AFIR LIFE MUNICH 2009

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ACTUARIAL ANALYSIS OF THE MULTIPLE LIFE ENDOWMENT INSURANCE CONTRACT

3rd International IAA Life Colloquium, Munich, Sept 6-9, 2009

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AGENDA

- **Modeling issues**
  - **general life status / single, joint, last-survivor & term**
  - **two-life status / survival probabilities**
- **Technical issues**
  - **premiums / term life / endowment / annuity**
  - **reserves / state dependent / state independent**
  - **premium components / state dependent / independent**
- **Computational issues**
  - **reduction formulas / term life / insurance / annuity**
  - **independence assumption / Höffding-Fréchet bounds**

Modeling / general life status

- **Random future lifetime of a general life status (u)**

consider group of g lives aged x(1), x(2), …, x(g)

T[k]=T[x(k)] : future lifetime of single life aged x(k)

T=T[u] : future lifetime of a status (u) on this group

p(t,u)=P(T[u]>t) : probability of survival to time t>0

q(t,u)=P(T[u]≤t) : probability of failure to time t>0

- single life status : u with T[u]=T[x] for single life aged x
- joint-life status : u with T[u]=min{T[1],…,T[g]}
- last survivorship : u with T[u]=max{T[1],…,T[g]}
- term certain : u with deterministic T[u]=n an integer

Modeling / two-life status

- **Survival probabilities**

  two lives aged x, y, with random age-at-deaths X, Y

  T[x]=X−x, T[y]=Y−y : random future lifetimes of (x), (y)

  S(x,y)=P(X>x,Y>y) : joint survival function of (X,Y)

- joint-life status : u=x:y with T[u]=min{T[x],T[y]}

  p(t,u)=P(T[u]>t) = S(x+t,y+t) / S(x,y)

- last survivorship : u'=(x:y)' with T[u']=max{T[x],T[y]}

  p(t,u')=P(T[u']>t) = {S(x+t,y) + S(x,y+t) – S(x+t,y+t)} / S(x,y)

- **independence : p(t,u) = p(t,x)·p(t,y)**
- **partial independence : p(t,u) = p(t,u') − p(t,x) − p(t,y)**

Technical / net single premiums

- **n-year term life insurance**

  D(m,u:n) : NSP for one unit of benefit payable at the end of the m-thly period of a year after failure of status u
- **n-year pure endowment**

  E(u:n) : NSP payable at survival of status u
- **n-year endowment**

  A(m,u:n) = D(m,u:n) + E(u:n) : NSP for status u
- **n-year life annuity**

  a(c,u:n) : NSP for one unit of benefit per year payable in installments of c fractional units at beginning of each payment cycle of length c as long as status u survives

Technical / level premiums

- **net level premium of n-year endowment (NLP)**

  NLP(m,c,u:n) = A(m,u:n)·SI / a(c,u:n) :

  NLP with benefit SI for status u
- **level premium of n-year endowment (LP)**

  acquisition costs : rate α of sum insured

  premium proportional operating costs : rate βv

  constant operating costs : fixed costs βf

  benefit proportional operating costs : rate γ

  LP with benefit SI for status u is determined by

  (1−βv)·LP(m,c,u:n) = NLP(m,c,u:n) + (α/a(c,u:n) + γ) + βf

Technical / reserves

- **random failure time of contract**

K[u] : curtate future lifetime, S[u]=T[u]−K[u] : fractional time of life in failure year, S[m,u]=m·int{S[u]/m+1} :

fractional portion S[u] rounded up to next m-th of a year

T[m,u]=K[u]+S[m,u] : moment of benefit payment by failure

- **random prospective loss of n-year endowment**

v=1/(1+i) : discount factor to technical interest rate i

random prospective loss at contract time t > 0 :

L(t;m,c,u:n) = v^min{T[m,u+t],n−t}·SI

− NLP(m,c,u:n)·a:(c,min{T[m,u+t],n−t}) with

a:(c,n)={1-v^n}/d(c), d(c)=i(c)/{1+c·i(c)}, i(c)={(1+i)^c−1}/c

Technical / reserves

- **states at time t>0 of couple (x,y) under mortality risk**

  X(t)=1 ⇔ (T[m,x]>t,T[m,y]>t) (x & y alive at t)

  X(t)=2 ⇔ (T[m,x]>t,T[m,y]≤t) (x alive & y dead at t)

  X(t)=3 ⇔ (T[m,x]≤t,T[m,y]>t) (x dead & y alive at t)

  X(t)=4 ⇔ (T[m,x]≤t,T[m,y]≤t) (x & y dead at t)

- **state dependent mathematical reserves**

  mathematical reserve in state X(t)=i at time t>0 =

  expected value of prospective loss conditional on state :

  V(t,i) = E[L(t;m,c,u:n)|X(t)=i]

- **joint-life status : V(t,1) ≠ 0, V(t,i) = 0 for i=2,3,4**
- **last survivorship : V(t,i) ≠ 0 for i=1,2,3, V(t,4) = 0**

Technical / reserves

- **state independent net premium reserve at time t>0**

  conditional expectation of prospective loss given survival :

  $V(t) = E[L(t;m,c,u:n)|T[m,u]>t] = \sum V(t,i)\cdot P(X(t)=i|T[m,u]>t)$
- **state dependent deferred acquisition costs (DAC)**

  $VE(t,i) = -\alpha\cdot\{SI - V(t,i)\}$, $i=1,2,\ldots$
- **state independent expense reserve**

  $VE(t) = -\alpha\cdot\{SI - V(t)\}$
- **state dependent actuarial reserve**
- **$VA(t,i) = V(t,i) + VE(t,i)$, $i=1,2,\ldots$**
- **state independent premium reserve**
- **$VA(t) = V(t) + VE(t)$**

Technical / premium components

- **state dependent components (special case m=c)**

  At the discrete times $t=k\cdot c$, $k=0,1,\ldots,n/c-1$, one has
- **saving premium**

  $SP(t,i) = v^c\cdot V(t+c,i) - V(t,i), \quad i=1,2,\ldots$
- **risk premium**

  $RP(t,i) = v^c\cdot q(c,u+t)\cdot\{SI-V(t+c,i)\}, \quad i=1,2,\ldots$

  $= NLP(c,c,u:n) - SP(t,i)$

  (saving premium + risk premium = net level premium)
- **expense premium**

  $EP(c,c,u:n) = LP(c,c,u:n) - NLP(c,c,u:n)$

  The expense premium splits into (similar net level premium)

Technical / premium components

- **risk component expense premium**

  $REP(t,i) = \alpha \cdot RP(t,i), \quad i=1,2,\ldots$

- **saving component expense premium**

  $SEP(t,i) = EP(c,c,u:n) - REP(t,i), \quad i=1,2,\ldots$

- **state independent components (similar decomposition)**

  saving premium : $SP(t) = v^c \cdot V(t+c) - V(t)$

  risk premium : $RP(t,i) = v^c \cdot q(c,u+t) \cdot \{SI - V(t+c)\}$

  $= NLP(c,c,u:n) - SP(t)$

  expense premium: $EP(c,c,u:n) = LP(c,c,u:n) - NLP(c,c,u:n)$

  risk component : $REP(t) = \alpha \cdot RP(t)$

  saving component: $SEP(t) = EP(c,c,u:n) - REP(t)$

Computational / reduction formulas

All technical values related to the multiple life endowment depend solely on functions A(m,u:n) and a(c,u:n). Under the assumption of uniform distribution of deaths (UDD) further reduction to D(u:n) and E(u:n):

- **n-year term life insurance**

  $D(m,u{:}n) = D(u{:}n)\cdot i/i(m)$ if $m>0$

  $D(m,u{:}n) = D(u{:}n)\cdot i/\delta$ if $m=0$, $\delta = \ln\{1+i\}$

- **n-year endowment**

  $A(m,u{:}n) = A(u{:}n) + \{i/i(m) - 1\}\cdot D(u{:}n)$

- **n-year life annuity**

  $a(c,u{:}n) = \{1 - A(u{:}n) - \{i/i(m)-1\}\cdot D(u{:}n)\}/d(c)$

Computational / independence

- **simplifiying assumption**

  tariff book is generated under independent future lifetimes:

  measure impact of assumption on actuarial calculations

- **Höffding - Fréchet upper bound**

  Fréchet class of **bivariate distributions F(s,t)** with fixed margins q(s,x)=P(T[x]≤s) & q(t,y)=P(T[y]≤t). Consider

  Fréchet upper bound : FU(s,t) = min{q(s,x),q(t,y)}

  Bivariate inequality : F(s,t) ≤ FU(s,t)

- **upper bound for joint-life survival distribution**

  pU(t,u)=min{p(t,x),p(t,y)}

Computational / independence

- **upper bound for last survivor survival distribution**
  pU(t,u')=max{p(t,x),p(t,y)}
- **joint-life survival distribution under independence**
  p(t,u) = p(t,x)·p(t,y)
- **last survivor survival distribution under independence**
  p(t,u') = p(t,x) + p(t,y) − p(t,u)
- **inequalities between survival distributions**
  pl(t,u) ≤ pU(t,u), pU(t,u') ≤ p(t,u')
  => random future lifetimes ordered in stochastic order
  => inequalities between NSP and NLP for status (u) & (u')
- **maximum deviations for two-lives endowment in Table:**

Computational / independence

interest i	male x	female y	term n	maximal deviations in per mill NSP(u)	NSP(u')	NLP(u)	NLP(u')
2%	30	30	10	0.4	-0.4	0.3	-0.3
			20	2.4	-2.4	0.5	-0.4
			30	7.4	-7.4	0.8	-0.7
			40	17.8	-17.8	1.3	-1.2
			50	34.5	-34.5	2.1	-1.9
2%	40	40	10	1.2	-1.2	0.7	-0.7
			20	6.4	-6.4	1.2	-1.2
			30	18.5	-18.5	2.1	-1.9
			40	38.5	-38.5	3.3	-2.8
			50	56.7	-56.7	4.5	-3.6
2%	50	50	10	3.2	-3.2	2.0	-2.0
			20	15.9	-15.9	3.4	-3.1
			30	39.6	-39.6	5.5	-4.5
			40	61.9	-61.9	7.5	-5.6
			50	66.9	-66.9	8.0	-5.9
2%	60	60	10	8.1	-8.1	5.7	-5.2
			20	34.2	-34.2	9.6	-7.5
			30	61.7	-61.7	13.5	-9.3
			40	68.1	-68.1	14.6	-9.7
			50	68.2	-68.2	14.6	-9.7

- technical interest : 2%
- Life Table: Gompertz survival distribution
- **Stress testing results:**

  1) joint life status **over-estimates NSP & NLP**

  2) last survivorship status **underestimates NSP & NLP**