- **title of the paper** Implementing a pension plan along with the age increase of the plan participants

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Implementing a pension plan along with the age increase of the plan participants

Abstract

In general, in order for the actuary to communicate the contribution amount to the employer, and/or the employees, it is normal to refer to a percentage applied to the salaries of the plan participants. However, not all of the plan participants are in the same age groups; for some the future working life is expected to be short, while for others it is expected to be longer. As a consequence, a proportion of the pension benefit for the older members will be funded from the younger age groups. In this case there is an issue of fairness, in a funded plan, that needs to be addressed.

On the other hand, and according to the living standards of the individuals, some may wish to have the flexibility to buy either higher or lower portions of their benefits as they get older. This may be an option that may be worth examining. We have approached the above issue by introducing the pension accrual density function, m(x), to the funding of defined benefit occupational pension schemes. We compare different approaches to the Normal Cost and Actuarial Liability from the viewpoint of the different accrual density functions. This leads to a discussion of which accrual density function should be chosen, and the criteria that should be used from choosing the most appropriate accrual function for the plan, taking into account the relationship between the level of the contribution rates and the age of the plan members.

Keywords: defined benefit pension plans; pension accrual density function; accrual function

1. Introduction

The age of the member of the pension plan may be considered in the implementation of the funding plan through the pension accrual function m(x). On this basis, the actuary has the flexibility to recommend a build up of the fund either at a higher or at a lower.

We work with m(x) and the accrual function, M(x), which represents that fraction of the actuarial value of future pensions accrued as an actuarial liability by age x under the actuarial cost method, and which is defined as:

$$M(x) = \begin{cases} 0, & x \prec a \\ \int_a^x m(t)dt, & a \leq x \prec r \\ 1, & x \geq r \end{cases}$$

The pension accrual density function may be regarded as a probability density function and the criteria for choosing it are summarised in terms of a) its mathematical properties, b) its profile with respect to age and c) the utility of the accrual function from the perspective of the actuary and the pension plan. On the basis of these criteria, we consider the following distributions as being possible candidates for application to pension funding methods, and these are defined in Appendix 1: a) the Power function, b) the Truncated Exponential and c) the Truncated Pareto.

This choice is based on the observation that m(x) and M(x), under the uniform distribution, coincide with the benefit accrued under the Normal Cost and Actuarial Liability respectively, for the Projected Unit Credit method, (see Appendix 1). Since the Uniform distribution is the special case of the Power function when p = 1, we examine the general case where $p \neq 1$ and then extend our choice to other related distributions. In addition, we categorise these distributions according to their relationship with either an accelerating or a decelerating cost method.

We consider defined benefit occupational pension plans, where the benefits promised in the event of various contingencies are defined by a formula, while the contributions are to be determined by the actuary by means of the valuation process. The funding method then represents the means by which the contribution rate is fixed at each valuation. We consider individual funding methods, and the case of annual valuations.

In the mathematical discussion we adopt a continuous time approach on the basis of a deterministic model and we make the following assumptions:

Assumption 1: In the model pension plan, we consider an active group which extends over the ages a to r, with all new entrants coming in at age a and all retirements occurring at age r; only retirement benefits are allowed. For both the active and the retired participants, survivorship is in accordance with the function l_x, which does not depend on the time variable t; and so the population is stationary from the start.

Assumption 2: The annual salary rate for a participant aged x, $a \leq x \leq r$, is represented by s_x. At time t = 0, it is equal to the salary at entry age, s_a , which remains as a base factor for $t \geq 0$. Growth in salaries over time are represented by means of a function g(t), defined as :

$g(t) = e^{\tau t}$, $0 \leq t < r - a$, where τ is the valuation force of salary rate of increase; the latter assumption establishes a year-of-experience pattern of growth for salaries.

Assumption 3: Initial pensions are a fixed percentage, b, of final salaries and increase during retirement by a factor $\beta(x)$ which is used to denote the adjustment of the initial pension at age of retirement r, of a retiree age x; $x > r$, $\beta(x) = e^{\beta*(x-r).}$

The symbol h(t) represents the density at time t of the amount of newly incurred age r pensions. Thus h(10) = 1,000 implies that in the moment (10,10 + dt) the amount of new age r pensions which come into effect under the model plan is 1,000dt.

Our population is stationary and thus the density of new retirees at time t + r-x from participants aged x at time t is l_r. Each of these will, at time t + r-x, has annual salary rate $e^{\tau*(t+r-1-x)}* s_{a}$. Then according to the definition above, the density of new pensions incurred, i.e. entering benefit status at time t + r-x, for those who at time t are aged x (x<r), or who at time t +r –x are aged r ($x \geq r$), can be expressed as: $h(t + r-x) = e^{\tau*(t+r-1-x)}* s_{a} * b*l_{r,}$ assuming that pensions are a flat percentage, b, of final salary.

Assumption 4: The Normal Cost rate NC_x in regard to a participant aged x for B_r units of initial pension from age r is given in the continuous case by the following formula:

$$NC_x = \begin{cases} B_r * m(x) * \dfrac{D_r}{D_x} * \bar{a}_r^{(\delta-\beta)}, & a \leq x \prec r \\ 0 & , x > r \end{cases} \qquad \textbf{(1.1)}$$

The annual rate of plan Normal Cost at time t, equals to:

$$NC(t) = \int_a^r h(t+r-x)*m(x) * e^{-\delta*(r-x)} * \bar{a}_r^{(\delta-\beta)}\, dx \qquad \textbf{(1.2)}$$

or

$$NC(t) = \int_a^r h(t+r-x) * \frac{l_x}{l_r} *\left(\frac{1}{B_r} \right)* NC_x \, dx = e^{\tau * t} * NC(0) \qquad \textbf{(1.3)}$$

Assumption 5: The accrued liability in regard to a participant aged x for B_r units of initial pension from age r is given in the continuous case by the following formula:

$$AL_x = \begin{cases} B_r * M(x) * \frac{D_r}{D_x} * \bar{a}_r^{(\delta-\beta)} , & a \le x \le r \\ B_r * \bar{a}_x^{(\delta-\beta)} * e^{\beta*(x-r)} , & x > r \end{cases} \qquad \textbf{(1.4)}$$

The annual rate of plan Accrued Liability at time t, equals to:

$$AL(t)= \int_a^r h(t+r-x)* \frac{l_x}{l_r} M(x)*e^{-\delta*(r-x)}* \bar{a}_r^{(\delta-\beta)} \, dx + \int_r^w h(t+r-x)* \frac{l_x}{l_r} * \bar{a}_x^{(\delta-\beta)} e^{\beta*(x-r)} dx \qquad \textbf{(1.5)}$$

where the second term represents the value of future pension payments for those participants already retired at time t.

Given that M(x) = 1, for $x \ge r$, then

$$AL(t) = \int_a^w h(t+r-x)* \frac{l_x}{l_r} * \left(\frac{1}{B_r} \right)* AL_x \, dx = e^{\tau * t} * AL(0) \qquad \textbf{(1.6)}$$

We use the following notation:

δ: continuous rate of Investment return,

β: continuous rate of pension adjustment,

$\bar{a}_r$: continuous life annuity payable from the normal retirement age r,

$\ddot{a}_r$: life annuity-due payable from the normal retirement age r,

$\bar{a}_r^{(\delta-\beta)}$ / $\ddot{a}_r^{(\delta-\beta)}$: continuous life annuity / life annuity-due, payable from the normal retirement age r and calculated at the force of interest $\delta - \beta$.

We focus on the levels of the Normal Cost and the Actuarial Liability, resulting from the introduction of a different pension accrual density function in each case. We compare the cost methods defined with the use of the accrual density functions on the basis of their Normal Cost and Actuarial Liability. On the same basis we compare these with the traditional methods; i.e. Unit Credit and Entry Age Normal as defined in the Appendix 1.

2. Categorisation of m(x)

The categorisation of m(x) with respect to its association with an accelerating or a decelerating cost method has been defined by Cooper and Hickman (1967), as follows:

Let $m'(x)$ be the first derivative of m(x).

If $M''(x) = m'(x) < 0,\ a \le x \le r$, the actuarial cost method defined by m(x) results in decelerating funding at age x, and the actuarial cost method associated with m(x) is a decelerating actuarial cost method.

If $M''(x) = m'(x) > 0,\ a \le x \le r$, the actuarial cost method defined by m(x) results in accelerating funding at age x, and the actuarial cost method associated with m(x) is an accelerating actuarial cost method.

On the basis of the above definition,

- for Power function distribution, $m'(x) = p*(p-1)*\dfrac{(x-a)^{p-2}}{(r-a)^{p}}$, $a \le x \le r$, $p \succ 0$,
- for Truncated Pareto, $m'(x) = -(k+1)*\dfrac{\frac{k}{a}*(\frac{a}{x})^{k+1}}{1-(\frac{a}{r})^{k}}$, $a \le x \le r$, $k \succ 0$

and

- for Truncated Exponential $m'(x) = -\dfrac{1}{\sigma^{2}}*\dfrac{e^{-\frac{x-a}{\sigma}}}{1-e^{-\frac{r-a}{\sigma}}}$, $a \prec x \prec r$, $\sigma \succ 0$

Regarding both Truncated Exponential and Truncated Pareto, it is clearly seen that $m'(x) \prec 0$ which implies that they are decelerating actuarial cost methods. The Power function as a result of the value of the parameter p, may be categorised as either a decelerating ($p \prec 1$) or an accelerating cost method ($p \succ 1$). Also, when p =1, i.e. under Uniform distribution**,** the actuarial cost method is characterised by $m'(x) = 0$. This result shows that it is neither a decelerating nor an accelerating cost method.

Table 2.1 below describes the development of m(x), at specific ages, under the different functions. The age interval runs from 30 (entry age) to 65 (age of retirement). For each distribution, we have investigated different parameter values in order to examine the trend of m(x) with increasing age and thus to reach a conclusion regarding the magnitude of the portion of the benefit purchased. Here, they have been set equal to:

Power function: p = 0.3, 0.8, 1, 1.5,

Truncated Pareto: k = 0.3, 0.8, 1.5

Truncated Exponential: σ = 30, 40, 50

<u>Table 2.1:</u> *m(x) development under different distributions, at specific ages*

age	**Power p = 0.3**	**Power p = 0.8**	**Tr. Exp. σ = 30**	**Tr. Exp. σ = 40**	**Tr. Exp. σ = 50**	**Pareto k = 0.3**	**Pareto k = 0.8**	**Pareto k = 1.5**	**Power p = 1**	**Power p = 1.5**
35	0.033	0.034	0.041	0.038	0.036	0.040	0.044	0.050	0.029	0.016
40	0.021	0.029	0.035	0.033	0.033	0.033	0.034	0.035	0.029	0.023
45	0.016	0.027	0.029	0.029	0.029	0.029	0.028	0.026	0.029	0.028
50	0.013	0.026	0.025	0.026	0.027	0.025	0.023	0.020	0.029	0.032
55	0.011	0.024	0.021	0.023	0.024	0.022	0.019	0.016	0.029	0.036
60	0.010	0.024	0.018	0.020	0.022	0.020	0.017	0.013	0.029	0.040
65	0.009	0.023	0.015	0.018	0.020	0.018	0.014	0.011	0.029	0.043

In table 2.1, we observe that the lowest values of the m(x) s associated with a decelerating cost method are calculated under the Power function p =0.3. Truncated Pareto, k = 1.5, gives very low m(x) values after age 50 and considerably higher values up to age 40. m(x) values under Truncated Exponential σ = 40 and Truncated Pareto k = 0.3 are very close to each other. As expected, since $\int_a^r$ m(x) dx = 1, the low values of m(x) s at the older ages imply that M(x) rapidly accumulates to 1. This is clearly seen in the case of Power function under p= 0.3, since the values of m(x) s are significantly lower than those under all other distribution functions from age 35 and thereafter.

3. The New Cost Methods

On the basis of the accrual density function, we consider the following new Cost Methods, defined accordingly through the Normal Cost and Actuarial Liability formulae 1.3, 1.6, given earlier:

- Under the Power function distribution:

$$NC(t)=\begin{cases} e^{-\tau} * e^{\tau*t+(\tau-\delta)*(r-a)} l_r s_a b \bar{a}_r^{(\delta-\beta)} \dfrac{p}{(r-a)^p} \displaystyle\int_0^{r-a} e^{(\delta-\tau)*y} * \left(\frac{x}{a}\right)^{p-1} dx & , \ p \neq 1 \\ e^{-\tau} * e^{\tau*t} * l_r * s_a * b * \bar{a}_r^{(\delta-\beta)} * \dfrac{1}{r-a} \bar{a}_{\overline{r-a|}}^{(\delta-\tau)} & , p = 1 \end{cases} \quad \textbf{(3.1)}$$

and

$$AL(t)=\begin{cases} \dfrac{e^{\tau*(t-1)+(\tau-\delta)*(r-\alpha)} l_r s_\alpha b * \bar{\alpha}_r^{(\delta-\beta)}}{(r-a)^p} \displaystyle\int_a^{r-a} e^{\frac{(\delta-\tau)*x}{a}} * \left(\frac{x}{a}\right)^p dx + \dfrac{l_r * e^{-\tau(1-t)} * s_\alpha * b}{\delta-\tau} \left(\bar{a}_r^{(\tau-\beta)} - \bar{a}_r^{(\delta-\beta)} \right), & p \neq 1 \\ \dfrac{e^{\tau*(t-1)+(\tau-\delta)*(r-\alpha)} l_r s_\alpha b * \bar{\alpha}_r^{(\delta-\beta)}}{(r-1-a)*(\delta-\tau)} * \left((r-a) - \bar{a}_{\overline{r-a|}}^{(\delta-\tau)}\right) + \dfrac{l_r * e^{-\tau(1-t)} * s_\alpha * b}{\delta-\tau} \left(\bar{a}_r^{(\tau-\beta)} - \bar{a}_r^{(\delta-\beta)} \right) & , p = 1 \end{cases} \quad \textbf{(3.2)}$$

- Under Truncated Pareto:

$$NC(t) = e^{\tau*(t-1)+(\tau-\delta)*(r-\alpha)} l_r s_\alpha b * \bar{\alpha}_r^{(\delta-\beta)} * \frac{k}{a} * \frac{a^{k+1}}{1-\left(\frac{a}{r}\right)^k} \int_1^{\frac{r}{a}} e^{a*(\delta-\tau)*} \left(\frac{x}{a}\right) * \left(\frac{x}{a}\right)^{-(k+1)} dx, \quad \textbf{(3.3)}$$

$$AL(t)= e^{\tau*(t-1)} l_r s_\alpha b * \bar{\alpha}_r^{(\delta-\beta)} \left(\bar{a}_{\overline{r-a|}}^{(\delta-\tau)} + \frac{a^k * e^{(\tau-\delta)*r}}{1-\left(\frac{a}{r}\right)^k} * \int_1^{(r/a)} e^{\frac{a*(\delta-\tau)x}{a}} * \left(\frac{x}{a}\right)^{-k} dx + \frac{l_r * e^{-\tau(1-t)} * s_\alpha * b}{\delta-\tau} \left(\bar{a}_r^{(\tau-\beta)} - \bar{a}_r^{(\delta-\beta)} \right) \right) \textbf{(3.4)}$$

- Under Truncated Exponential

$$NC(t) = e^{-\tau} * e^{\tau * t} * l_r\, s_a\, b\, \bar{a}_r^{(\delta-\beta)} \frac{1}{\sigma * (1-e^{-(\frac{r-a}{\sigma})})} * e^{-\frac{r-a}{\sigma}} * \bar{a}_{\overline{r-a}|}^{(\delta-\tau-\frac{1}{\sigma})} \tag{3.5}$$

$$AL(t) = e^{\tau*(t-1)} l_r s_r b * \bar{\alpha}_r^{(\delta-\beta)} \frac{1}{1-e^{-\frac{r-a}{\sigma}}} \left(\bar{a}_{\overline{r-a}|}^{(\delta-\tau)} - e^{-\frac{r-a}{\sigma}} * \bar{a}_{\overline{r-a}|}^{(\delta-\tau-\frac{1}{\sigma})}\right) + \frac{l_r * e^{-\tau(1-t)} * s_\alpha * b}{\delta-\tau} \left(\bar{a}_r^{(\tau-\beta)} - \bar{a}_r^{(\delta-\beta)}\right) \tag{3.6}$$

4. Comparison between the traditional and the new defined cost methods in terms of Normal Cost and Actuarial Liability at age x.

We compare the traditional and the new defined cost methods, for a participant aged x, on the basis of the Normal Cost and Actuarial Liability according to equations 1.1 and 1.4. Our calculations concern a participant who enters the scheme at age 30 and retires at age 65 receiving a benefit B_r of one unit; b_x is then defined accordingly. We consider three types of standard actuarial cost method: Current Unit Credit, Projected Unit Credit and Entry Age Normal. Regarding the distinction between the 2 types of Unit Credit cost method, i.e. Current and Projected, we point out that, in the Current UC, the benefit is defined as a percentage b of the member’s current salary. In the Projected UC it is defined as a percentage of the member’s final salary, s_{r-1}. The formulae used for our representations of these cost methods are presented below in table 4.1.

<u>Table 4.1</u>*: Formulae used in the illustrative examples*

Actuarial Cost Method	**Normal Cost, NC$_x$**	**Actuarial Liability, AL$_x$**
Current Unit Credit	$\frac{s_x}{S_r} * \frac{D_r}{D_x} \ddot{a}_r^{(\delta-\beta)}$	$\frac{1}{S_r} * S_x * \frac{D_r}{D_x} * \ddot{a}_r^{(\delta-\beta)}$
Projected Unit Credit	$\frac{1}{(r-a)} * \frac{D_r}{D_x} \ddot{a}_r^{(\delta-\beta)}$	$\frac{x-a}{r-a} * \frac{D_r}{D_x} * \ddot{a}_r^{(\delta-\beta)}$
Entry Age Normal	$\frac{s_x}{s_a * {}^s\ddot{s}_{\overline{a:r-a}\rvert}} * \frac{D_x}{D_a} * \frac{D_r}{D_x} \ddot{a}_r^{(\delta-\beta)}$	$\frac{{}^s\ddot{a}_{\overline{a:x-a}\rvert}}{{}^s\ddot{a}_{\overline{a:r-a}\rvert}} * \frac{D_r}{D_x} * \ddot{a}_r^{(\delta-\beta)}$
Power function	$p * \frac{(x-a)^{p-1}}{(r-a)^p} * \frac{D_r}{D_x} * \bar{a}_r^{(\delta-\beta)} \quad , \; a \le x \le r$	$\frac{(x-a)^p}{(r-a)^p} * \frac{D_r}{D_x} * \bar{a}_r^{(\delta-\beta)} \quad , a \le x \le r$
Truncated Exponential	$\frac{1}{\sigma} * \frac{1}{1-e^{(-\frac{r-a}{\sigma})}} * e^{-\frac{x-a}{\sigma}} \frac{D_r}{D_x} \bar{a}_r^{(\delta-\beta)}$	$\frac{1-e^{-\frac{x-a}{\sigma}}}{1-e^{-\frac{r-a}{\sigma}}} * \frac{D_r}{D_x} * \bar{a}_r^{(\delta-\beta)} \quad , a \le x \le r$
Truncated Pareto	$\frac{\frac{k}{a} * (\frac{a}{x})^{k+1}}{1-(\frac{a}{r})^k} * \frac{D_r}{D_x} * \bar{a}_r^{(\delta-\beta)}$	$\frac{1-(\frac{a}{x})^k}{1-(\frac{a}{r})^k} * \frac{D_r}{D_x} * \bar{a}_r^{(\delta-\beta)} \quad , a \le x \le r$

In the formulae of table 4.1, the salary, s$_x$, increases exponentially at a force of rate τ.

$S_x = \sum_{t=a}^{x-1} s_t$, $\ddot{s}_{\overline{a:r-a}\rvert}$ the accumulated value of n annuity due payments and

${}^s\ddot{a}_{\overline{x:r-x}\rvert} = \sum_{t=0}^{r-x-1} \frac{s_{x+t}}{s_x} e^{-\delta * t} * {}_tp_x$ is the salary based annuity.

In the calculations that follow, the other parameters take the values:

a = entry age = 30,

r = retirement age = 65,

i = rate of investment return =0.05, corresponding to δ,

β = continuous rate of pension adjustment = 0.015,

τ = continuous rate force of salary increase 0.03,

γ = continuous rate of price inflation = 0,

s_a = salary at entry age =1 unit

For the service table, (assuming x $\geq$ 30), the illustrative life table quoted in Bowers et al (1986) has been used.

The results concerning the Actuarial Liability, presented in the following table 4.2 are summarised in the following inequalities**:**

$$ {}^{CUC}AL_x < {}^{PUC}AL_x \equiv AL_{x\ (\text{Uniform})} < AL_{x\ (\text{Truncated Exponential})} \qquad \textbf{(4.1)} $$

$$ {}^{CUC}AL_x < {}^{PUC}AL_x \equiv AL_{x\ (\text{Uniform})} < AL_{x\ \left(\underset{Trunc.Pareto}{\overset{k<1,\,k<\frac{p}{d}}{}}\right)} \qquad \textbf{(4.2)} $$

$$ {}^{EAN}AL_x > {}^{PUC}AL_x \equiv AL_{x\ (\text{Uniform})} > AL_{x\ (\text{Power}\ ,\ p>1)} \qquad \textbf{(4.3)} $$

These inequalities show that the Actuarial Liability of the accelerating cost methods (i.e. Current Unit Credit, Power function, p > 1) is less than the Actuarial Liability of the decelerating cost methods (i.e. Truncated Exponential, Truncated Pareto, Entry Age Normal (EAN)).

These conclusions verify the proposition proved by Bowers et al, according to which: “ if m(x) is associated with a decelerating cost method ($m'(x)$<0) and $m_1(x)$ is associated with an accelerating cost method ($m_1'(x)$>0), then $M(x) > M_1(x)$, $a<x<r$ ”

<u>Table 4.2</u>: *Actuarial liability under the new & traditional cost methods*

age	**C.U.C.**	**P.U.C. Uniform**	**Power p = 1.5**	**Tr. Exp. σ = 30**	**Tr. Exp. σ = 40**	**Tr. Exp. σ = 50**	**Pareto k = 0.3**	**Pareto k = 0.8**	**E.A.N**
35	0.19	0.31	0.12	0.49	0.44	0.42	0.48	0.55	0.43
40	0.54	0.81	0.43	1.17	1.08	1.02	1.13	1.27	1.06
45	1.13	1.58	1.03	2.10	1.98	1.90	2.04	2.21	1.96
50	2.14	2.75	2.08	3.40	3.25	3.15	3.30	3.50	3.24
55	3.84	4.54	3.84	5.22	5.07	4.97	5.11	5.30	5.08
60	6.74	7.35	6.80	7.87	7.75	7.68	7.77	7.91	7.77
64	10.60	10.78	10.63	10.93	10.90	10.88	10.90	10.94	10.91

The results concerning Normal Cost are presented in the following table 4.3:

Table 4.3: *Normal Cost under the new & traditional cost methods*

Age	**C.U.C.**	**P.U.C. Uniform**	**Power p = 1.5**	**Tr. Exp. σ = 30**	**Tr. Exp. σ = 40**	**Tr. Exp. σ = 50**	**Pareto k = 0.3**	**Pareto k = 0.8**	**E.A.N**
35	0.04	0.06	0.04	0.09	0.08	0.08	0.09	0.10	0.08
40	0.06	0.08	0.07	0.10	0.09	0.09	0.09	0.10	0.09
45	0.09	0.11	0.10	0.11	0.11	0.11	0.11	0.10	0.11
50	0.14	0.14	0.16	0.12	0.13	0.13	0.12	0.11	0.13
55	0.22	0.18	0.23	0.13	0.15	0.15	0.14	0.12	0.15
60	0.34	0.24	0.34	0.15	0.17	0.19	0.17	0.14	0.17
64	0.50	0.32	0.47	0.17	0.20	0.22	0.20	0.16	0.19

In table 4.3, we note that the figures provided by the Current Unit Credit approach those under the Power function when $p>1$. For this reason we may consider that they form one group. This result is to be expected since under both methods, the benefit is allocated in higher proportions as age increases.

The figures provided under Entry Age Normal are lower than the corresponding ones provided under the Truncated Exponential and Truncated Pareto. As above, we may also consider that Entry Age Normal, Truncated Exponential and Truncated Pareto form one group, where the benefit is allocated in lower proportions as age increases. We conclude from this comparative analysis that the progress with time of both the Normal Cost and Actuarial Liability is determined (ceteris paribus) by the choice of the function, $m(x)$.

In order to examine the sensitivity of the results obtained, we have tested the underlying parameters, changing consecutively the entry age, α, the age of retirement, r, and the valuation rate of investment return, i.

In particular, we consider, in the light of the above discussion:

a) $\alpha=25$ & $r = 65$, $\alpha=40$ & $r=70$, $\alpha=30$ & $r=70$.

b) $i = 0.03$ and $i = 0.07$

When we assume that the participant either enters the scheme at age 25 and retires at age 65 or enters at age 30 and retires at age 70, we observe that the trend followed by the Normal

Cost and Actuarial Liability throughout his/her active years does not change. As expected, however, the Normal Cost values are lower than those where α =30 and r=65 because the cost is spread over 40 instead of 35 years. We point out that, as the retirement age increases the Normal Cost and Actuarial Liability values decrease due to the change in post-retirement life expectancy.

On the other hand, when we assume that the participant enters the scheme at age 40, the Normal Cost increases since it is spread over 25 years and the Actuarial Liability decreases slightly up to the age of retirement.

We conclude that the difference between the entry and retirement age is the key assumption that affects their values. Specifically, given that the age of retirement is kept constant, the higher the number of years in service, the lower the Normal Cost and the higher the liability as the plan members approach retirement.

The Normal Cost and Actuarial Liability patterns remain also unchanged if the valuation rate of investment return either increases or decreases. The effect of its change is seen in the cost and liability values, which as expected, increase as it decreases, and vice versa.

Summarising the above, we may conclude that the overall pattern with which the Normal Cost and Actuarial Liability progress with time does not change when the entry age or the age of retirement or the valuation rate of investment return (which is here assumed to be a deterministic variable) change.

5. Comparison of the new defined cost methods in terms of the Accrued Liability at time t

Bowers et al (1986) prove the following proposition:

Proposition : Consider two accrual functions $M_I(x)$,$M_{II}(x)$. If $D(x) = M_I(x) - M_{II}(x)$ is such that $D'(a) > 0$ and $D'(x) = 0$ has exactly one solution, $a \prec x \prec r$, then $AL_I(t) > AL_{II}(t)$.

We apply this proposition for the following pairs of accrual functions ($M_I(x)$,$M_{II}(x)$): (Truncated Exponential, Uniform), (Uniform, Power function), (Truncated Pareto, Power function), (Projected Unit Credit, Current Unit Credit), (Entry Age Normal, Projected Unit Credit). From the upcoming results we conclude the following:

$$^{CUC}AL(t) < {}^{PUC}AL(t) \equiv AL(t)_{(\text{Uniform})} < AL(t)_{(\text{Truncated Exponential})} \qquad \mathbf{(5.1)}$$

$$^{CUC}AL(t) < {}^{PUC}AL(t) \equiv AL(t)_{(\text{Uniform})} < AL(t)_{(Trunc.Pareto)}^{k<1, k<\frac{p}{d}} \qquad \mathbf{(5.2)}$$

$$AL(t)_{(\text{Power}\ ,\ p>1)} < {}^{PUC}AL(t) \equiv AL(t)_{(\text{Uniform})} < {}^{EAN}AL(t) \qquad \mathbf{(5.3)}$$

These inequalities show that the Actuarial Liability of the accelerating cost methods (i.e. Current Unit Credit, Power function, p > 1) is less than the Actuarial Liability of the decelerating cost methods (i.e. Truncated Exponential, Truncated Pareto, Entry Age Normal). These conclusions derived from the comparison in terms of the Accrued Liability at time t, are the same as those derived from the comparison in terms of the Actuarial Liability at age x, (table 4.2).

6. Concluding Comments

We have presented an analysis in terms of appropriate accrual functions and appropriate parameters, in order to give the actuary the tools to proceed on the basis of a portion of benefit purchased which varies with the age of the scheme participant. In addition, we offer the option to the scheme participants to buy either higher or lower portions of their benefit as they get older.

More specifically, we work with the pension accrual function, m(x), choosing the Power function, the Truncated Pareto and the Truncated Exponential distributions.

We show that if the benefit is allocated in higher proportions as age increases, the Normal Cost values are very similar when they are calculated either under the Current Unit Credit method or using the Power function. On the other hand, if it is allocated in lower proportions as age increases, they are very similar under the Entry Age Normal the Truncated Exponential and the Truncated Pareto methods.

In addition, we show that among the different accrual functions, a lower Actuarial Liability may be expected from the accelerating cost methods than from the decelerating ones while the Normal Cost follows the opposite trend.

Appendix 1

1. Distribution functions

- Suppose X ~ Power function, then

$$m(x) = \begin{cases} p * \dfrac{(x-a)^{p-1}}{(r-a)^{p}} & , \ a \le x \le r \ , \ p \succ 0 \\ 0, \text{ otherwise} & \end{cases}$$

$$M(x) = \begin{cases} 0 & , x \prec a \\ \dfrac{(x-a)^{p}}{(r-a)^{p}} & , a \le x \le r \ , \ p \succ 0 \\ 1 & , x \succ r \end{cases}$$

In case p = 1, X ~ Uniform distribution and

$$m(x) = \begin{cases} \dfrac{1}{(r-a)} & , \ a \le x \le r \\ 0 & , \text{ otherwise} \end{cases}$$

$$M(x) = \begin{cases} 0 & , x \prec a \\ \dfrac{(x-a)}{(r-a)} & , a \le x \le r \ , \\ 1 & , x \succ r \end{cases}$$

- Suppose X ~ Truncated Pareto distribution, then

$$m(x) = \begin{cases} \dfrac{\frac{k}{a} * (\frac{a}{x})^{k+1}}{1-(\frac{a}{r})^{k}} & , \ a \le x \le r \, , \ k \succ 0 \\ 0 & , \text{ otherwise} \end{cases}$$

$$M(x) = \begin{cases} 0 & , x \le a \\ \dfrac{1-(\frac{a}{x})^k}{1-(\frac{a}{r})^k} & , \quad a \prec x \prec r \ , \ k \succ 0 \\ 1 & , \quad x \ge r \end{cases}$$

- Suppose X ~ Truncated Exponential distribution, then

$$m(x) = \begin{cases} \dfrac{1}{\sigma} * \dfrac{1}{1-e^{(-\frac{r-a}{\sigma})}} * e^{-\frac{x-a}{\sigma}} \ , \ a \prec x \prec r \ , \ \sigma \succ 0 \\ 0, \qquad \text{otherwise} \end{cases}$$

$$M(x) = \begin{cases} 0 & , x \le a \\ \dfrac{1-e^{-\frac{x-a}{\sigma}}}{1-e^{-\frac{r-a}{\sigma}}} & , \ a \prec x \prec r \ , \ \sigma \succ 0 \\ 1 & , \quad x \ge r \end{cases}$$

Obviously, in all the above, M(a) = 0 and M(r)=1, i.e. M(x) has the properties of an accrual function.

2. Traditional Cost Methods

Current (or Traditional) Unit Credit: It aims to maintain a fund equal to the value of accrued benefits by reference to their amount as at the calculation date.

Projected Unit Credit: It aims to maintain a fund equal to the value of accrued benefits by reference to their projected amount at the date of retirement.

The distinction between Current and Projected Unit Credit is not strict. However, in order to make a distinction, we thought to consider as “Current Unit Credit” the method where the units are defined as a flat percentage benefit of the current salary and as “Projected Unit Credit” the method where the units are defined as the projected benefit at retirement divided by the number of active years before retirement.

Entry Age Normal: It aims to establish the level contributions rate which, when payable over the active lifetime of the employee, is sufficient to finance the benefits being provided.

References

Bowers N., Hickman J., Nesbitt C.: Introduction to the Dynamics of Pension Funding, Transaction of the Society of Actuaries, 1976, pp. 177-213

Bowers N., Hickman J., Nesbitt C.: The Dynamics of Pension Funding: Contribution Theory, Transaction of the Society of Actuaries, 1979, pp. 93-136

Bowers N., Gerber, Hickman J., Jones, Nesbitt C.: Actuarial Mathematics, 1986, pp.560-561 Transaction of the Society of Actuaries, 1976, pp. 177-213

Cooper S. and Hickman J.: A Family of Accrued Benefit Actuarial Cost Methods, Transaction of the Society of Actuaries, 1967,Vol.14, pp.53-65

Johnson N., Kotz S., Balakrishan N. : Continuous Univariate Distributions John Willey & Sons Inc., 1995,V 1, p.554, pp. 608, V 2, p.210