

**STOCHASTIC MORTALITY:
EXPERIENCE-BASED MODELING
AND
APPLICATION ISSUES CONSISTENT WITH
SOLVENCY 2**

ANNAMARIA OLIVIERI

University of Parma (Italy)

`annamaria.olivieri@unipr.it`

IAA Life Colloquium 2009 – Munich

September 8, 2009

Introduction

Aim: to perform a stochastic assessment of mortality risks (aggregate mortality risk included), calibrating the stochastic mortality model to market or portfolio experience

Reference is to a life annuity portfolio

Mortality risks only are focused

- ↳ risk of random fluctuations
- ↳ longevity risk

other risks (i.e.: market, operational, and so on) are disregarded

Application: within an internal model, for the assessment of the capital required to face the specific risks of an insurance portfolio

The traditional actuarial tool-kit for the valuation of mortality in a life annuity portfolio is based on the best estimate life table

a projected life table



it provides a deterministic representation of future mortality

but: a stochastic approach is required for risk modeling

constructed by some independent institution (thus, reflecting some general experience)

validation issues vs the need for a risk assessment consistent with the specific risk profile

The idea: to extend some classical results about the modeling of the number of deaths joint to the modeling of parameter uncertainty. Calibration based on the best estimate life table and, through inference, on portfolio experience

What follows is based on joint work with Ermanno Pitacco, University of Trieste (Italy)

Basic assumptions, notation, terminology

Portfolio of life annuities:

annuities are immediate, in arrears and with fixed benefits

Time of issue of the portfolio: t_0

Entry age: x_0 ; maximum age: ω (known)

Annual amount: b

Annual payout: $b N_t$

↪ number of annuitants at time t

Number of annuitants

		time (since issue)				
		1	2	...	t	...
current age	x_0					
	⋮					
	x				$N_{x,t}$	
	⋮					
	ω					
total # annuitants		N_1	N_2	...	N_t	...

$$N_t = N_{t-1} - D_t$$

Number of deaths

		time (since issue)				
		1	2	...	t	...
current age	x_0					
	⋮					
	x				$D_{x,t}$	
	⋮					
	ω					
total # deaths		D_1	D_2	...	D_t	...

Basic assumptions, notation, terminology (*cont*)

The annual number of deaths is random because of
random fluctuations
systematic deviations

Random fluctuations

If the size of the portfolio is large enough, then with high probability

random number of deaths

mortality rate: $\frac{D_{x,t}}{n_{x,t-1}} \approx q_{x,t}^*$: best estimate (BE) mortality rate

number of annuitants (observed),
beginning of the year

Due to the actual size of the portfolio: $\frac{D_{x,t}}{n_{x,t-1}} \begin{matrix} \geq \\ < \end{matrix} q_{x,t}^*$

Basic assumptions, notation, terminology (*cont*)

Representation

One cohort

Assuming independence among the lifetimes (conditional on the BE mortality rate):

$$[D_{x,t} | q_{x,t}^*; n_{x,t-1}] \sim \text{Bin}(n_{x,t-1}, q_{x,t}^*)$$

Possibly approximated as: $[D_{x,t} | q_{x,t}^*; n_{x,t-1}] \sim \text{Poi}(n_{x,t-1} q_{x,t}^*)$

More than one cohort

If we accept the Poisson approximation (and independence among cohorts, conditional on the BE life table):

$$[D_t | \{q_{x,t}^*\}; \{n_{x,t-1}\}] \sim \text{Poi}(\sum_{x=x_0}^{\omega} n_{x,t-1} q_{x,t}^*)$$

Systematic deviations

High probability that $\frac{D_{x,t}}{n_{x,t-1}}$ is not close to $q_{x,t}^*$ also in very large portfolios \Rightarrow *deviation in aggregate mortality*

Representation: random mortality rate, $Q_{x,t}$

The deviation in aggregate mortality can be ...

... temporary

Typically an upward shock, reasonably independent of previous ones

... **permanent**

The underlying trend, for the whole population or for some cohorts, is other than what described by the $q_{x,t}^*$'s

Reasonably, *deviations are (positively) correlated in time*

The mortality rate

Refer to one cohort only

Two approaches (at least)

1. model the mortality rate $Q_{x,t}$ directly
2. model an adjustment to the BE mortality rate

Example of a **direct modeling of the mortality rate**

Let $Q_{x,t} \sim \text{Beta}(a_{x,t}, b_{x,t}) \Rightarrow 0 \leq Q_{x,t} \leq 1$

Setting $Q_{x,t} = q$, we take: $[D_{x,t} | q; n_{x,t-1}] \sim \text{Bin}(n_{x,t-1}, q)$

Then: $[D_{x,t} | n_{x,t-1}] \sim \text{Beta-Binomial law}$

with age- and time-dependent parameters

Example of a stochastic adjustment to the BE mortality rate

Take the multiplicative model: $Q_{x,t} = q_{x,t}^* Z_{x,t}$

where $Z_{x,t}$ ($Z_{x,t} > 0$, but such that $0 \leq Q_{x,t} \leq 1$) is a (random) coefficient expressing deviations in aggregate mortality

A particular choice:

$$Z_{x,t} \sim \text{Gamma}(\alpha_{x,t}, \beta_{x,t}) \Rightarrow Q_{x,t} \sim \text{Gamma}\left(\alpha_{x,t}, \frac{\beta_{x,t}}{q_{x,t}^*}\right)$$

Setting $Q_{x,t} = q$ we take: $[D_{x,t} | q; n_{x,t-1}] \sim \text{Poi}(n_{x,t-1} q)$

Then: $[D_{x,t} | n_{x,t-1}] \sim \text{NBin}\left(\alpha_{x,t}, \frac{\theta_{x,t}}{\theta_{x,t} + 1}\right)$

$$\theta_{x,t} = \frac{\beta_{x,t}}{n_{x,t-1} q_{x,t}^*}$$

Beta-Binomial vs Poisson-Gamma model

Against the Poisson-Gamma model

approximation at old ages (from Poisson assumption) \rightsquigarrow possibly of poor impact when more than one cohort are referred to the range of possible values of the mortality rate (Gamma distribution) \rightsquigarrow possibly negligible when multiple cohorts are addressed

Advantages of the Poisson-Gamma model

generalizations to the case of more than one cohort are straightforward

correlation in time among the mortality rates can be modeled naturally, without explicit assumptions

possible extension: accounting for the rate of decrease of the annual payout (rather than simply the mortality rate)

\rightsquigarrow *In the following: Poisson-Gamma model*

The Poisson-Gamma model

For one cohort only

Mortality rate: $Q_{x,t} = q_{x,t}^* Z_{x,t}$

$$Z_{x,t} \sim \text{Gamma}(\alpha_{x,t}, \beta_{x,t}) \Rightarrow Q_{x,t} \sim \text{Gamma}\left(\alpha_{x,t}, \frac{\beta_{x,t}}{q_{x,t}^*}\right)$$

Expected mortality rate: $\mathbb{E}[Q_{x,t}] = \underbrace{\frac{\alpha_{x,t}}{\beta_{x,t}}}_{\text{magnitude of the systematic deviation}} q_{x,t}^*$

magnitude of the
systematic deviation

Coefficient of variation: $\text{CV}[Q_{x,t}] = \frac{\sqrt{\text{Var}[Q_{x,t}]}}{\mathbb{E}[Q_{x,t}]} = \frac{1}{\sqrt{\alpha_{x,t}}}$

↪ Idea: set (initially) $\frac{\alpha_{x,t}}{\beta_{x,t}}$ according to a benchmark for the magnitude of systematic deviations (e.g.: $\frac{\alpha_{x,t}}{\beta_{x,t}} = 1$ for portfolio valuation; $\frac{\alpha_{x,t}}{\beta_{x,t}} < 1$ for capital allocation), with $\alpha_{x,t}$ consistent with data on volatility (if available)

The Poisson-Gamma model (*cont*)

Number of deaths

disregarding deviations in aggregate mortality

$$[D_{x,t} | q_{x,t}^*; n_{x,t-1}] \sim \text{Poi}(n_{x,t-1} q_{x,t}^*)$$

allowing for deviations in aggregate mortality

$$[D_{x,t} | n_{x,t-1}] \sim \text{NBin}\left(\alpha_{x,t}, \underbrace{\frac{\theta_{x,t}}{\theta_{x,t} + 1}}_{\frac{\beta_{x,t}}{n_{x,t-1} q_{x,t}^*}}\right)$$

Expected number of deaths

disregarding deviations in aggregate mortality

$$\mathbb{E}[D_{x,t} | q_{x,t}^*; n_{x,t-1}] = n_{x,t-1} q_{x,t}^*$$

allowing for deviations in aggregate mortality

$$\mathbb{E}[D_{x,t} | n_{x,t-1}] = \underbrace{\frac{\alpha_{x,t}}{\beta_{x,t}}}_{\text{magnitude of the systematic deviation (over the cohort)}} n_{x,t-1} q_{x,t}^*$$

magnitude of the systematic deviation (over the cohort)

The Poisson-Gamma model (*cont*)

More than one cohort

We let: $[D_t | \{q_{x,t}^*\}; \{n_{x,t-1}\}] \sim \text{Poi}(\sum_{x=x_0}^{\omega} n_{x,t-1} q_{x,t}^*)$

We assume: $Q_{x,t} = q_{x,t}^* Z_t$ for any age x

REMARK: the adjustment coefficient Z_t is common to all the cohorts in-force at time $t \Rightarrow$ it accounts for period effects only; cohort effects are missed

Then: $[D_t | \{z q_{x,t}^*\}; \{n_{x,t-1}\}] \sim \text{Poi}(z \sum_{x=x_0}^{\omega} n_{x,t-1} q_{x,t}^*)$

Taking $Z_t \sim \text{Gamma}(\alpha_t, \beta_t)$ we have

$$[D_t | \{n_{x,t-1}\}] \sim \text{NBin} \left(\alpha_t, \frac{\theta_t}{\theta_t + 1} \right)$$

$$\theta_t = \frac{\beta_t}{\sum_{x=x_0}^{\omega} n_{x,t-1} q_{x,t}^*}$$

The Poisson-Gamma model (*cont*)

Expected number of deaths

disregarding deviations in aggregate mortality

allowing for deviations in aggregate mortality

$$\mathbb{E}[D_t | \{q_{x,t}^*\}; \{n_{x,t-1}\}] = \sum_{x=x_0}^{\omega} n_{x,t-1} q_{x,t}^*$$

$$\mathbb{E}[D_{x,t} | \{n_{x,t-1}\}] = \underbrace{\frac{\alpha_t}{\beta_t}}_{\text{magnitude of the systematic deviation (over the whole population)}} \sum_{x=x_0}^{\omega} n_{x,t-1} q_{x,t}^*$$

magnitude of the systematic deviation (over the whole population)

Accounting for correlation in time among mortality rates: updating parameters to experience

Assumption: the $Q_{x,t}$'s are correlated in time

Further assumption: the mortality experience from the portfolio is reliable as an evidence of the trend of the cohort (or the population)

⇒ *An inferential procedure can be defined for updating the parameters of the pdf of $Q_{x,t}$ (or the parameters of the number of deaths) to experience*

Updating parameters to experience (*cont*)

Reference to one cohort

Valuation at time 0

Issue time; no previous experience available

$n_{x_0,0}$ annuitants at time 0

$$Z_{x,t} \sim \text{Gamma}(\bar{\alpha}, \bar{\beta}) \quad \text{for all times } t \text{ (and ages } x = x_0 + t)$$

$$D_{x_0,1} \sim \text{NBin} \left(\alpha_{x_0,1}, \frac{\theta_{x_0,1}}{\theta_{x_0,1} + 1} \right)$$

$$\alpha_{x_0,1} = \bar{\alpha}$$

$$\theta_{x_0,1} = \frac{\bar{\beta}}{n_{x_0,0} q_{x_0,1}^*}$$

Updating parameters to experience (*cont*)

Valuation at time 1

Let $D_{x_0,1} = d_{x_0,1}$ be the observed number of deaths in $(0, 1)$

Then $n_{x_0+1,1} = n_{x_0,0} - d_{x_0,1}$

We can calculate the posterior pdf of $Q_{x_0,1}$, conditional on $D_{x_0,1} = d_{x_0,1}$. It turns out

$$[Q_{x_0,1} | D_{x_0,1} = d_{x_0,1}] \sim \text{Gamma} \left(\bar{\alpha} + d_{x_0,1}, \frac{\bar{\beta}}{q_{x_0,1}^*} + n_{x_0,0} \right)$$

and hence:

$$[Z_{x,t} | D_{x_0,1} = d_{x_0,1}] \sim \text{Gamma}(\bar{\alpha} + d_{x_0,1}, \bar{\beta} + n_{x_0,0} q_{x_0,1}^*)$$

Updating parameters to experience (cont)

Then we have

$$[D_{x_0+1,2} | n_{x_0,0}, d_{x_0,1}] \sim \text{NBin} \left(\alpha_{x_0+1,2}, \frac{\theta_{x_0+1,2}}{\theta_{x_0+1,2} + 1} \right)$$

$$\alpha_{x_0+1,2} = \bar{\alpha} + d_{x_0,1} \quad \theta_2 = \frac{\bar{\beta} + n_{x_0,0} q_{x_0,1}^*}{n_{x_0+1,1} q_{x_0+1,2}^*}$$

For the expected number of deaths, we have

$$\mathbb{E}[D_{x_0+1,2} | n_{x_0,0}, d_{x_0,1}] = \frac{\bar{\alpha} + d_{x_0,1}}{\underbrace{\bar{\beta} + n_{x_0,0} q_{x_0,1}^*}_{\text{updated (previously: } \frac{\bar{\alpha}}{\beta})}} n_{x_0+1,1} q_{x_0+1,2}^*$$

Depending on experience:

$$\frac{\bar{\alpha} + d_{x_0,1}}{\bar{\beta} + n_{x_0+0,0} q_{x_0,1}^*} \begin{matrix} \geq \\ < \end{matrix} \frac{\bar{\alpha}}{\beta} \Rightarrow \mathbb{E}[D_{x_0+1,2} | n_{x_0,0}, d_{x_0,1}] \begin{matrix} \geq \\ \leq \end{matrix} \mathbb{E}[D_{x_0+1,2} | n_{x_0,0}]$$

Valuation at time t : ... (similar results follow)

Updating parameters to experience (*cont*)

Reference to more than one cohort

We accept the assumptions

$$Q_{x,t} = q_{x,t}^* Z_t$$

$$[D_t | \{z q_{x,t}^*\}; \{n_{x,t-1}\}] \sim \text{Poi}(z \sum_{x=x_0}^{\omega} n_{x,t-1} q_{x,t}^*)$$

$$\text{at time 0: } Z_t \sim \text{Gamma}(\bar{\alpha}, \bar{\beta}) \quad \text{for all } t$$

An **inferential procedure** similar to the case of one cohort can be defined, with the advantage of relying on a wider data set (i.e. on the number of deaths observed in the population). In particular

Updating parameters to experience (cont)

At time 0

$$[D_1 | \{n_{x,0}\}] \sim \text{NBin} \left(\alpha_1, \frac{\theta_1}{\theta_1 + 1} \right)$$

$$\alpha_1 = \bar{\alpha} \qquad \theta_1 = \frac{\bar{\beta}}{\sum_{x=x_0}^{\omega} n_{x,0} q_{x_0,1}^*}$$

At time 1

Let $D_1 = \underbrace{d_1}_{\sum_{x=x_0}^{\omega} d_{x,1}}$ be the observed number of deaths in $(0, 1)$

Then

$$[D_2 | \{n_{x,0}\}; D_1 = d_1] \sim \text{NBin} \left(\alpha_2, \frac{\theta_2}{\theta_2 + 1} \right)$$

$$\alpha_2 = \bar{\alpha} + d_1 \qquad \theta_2 = \frac{\bar{\beta} + \sum_{x=x_0}^{\omega} n_{x,0} q_{x,1}^*}{\sum_{x=x_0}^{\omega} n_{x,1} q_{x_0,2}^*}$$

At time t : ... (similar results follow)

Application

The mortality model is experience-based

⇒ suitable for internal models

Particular application: **capital allocation**

Implementation through a

deterministic . . .

the expected numbers of deaths

(or the expected mortality rates) only are involved

stochastic . . .

the numbers of deaths are stochastically
simulated

deterministic via stochastic . . .

short-cut formulae, constructed according to
simulated findings

. . . approach

Deterministic implementation

Refer to the Solvency 2 standard formula

Capital charge for longevity risk (in terms of the SCR – Solvency Capital Requirement): change in the net value of assets minus liabilities (ΔNAV) against a permanent 25% decrease in mortality rates for each age

Under our assumptions, this reduces to

$$\text{Life}_{\text{long},t} = V_t^{(\Pi)[-25\%]} - V_t^{(\Pi)[\text{BE}]}$$

expected present value of future payments in portfolio II, under the shock assumption (BE – 25%)

expected present value of future payments in portfolio II, under BE assumptions (or: portfolio reserve, net of the risk margin)

At time $t = 0$: shock scenario -25%

$$\underbrace{\mathbb{E}[Z_{x,t}]}_{\text{one cohort}} = \underbrace{\mathbb{E}[Z_t]}_{\text{multiple cohorts}} = \frac{\bar{\alpha}}{\bar{\beta}} = 0.75$$

set $\bar{\alpha}$ according to data on trend volatility, if available, or experts' opinion

At time $t = 1$: shock scenario updated to experience

$$\alpha_1 = \bar{\alpha} \rightarrow \alpha_2 = \bar{\alpha} + \begin{cases} d_{x_0,1} & \text{one cohort} \\ d_1 = \sum_x d_{x,1} & \text{multiple cohorts} \end{cases}$$

$$\beta_1 = \bar{\beta} \rightarrow \beta_2 = \bar{\beta} + \begin{cases} n_{x_0,0} q_{x_0,1}^* & \text{one cohort} \\ \sum_x n_{x,0} q_{x,1}^* & \text{multiple cohorts} \end{cases}$$

$$\Rightarrow \text{shock scenario: } 1 - \begin{cases} \mathbb{E}[Z_{x,t} | d_{x_0,1}] & = 1 - \frac{\alpha_{x_0+1,2}}{\beta_{x_0+1,2}} \\ \mathbb{E}[Z_t | d_1] & = 1 - \frac{\alpha_2}{\beta_2} \end{cases}$$

At time $t \dots$

Example:

One cohort; initial age: $x_0 = 65$; males

Best estimate life table: IPS55 (projected life table for Italian males, cohort 1955)

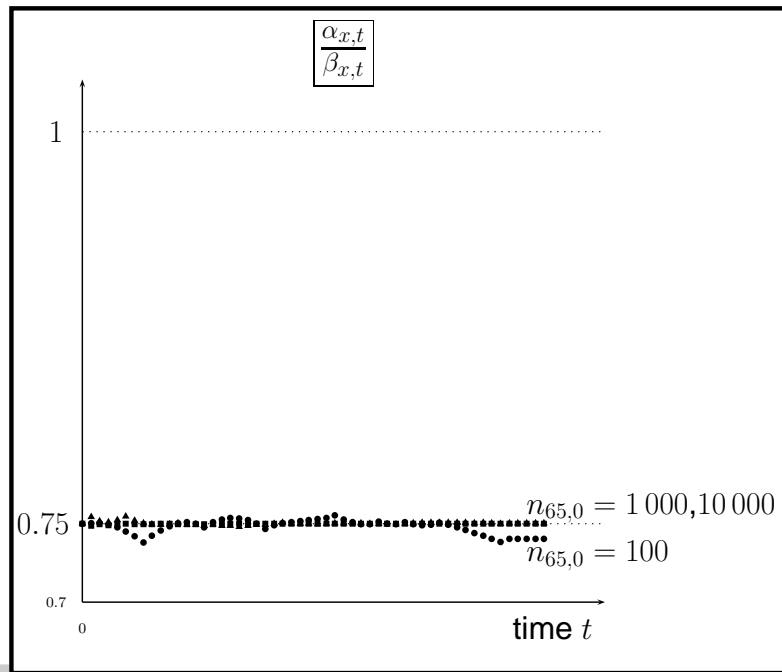
(Initial) parameters of the pdf of $Z_{x,t}$: $\bar{\alpha} = 0.75\bar{\beta}$, $\bar{\beta} = 100$, so that

$$\text{CV}[Q_{x,t}] = \frac{\sqrt{\text{Var}[Q_{x,t}]}}{\mathbb{E}[Q_{x,t}]} = 11.52\%$$

Experienced mortality

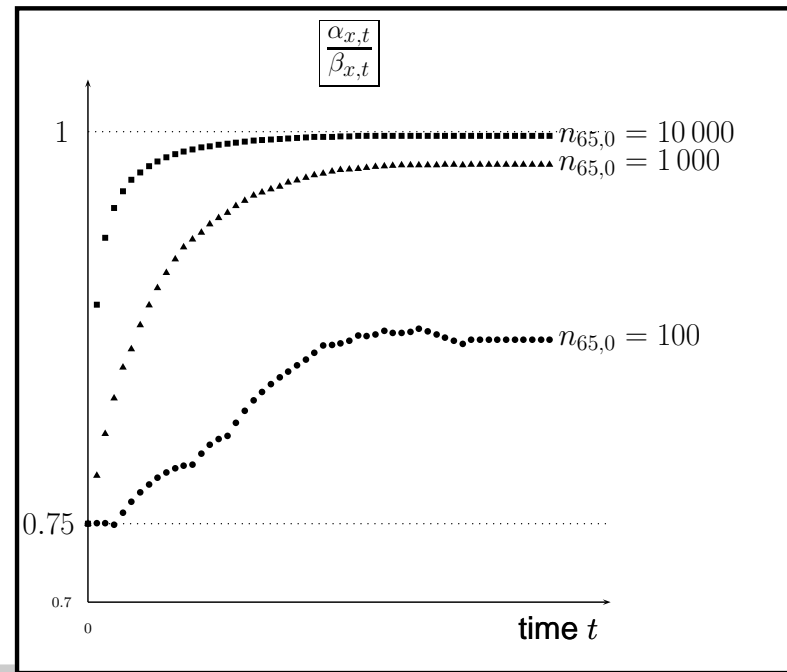
as the shock scenario

(# deaths $\simeq 75\%$ of what expected)



as the BE scenario

(# deaths $\simeq 100\%$ of what expected)



Stochastic implementation: a possible rule for an internal model

Reference to one cohort only

Let A_t be the amount of portfolio assets at time t

$$A_t = A_{t-1} (1 + i) - b N_{x_0+t,t} \quad (t = z + 1, z + 2, \dots)$$

with A_z given at the valuation time z and i the investment yield (assumed to be the risk-free rate)

Then

$$M_t = A_t - V_t^{(\Pi)[BE]}$$

portfolio reserve, net of
the risk margin

represents the assets available to meet risks (to be split into risk margin and required capital)

A reasonable solvency rule: set M_z such that

$$[R1] \quad \mathbb{P}[(M_{z+1} \geq 0) \cap (M_{z+2} \geq 0) \cap \dots \cap (M_{z+T} \geq 0) | n_{x_0+z,z}] = 1 - \varepsilon$$

where

ε accepted default probability

T time-horizon for solvency ascertainment

Requirement [R1] needs a stochastic model, and stochastic simulations

advantage: the capital required is consistent with risks

disadvantage: stochastic simulations are time-consuming \Rightarrow
possibly, look for short-cut formulae, consistent with the output of
the stochastic model

Example

One cohort; initial age: $x_0 = 65$; males

Best estimate life table: IPS55 (projected life table for Italian males, cohort 1955)

Maximum age: $\omega = 119$, whence the maturity of the portfolio at time z is:

$$m = 119 - 65 - z$$

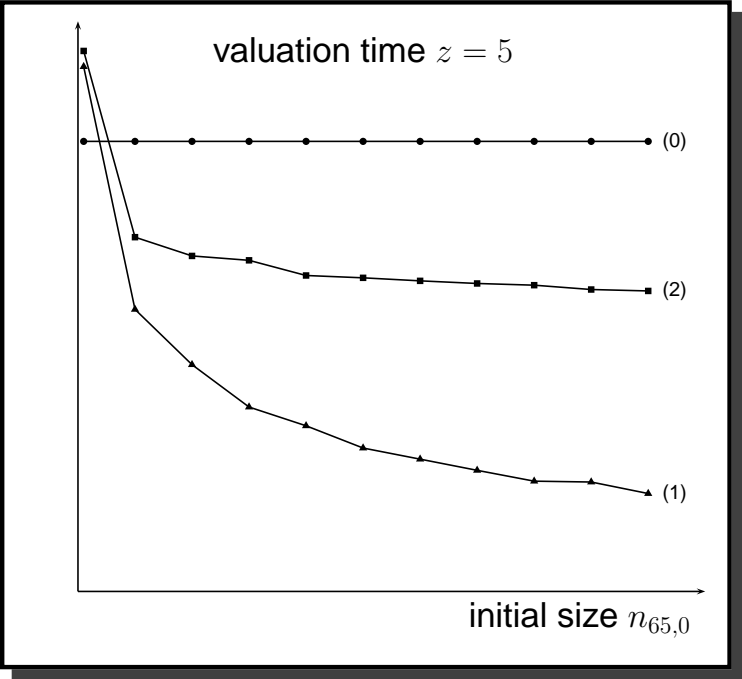
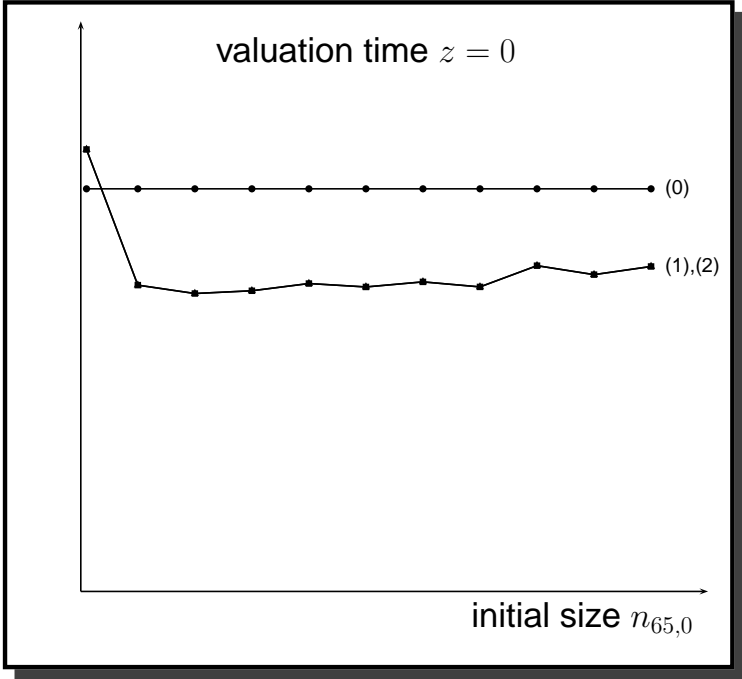
Initial parameters of the pdf of $Z_{x,t}$: $\bar{\alpha} = 0.75\bar{\beta}$, $\bar{\beta} = 100$

Risk-free rate and investment yield: 3% p.a.

Annual amount: $b = 1$

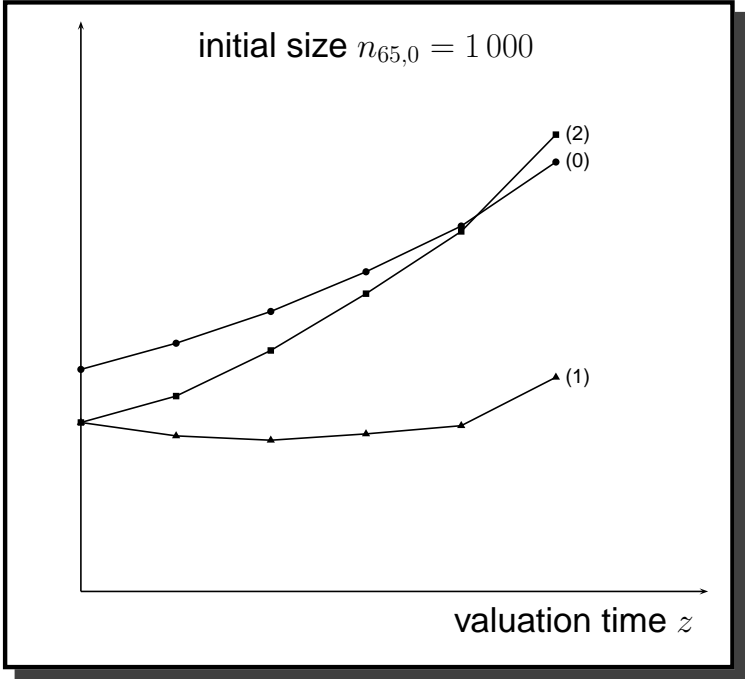
Internal model with default probability: $\varepsilon = 0.005$, time-horizon: $T = m$ (for consistency with Solvency 2)

Application (cont)



- (0) Solvency 2: $\frac{M_z^{[Solv2]}}{V_z^{(II)[BE]}} = \frac{Life_{long,z} + RM_z}{V_z^{(II)[BE]}}$
- (1)–(2) Rule [R1], with $T = m$: $\frac{M_z^{[R1]}}{V_z^{(II)[BE]}}$
 - (1) with updated parameters, experience as the best estimate life table
 - (2) with updated parameters, experience as the Solvency 2 shock scenario (BE–25%)

Application (cont)



- (0) Solvency 2: $\frac{M_z^{[Solv2]}}{V_z^{(\Pi)[BE]}} = \frac{Life_{long,z} + RM_z}{V_z^{(\Pi)[BE]}}$
- (1)–(2) Rule [R1], with $T = m$: $\frac{M_z^{[R1]}}{V_z^{(\Pi)[BE]}}$
- (1) with updated parameters, experience as the best estimate life table
- (2) with updated parameters, experience as the Solvency 2 stress scenario (BE–25%)

Concluding remarks

The mortality models described can be useful within an internal valuation model allowing for mortality risks

Even though the insurer does not have the expertise to deal with the methodologies underlying the best estimate table or does not have access to the relevant data set, a simple structure may lead to a satisfactory assessment of the impact of mortality risks, including both random fluctuations and systematic deviations

Statistical tests required, especially for setting the initial volatility parameter

Possible extensions (for example: rate of reduction in the total payout instead of the number of deaths, in order to account for adverse-selection)

Short-cut (factor-based) formulae for capital requirements within an internal model

....

Selected references

BIFFIS, E. (2005) Affine processes for dynamic mortality and actuarial valuations. *Insurance: Mathematics & Economics*, **37(3)**, 443–468.

BIFFIS, E. and MILLOSOVICH, P. (2006) A bidimensional approach to mortality risk. *Decisions in Economics and Finance*, **29**, 71–94.

CAIRNS, A.J.G., BLAKE, D. and DOWD, K. (2006) Pricing death: frameworks for the valuation and securitization of mortality risk. *ASTIN Bulletin*, **36(1)**, 79–120.

CEIOPS (2009) Draft CEIOPS' Advice for Level 2 Implementing Measures on Solvency 2. Standard formula SCR – Article 109 c. Life underwriting risk. *Consultation Paper No. 49*.

CMI (2006) *Stochastic projection methodologies: Further progress and P-Spline model features, example results and implications*. Working Paper 6, The Faculty of Actuaries and Institute of Actuaries.

DAHL, M. (2004) Stochastic mortality in life insurance. Market reserves and mortality-linked insurance contracts. *Insurance: Mathematics & Economics*, **35(1)**, 113–136.

Selected references (*cont*)

DAHL, M. and MØLLER, T. (2006) Valuation and hedging of life insurance liabilities with systematic mortality risk. *Insurance: Mathematics & Economics*, **39(2)**, 193–217.

OLIVIERI, A. (2001) Uncertainty in mortality projections: an actuarial perspective. *Insurance: Mathematics & Economics*, **29(2)**, 231–245.

OLIVIERI, A. and PITACCO, E. (2009) Stochastic mortality: the impact on target capital. *ASTIN Bulletin*, forthcoming.

PITACCO, E., DENUIT, M., HABERMAN, S. and OLIVIERI, A. (2009) *Modelling longevity dynamics for pensions and annuity business*. Oxford University Press.

WILLETS, R.C. (2004) The cohort effect: insights and explanations. *British Actuarial Journal*, **10**, 833–877.



Many thanks for your kind attention