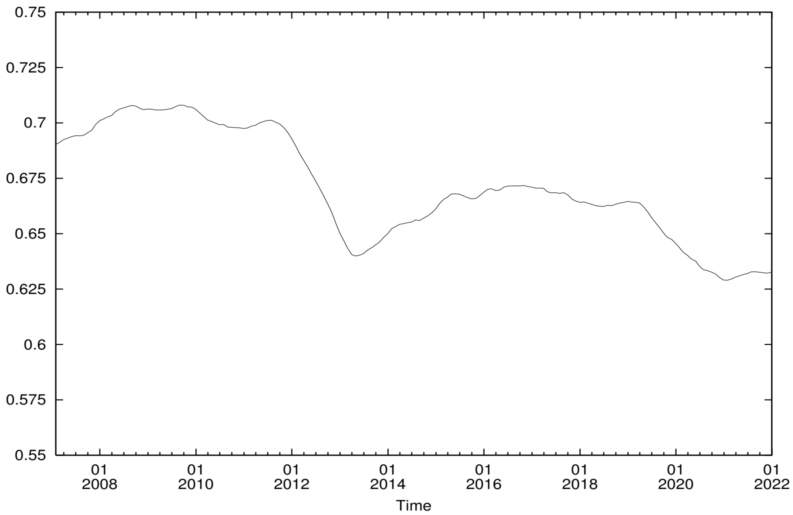

Figure 25: Simulated employment rate

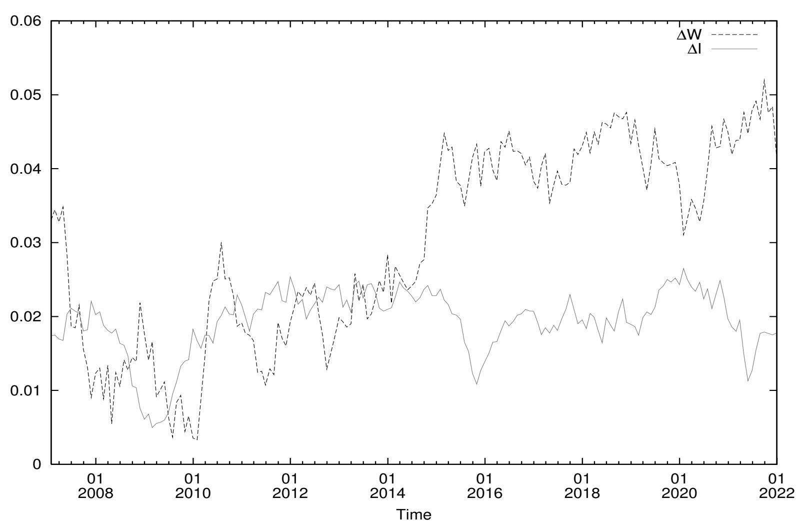

Figure 26: Simulated $\Delta W, \Delta i$