AFIR LIFE MUNICH 2009

The group balanced concept of long term saving: A continuous time model

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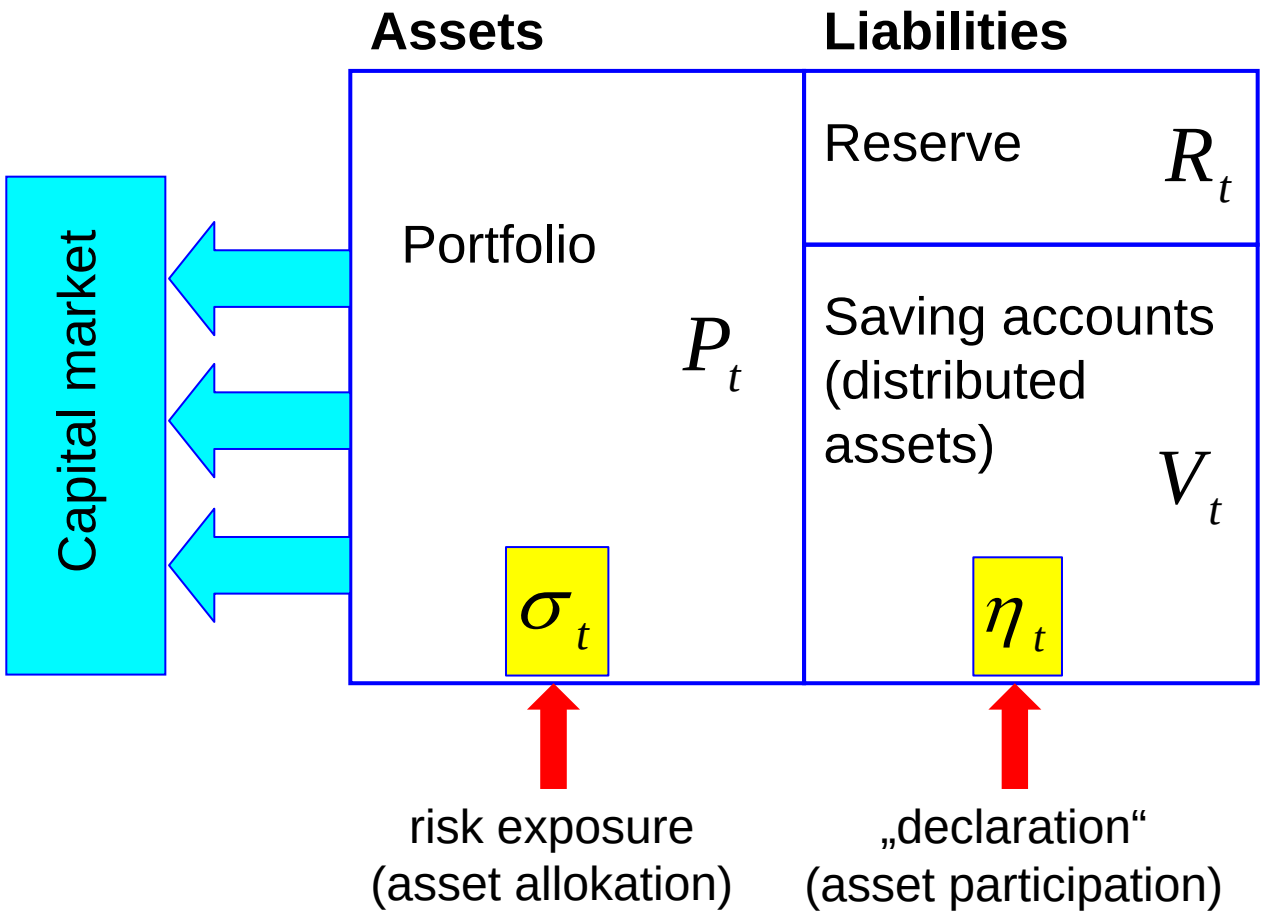
AFIR Conference, Munich 2009

Overview

- Model design
- Results
- Outlook

Saving plan with risk redistribution among generations



$$P_t = R_t + V_t$$

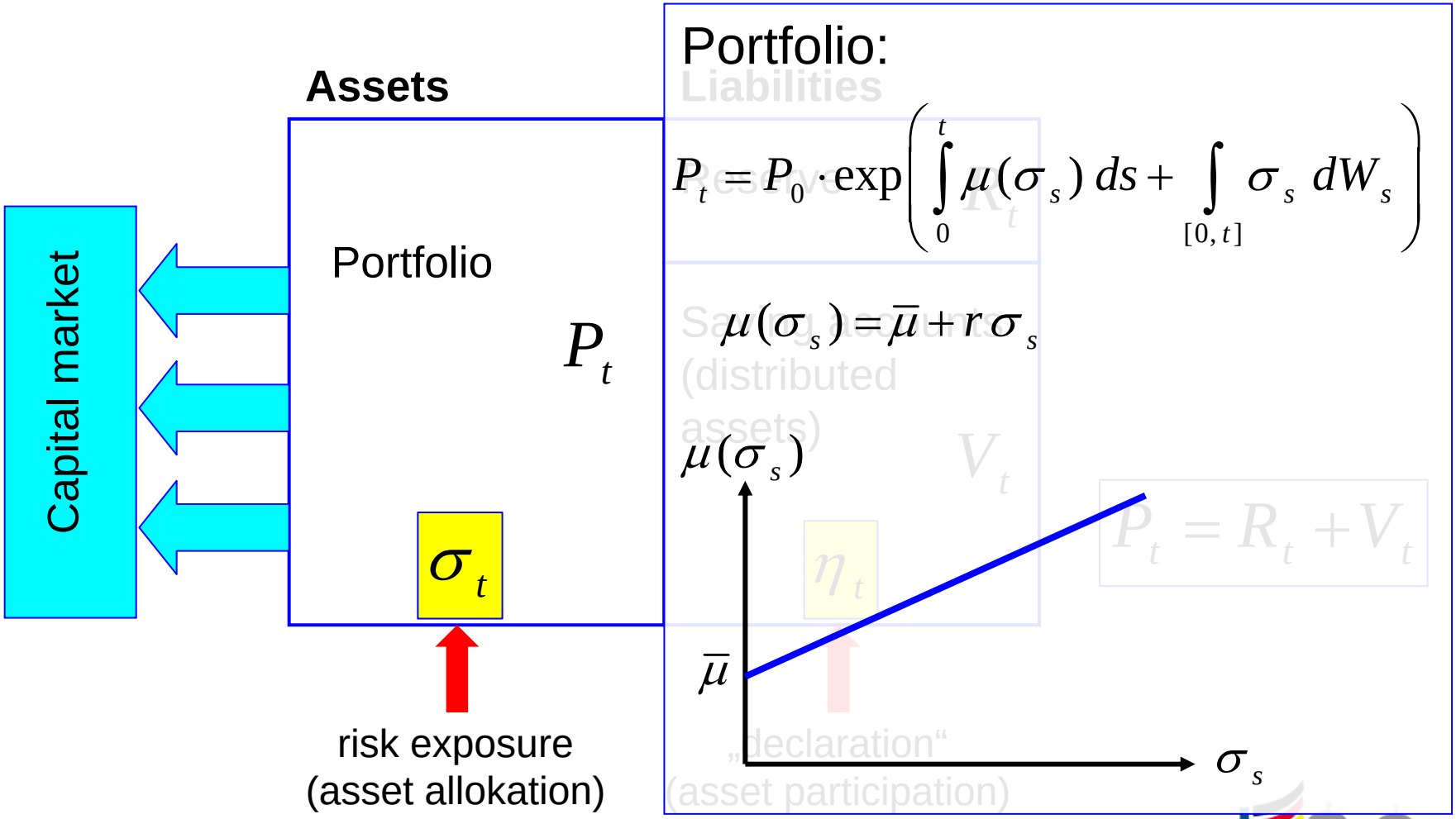
Questions

- Is risk sharing among generations possible?
- If so – can one measure “welfare” effects?
- What is the best way of risk sharing among generations?

Literature (among others)

- Christian Gollier: Intergenerational risk sharing and risk taking of a pension fund (J. of Public Economics, June 2008)
- Peter Løchte Jørgensen: Lognormal Approximation of Complex Path-dependent Pension Scheme Payoffs (European J. of Finance, Vol.13, 2007).
- Peter Albrecht: Die Kapitalanlageperformance der deutschen Lebensversicherer 1980-2007, Mannheimer Manuskripte 07/2008.

Capital market



Asset participation

Saving accounts:

$$V_t = V_0 \cdot \exp\left(\int_0^t \eta_s \, ds\right)$$

(log-)reserve ratio:

$$\rho_t := \ln\left(\frac{P_t}{V_t}\right) = -\ln\left(1 - \frac{R_t}{V_t}\right) \approx \frac{R_t}{V_t}$$

$$\rho_t = \rho_0 + \int_0^t \mu(\sigma_s) - \eta_s \, ds + \int_{[0,t]} \sigma_s \, dW_s$$

Assets

Portfolio

P_t

σ_t

risk exposure
(asset allokation)

Liabilities

Reserve	R_t
Saving accounts (distributed assets)	V_t

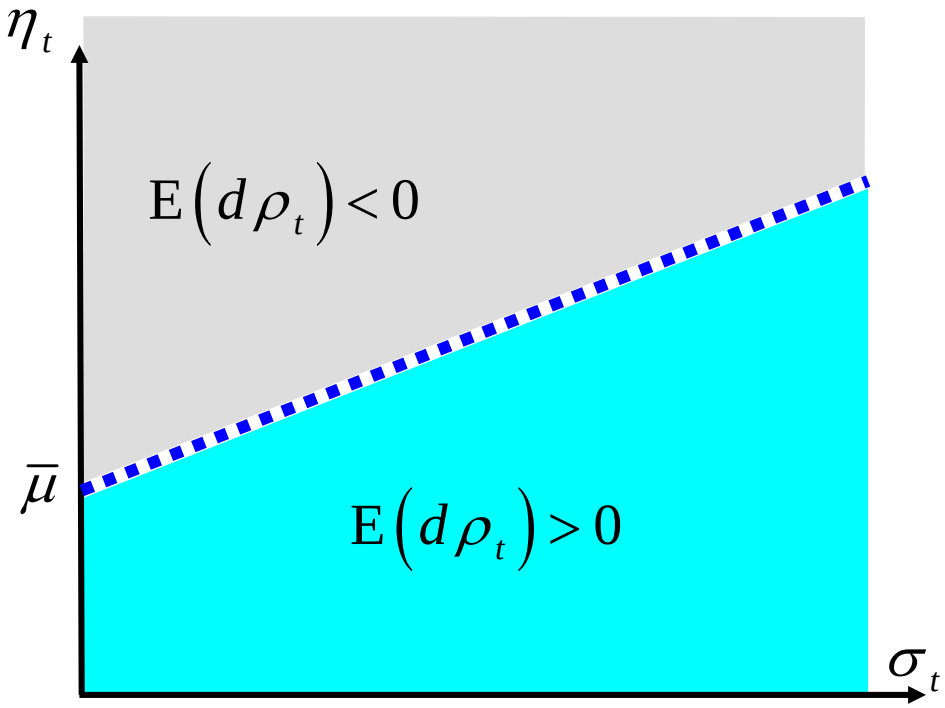
η_t

„declaration“
(asset participation)

Asset-Liability-Management

$$\rho_t = \rho_0 + \int_0^t \mu(\sigma_s) - \eta_s \, ds + \int_{[0,t]} \sigma_s \, dW_s$$

$$\Rightarrow \quad d\rho_t = \left(\mu(\sigma_t) - \eta_t\right) dt + \sigma_t \, dW_t$$



Overview

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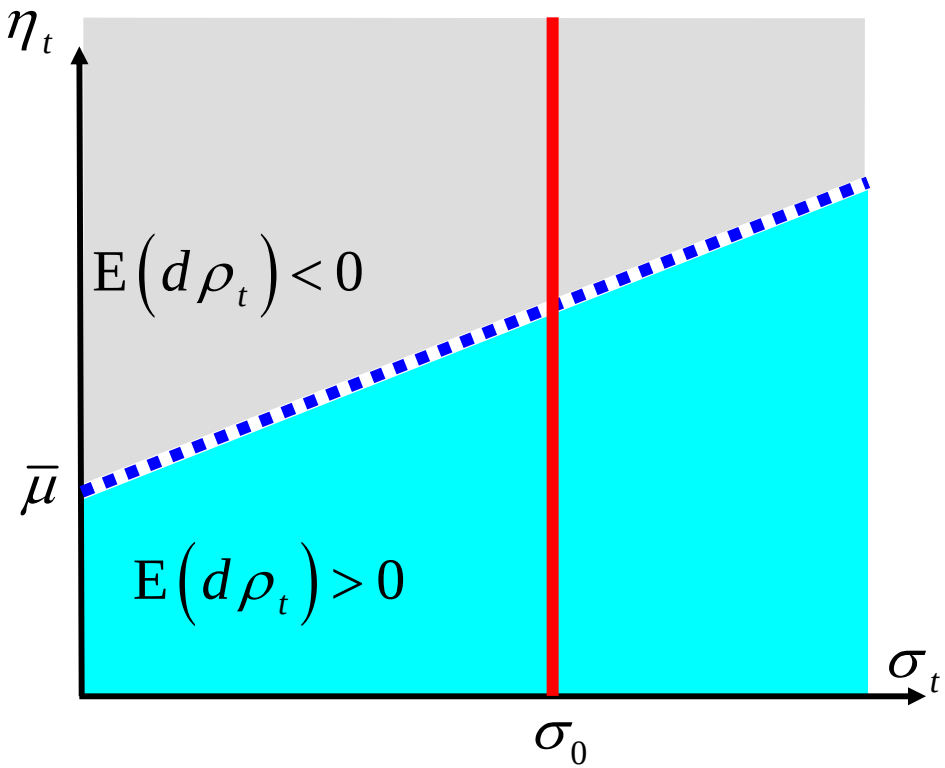
Constant mix strategy

for all $t \in [0,T]$ $\sigma_t := \sigma_0$ (const.)

$$\eta_t := \mu(\sigma_0) + \theta\left(\rho_t - \rho_{\text{target}}\right)$$



Constant mix strategy

for all $t \in [0,T]$ $\sigma_t := \sigma_0$ (const.) $\qquad \eta_t := \mu(\sigma_0) + \theta\left(\rho_t - \rho_{\text{target}}\right)$

$$\rho_t - \rho_{\text{target}} = \exp(-\theta t)\left(\rho_0 - \rho_{\text{target}} + \sigma_0 \int_{[0,t]} \exp(-\theta s)\, dW_s\right)$$

$$\eta_t = \mu(\sigma_0) + \theta \exp(-\theta t)\left(\rho_0 - \rho_{\text{target}} + \sigma_0 \int_{[0,t]} \exp(-\theta s)\, dW_s\right)$$

$$\mathrm{E}\left(\eta_t\right) = \mu(\sigma_0) + \theta \exp(-\theta t)\left(\rho_0 - \rho_{\text{target}}\right) \xrightarrow{t \to +\infty} \mu(\sigma_0)$$

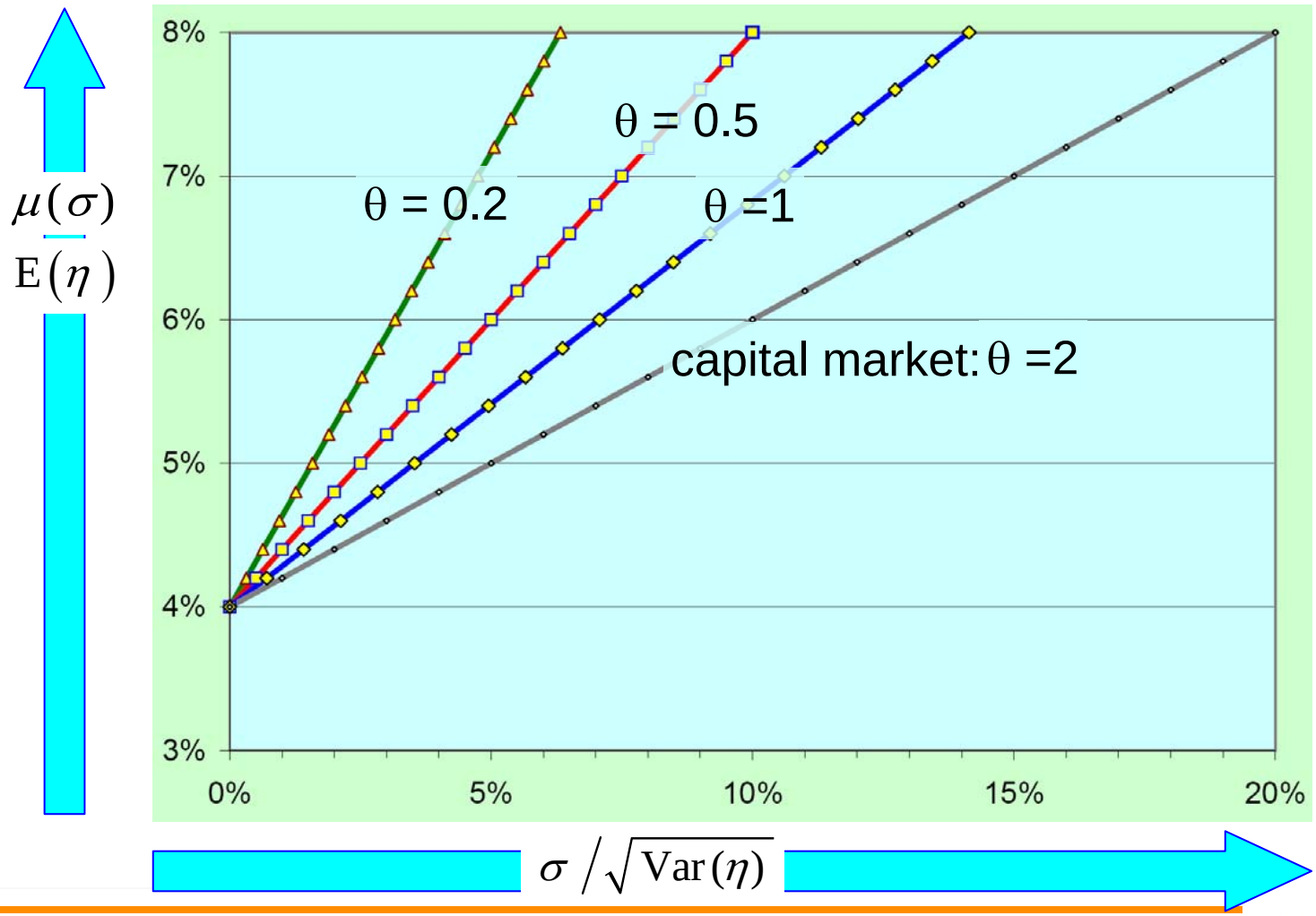
$$\mathrm{Var}\left(\eta_t\right) = \frac{\theta}{2}\sigma_0^2\left(1 - \exp(-2\theta t)\right) \xrightarrow{t \to +\infty} \frac{\theta}{2}\sigma_0^2$$

risk-return profile

$$\mathrm{E}\left(\eta_t\right) = \mu(\sigma_0) + \theta \exp(-\theta t)\left(\rho_0 - \rho_{\text{target}}\right) \approx \mu(\sigma_0)$$

$$\mathrm{Var}\left(\eta_t\right) = \frac{\theta}{2}\sigma_0^2\left(1 - \exp(-2\theta t)\right) \approx \frac{\theta}{2}\sigma_0^2$$

risk-return profile



Solvency condition

Given $\alpha > 0$, $\rho_{\min} > 0$ we stipulate: $\mathrm{P}\left(\rho_T < \rho_{\min}\right) \leq \alpha$

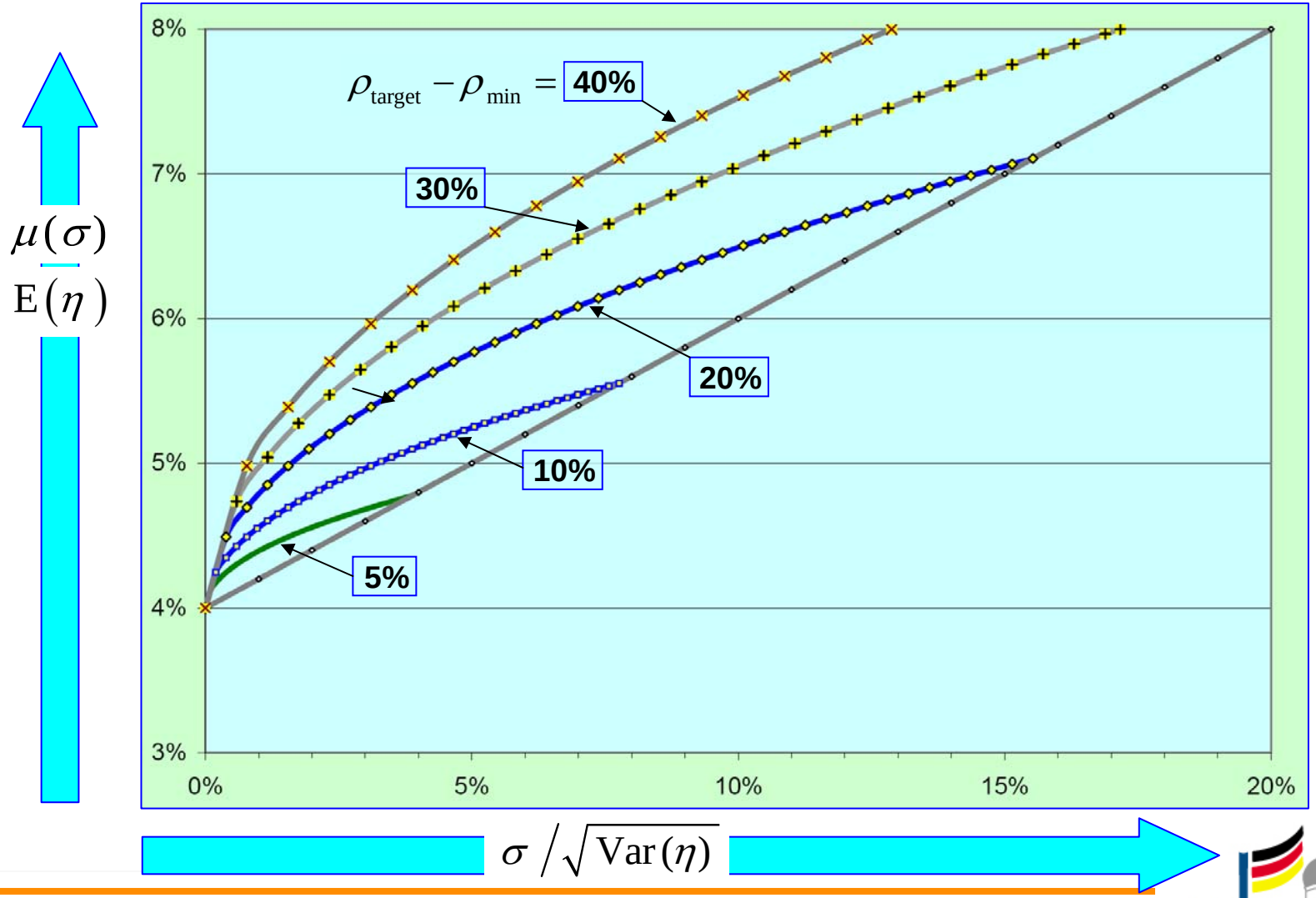
$$\sigma_0 \leq \frac{\rho_{\text{target}} - \rho_{\min} + \exp(-\theta T)\left(\rho_0 - \rho_{\text{target}}\right)}{u_\alpha \sqrt{\dfrac{1-\exp(-2\theta T)}{2\theta}}} \approx \sqrt{2\theta}\,\frac{\rho_{\text{target}} - \rho_{\min}}{u_\alpha}$$

$$\hat{\sigma}(\theta) := \sqrt{2\theta}\,\frac{\rho_{\text{target}} - \rho_{\min}}{u_\alpha}$$

"strategic risk exposure"

e.g. $\theta = 0.5; 1-\alpha = 99{,}5\% \Rightarrow \hat{\sigma} = \dfrac{\rho_{\text{target}} - \rho_{\min}}{2.5758}$

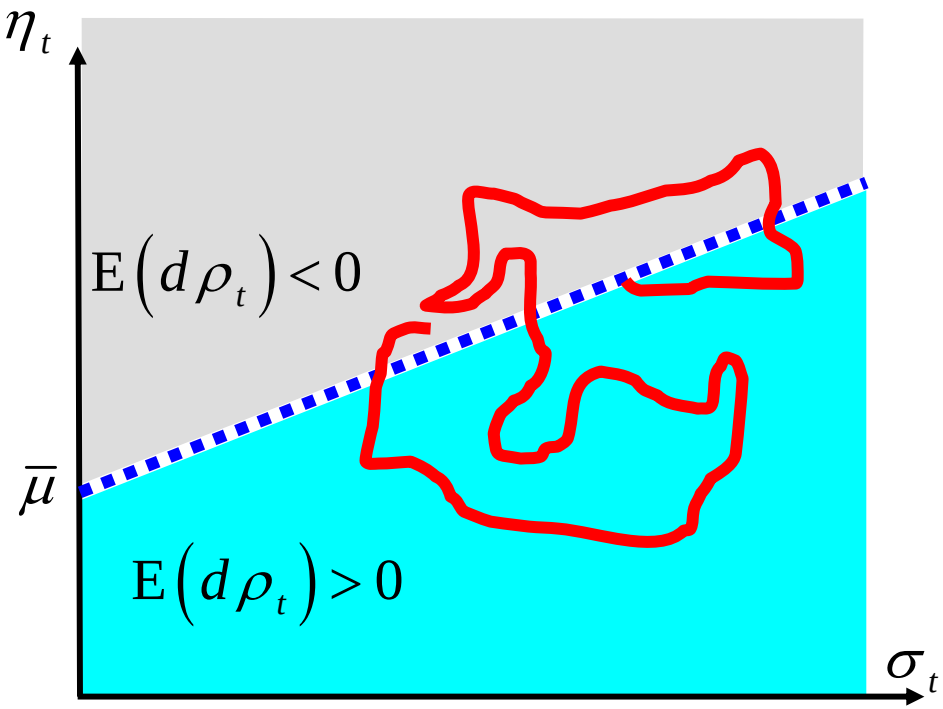
risk-return profile (“optimal” risk exposure)



ALM-Strategy

for all $t \in [0,T]$ $\eta_t := \mu(\sigma_t) + \theta\left(\rho_t - \rho_{\text{target}}\right)$

chosse σ_t such that: $\mathrm{P}\left(\rho_{t+1}^{t,\rho_t} < \rho_{\min}\right) = \alpha$



ALM-Strategie

$$\eta_t := \mu(\sigma_0) + \theta\left(\rho_t - \rho_{\text{target}}\right)$$

$$\sigma_t := \frac{1}{u_\alpha - r}\left(\bar{\mu} + \rho_t - \rho_{\min} - \eta_t\right)$$

Define: $a := \dfrac{1-\theta}{u_\alpha}$ $\quad \hat{\sigma} := \dfrac{\rho_{\text{target}} - \rho_{\min}}{u_\alpha}$ $\quad E_t := \exp\left(aW_t - \tfrac{1}{2}a^2\right)$

$$d\sigma_t = -\theta(\sigma_t - \hat{\sigma})\,dt + a\sigma_u\,dW_s$$

Solution: $\sigma_t = \exp(-\theta t)E_t\left(\sigma_0 + \theta\hat{\sigma}\int_0^t \frac{\exp(\theta u)}{E_u}\,du\right)$

ALM-Strategie

$$\mathrm{E}\left(\sigma_t\right) = \hat{\sigma} + \exp(-\theta t)(\sigma_0 - \hat{\sigma}) \xrightarrow{t \to +\infty} \hat{\sigma}$$

$$\mathrm{Var}\left(\sigma_t\right) = \hat{\sigma}^2 a^2 \frac{1 - \exp\left(-(2\theta - a^2)t\right)}{2\theta - a^2} + Function(\sigma_0 - \hat{\sigma})$$

$$\lim_{t \to \infty} \mathrm{Var}\left(\sigma_t\right) = \hat{\sigma}^2 \frac{a^2}{2\theta - a^2} \quad \text{iff} \quad 2\theta - a^2 > 0$$

ALM-Strategie

$$\mathrm{E}\left(\eta_t\right) = \mu(\hat{\sigma}) + \left(r + \frac{\theta}{a}\right)(\sigma_t - \hat{\sigma})$$

$$\mathrm{Var}\left(\eta_t\right) = (ar+\theta)^2 \frac{1-\exp\left(-(2\theta - a^2)t\right)}{2\theta - a^2}\hat{\sigma}^2 + Function(\sigma_0 - \hat{\sigma})$$

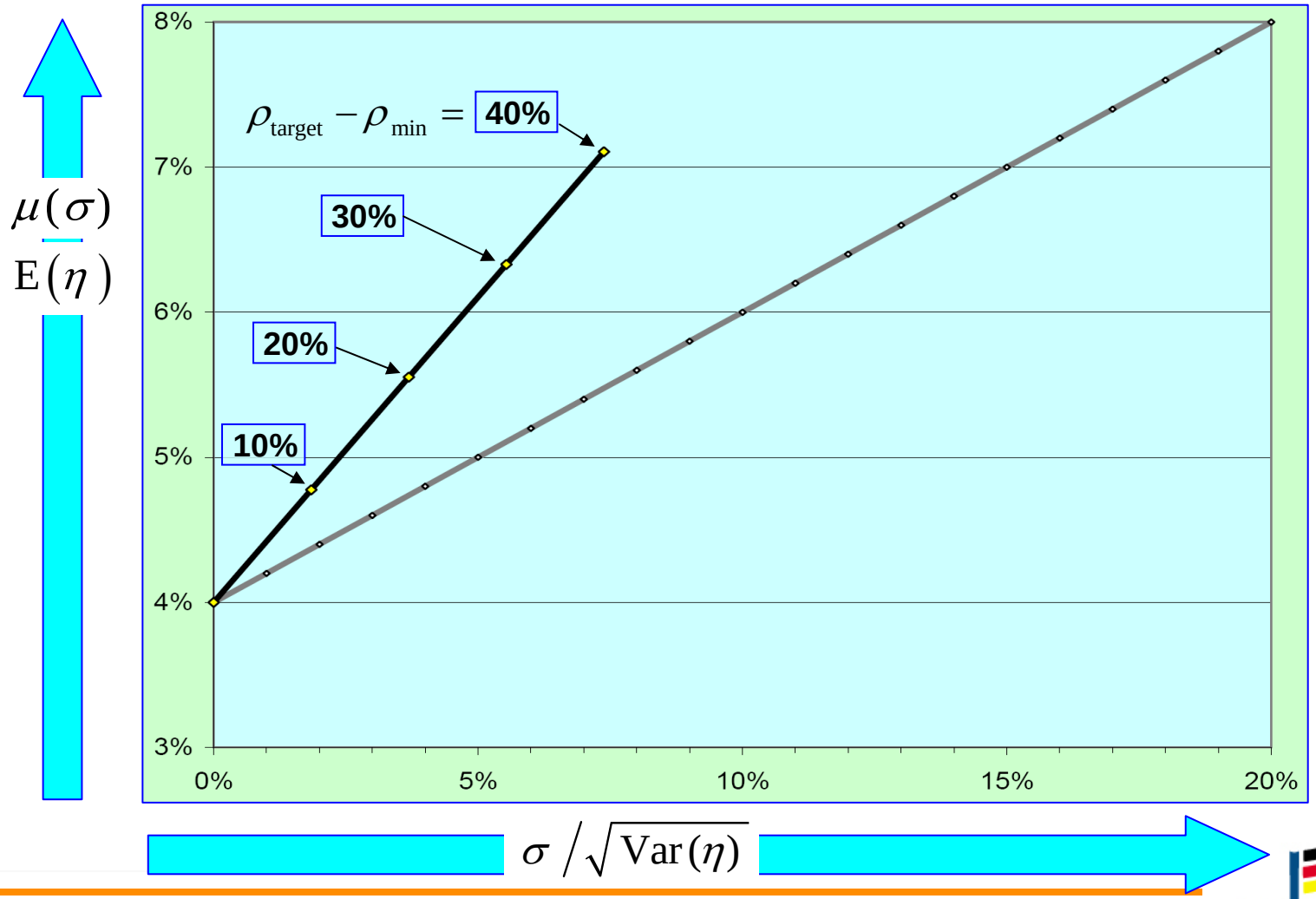
$$\lim_{t\to\infty} \mathrm{Var}\left(\eta_t\right) = \frac{(ar+\theta)^2}{2\theta - a^2}\hat{\sigma}^2 \quad \text{iff} \quad 2\theta - a^2 > 0$$

$$a = a(\theta) = \frac{1-\theta}{u_\alpha}$$

$$2\theta - a^2 > 0 \text{ iff } \left(1 + u_\alpha^2 + u_\alpha\sqrt{2+u_\alpha^2}\right)^{-1} < \theta < 1 + u_\alpha^2 + u_\alpha\sqrt{2+u_\alpha^2}$$

$$\min\left\{\frac{a(\theta)r+\theta}{\sqrt{2\theta - a(\theta)^2}}\hat{\sigma} : \theta \text{ feasible}\right\} \Rightarrow \theta_{\text{opt}} = \frac{1+ru_\alpha}{1-ru_\alpha+u_\alpha^2}$$

risk-return profile (“optimal” risk exposure)



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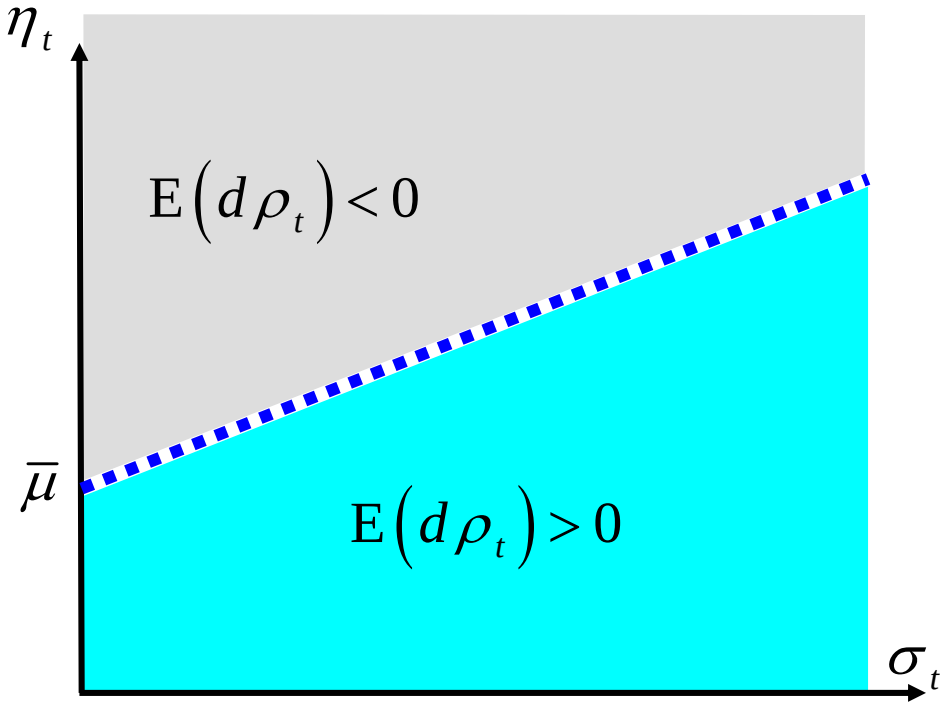
Asset-Liability-Management = Stochastic control

Optimization problem:

$$\max\left\{ U\left((\eta_t)_{0\le t\le T} \right) : \left(\sigma_t, \eta_t\right)_t \text{ is a feasible ALM-strategy subject to ....} \right\}$$



Value of the contract

Risk-neutral evaluation: $\mathrm{E}_Q\left(\exp(-\bar{\mu}T)V_T\right) = ?$

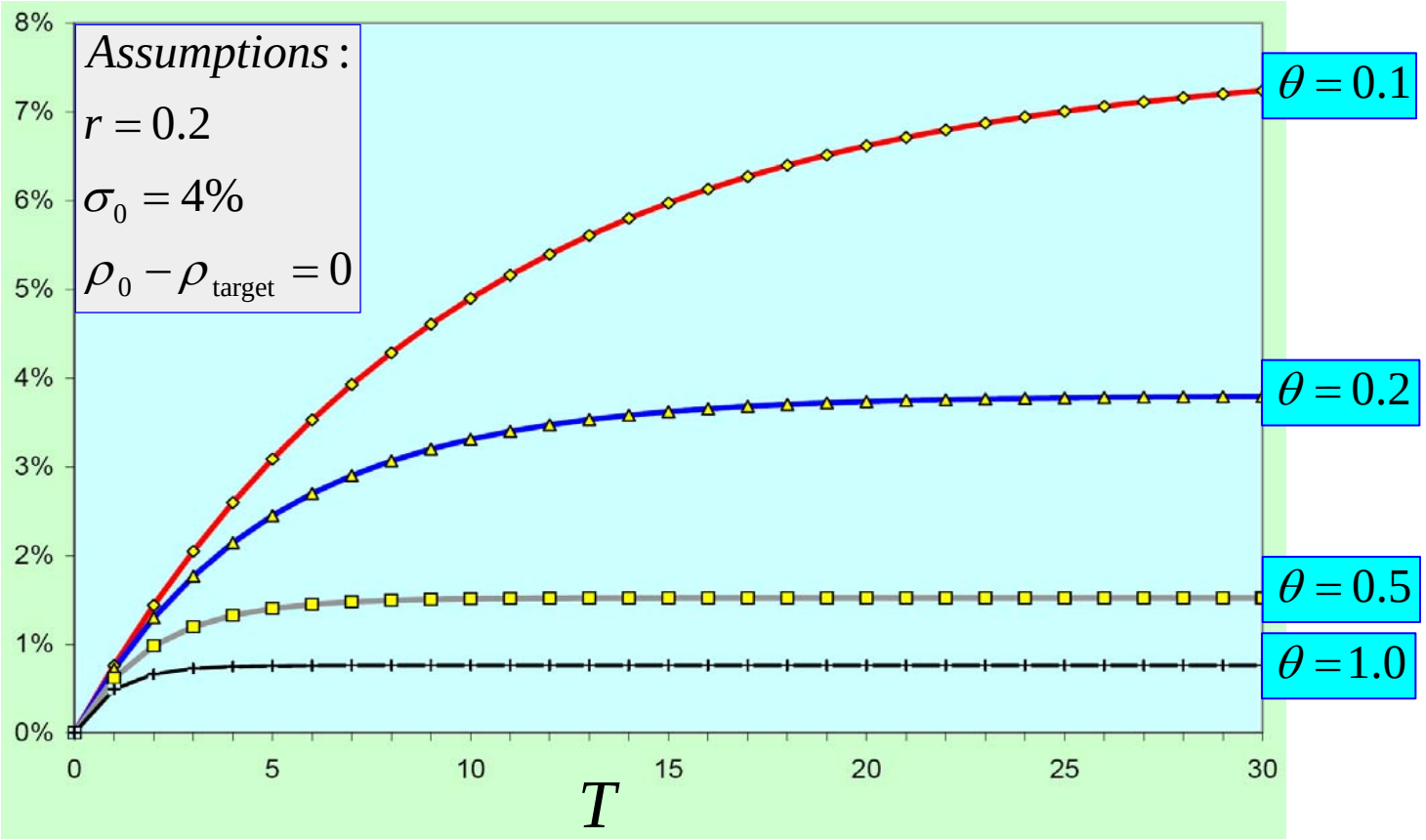
Const-Mix-Strategy:

$$\mathrm{E}_Q\left(\exp(-\bar{\mu}T)V_T\right) = V_0 \exp\left[(\rho_0 - \rho_{\text{target}})\left(1-\exp(-\theta T)\right) + F(\theta,\sigma_0,T)\right]$$

$$F(\theta,\sigma_0,T) := \frac{r\sigma_0 - \frac{1}{4}\sigma_0^2}{\theta} - \exp(-\theta T)\frac{r\sigma_0 - \frac{1}{2}\sigma_0^2 + \frac{1}{4}\sigma_0^2\exp(-\theta T)}{\theta}$$

Value of the contract

$F(\theta, \sigma_0, T)$



Résumé

Intergenerational risk sharing works!

One can measure the benefit!

Fund manager can communicate his/her strategy
➔ transparent product design

New way to calculate surrender values?

With-profit life contracts can do better!

Thank you for your kind attention!

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