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DYNAMIC LIFE TABLES: CONSTRUCTION AND APPLICATIONS

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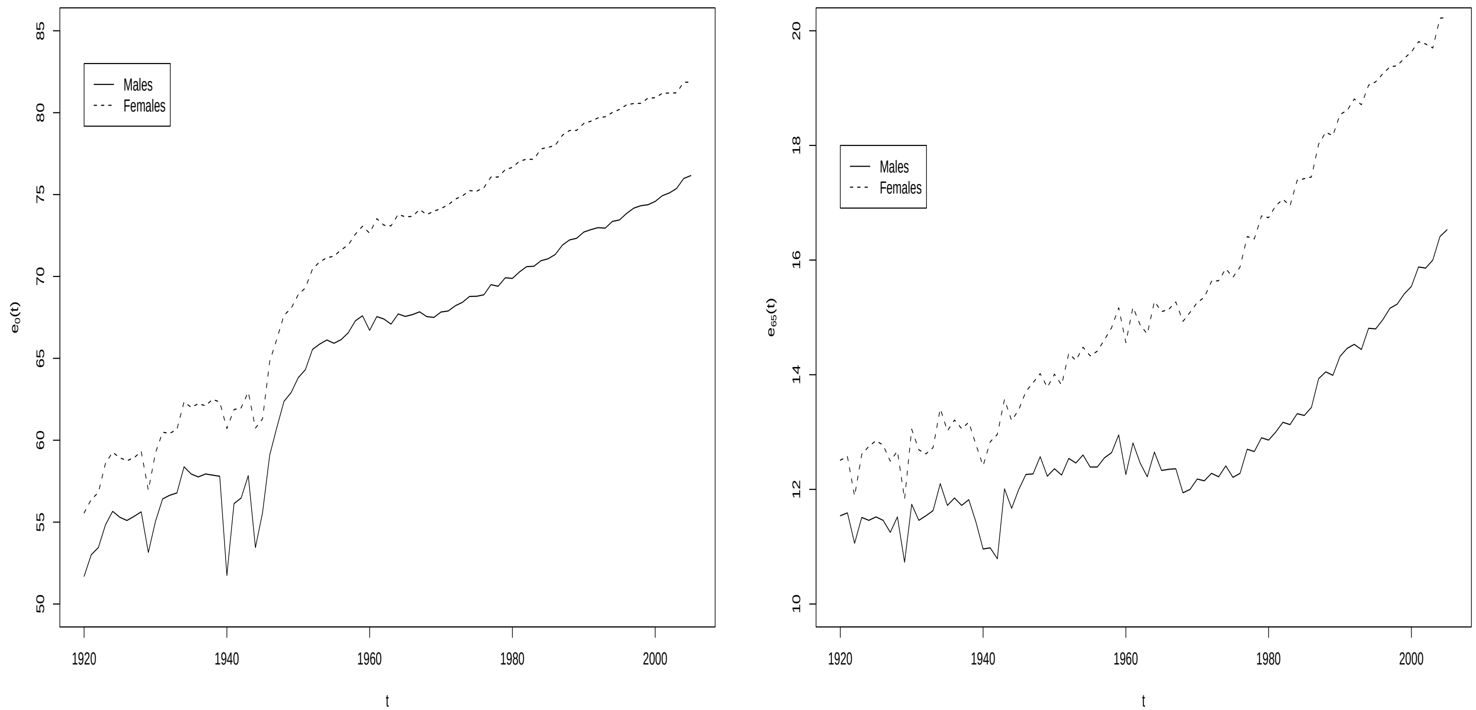
Summary

- Mortality projection models
- Type of dependence induced by projected life tables
- Systematic risk in large portfolios
- Risk sharing mechanisms for life annuities
- Model risk
- Some references

MORTALITY PROJECTION MODELS

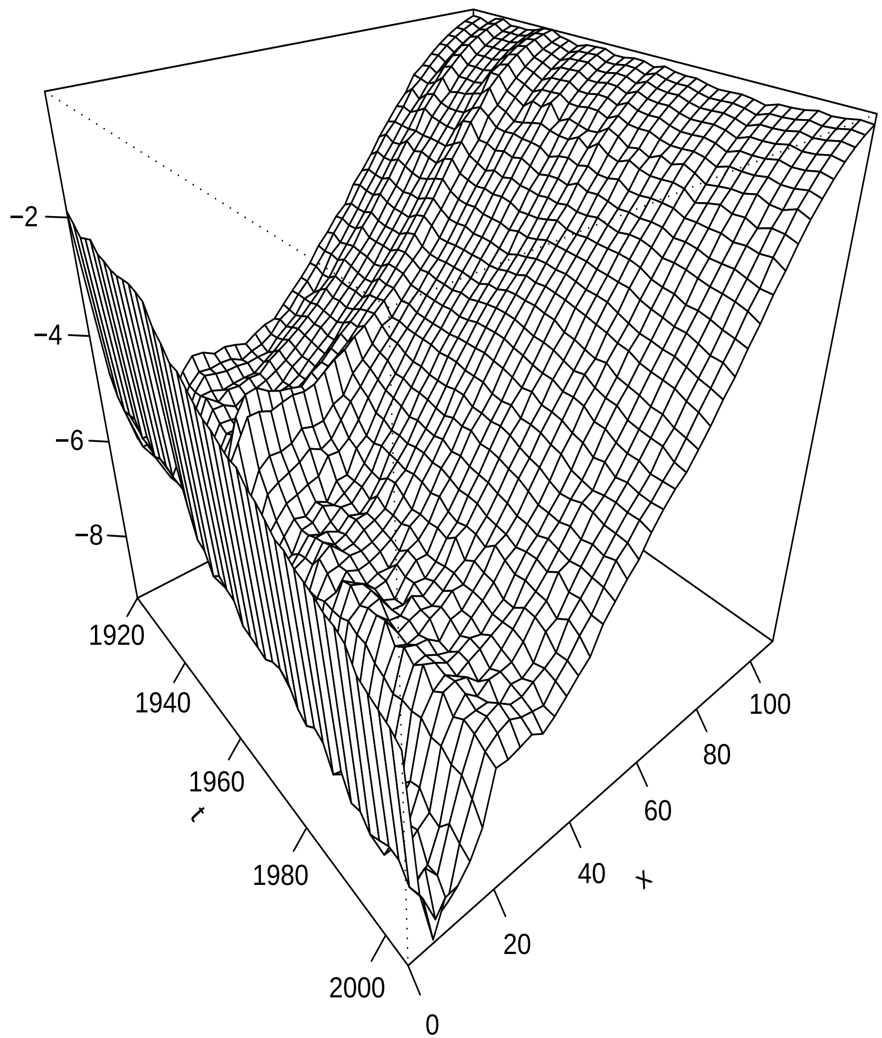
Preamble



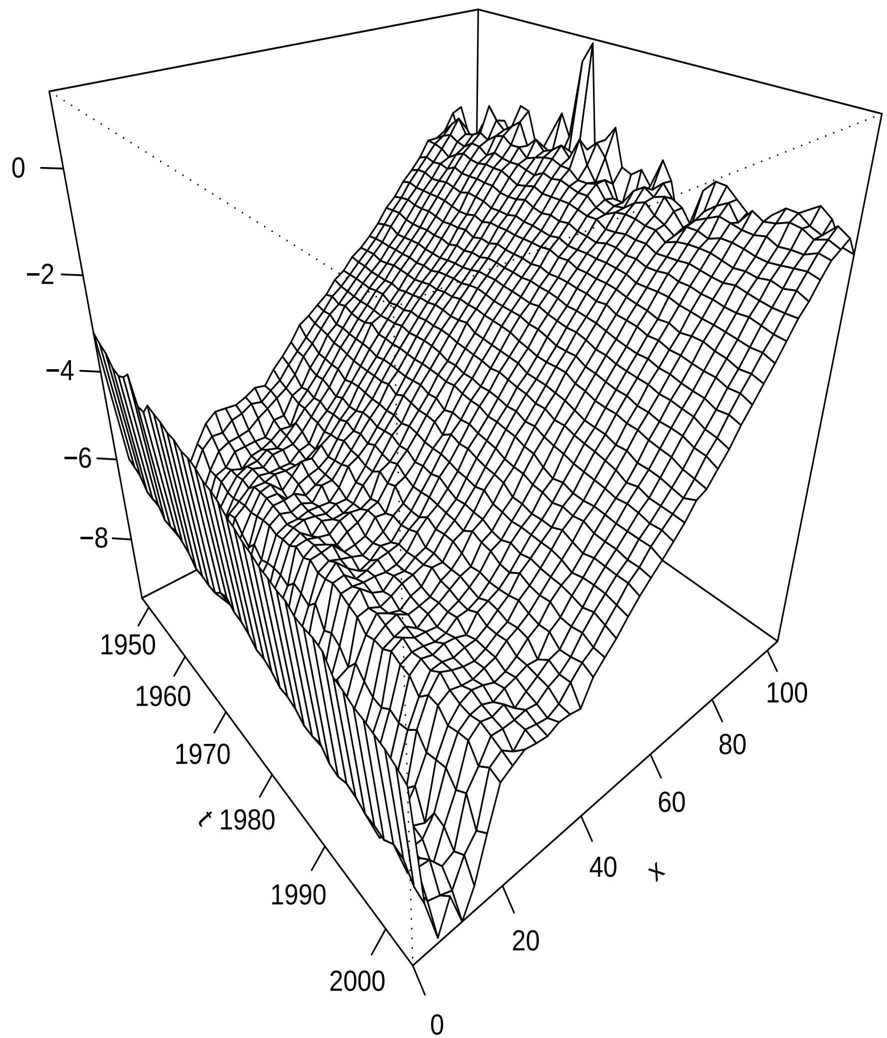
Life expectancies at birth (on the left) and at age 65 (on the right), Belgian general population.

Source: HMD, www.mortality.org.

Belgian males, HMD data



Belgian males, official FPB data



Mortality projection models

- LC model by Lee & Carter (1992).
- APC extension by Renshaw & Haberman (2006)
- CBD model by Cairns, Blake & Dowd (2006)
- Multivariate time series by Lazar & D (2009)
- P-splines by Currie, Durban & Eilers (2004)
- Also,
  - continuous-time models inherited from the interest rate management and credit risk literature (see, e.g. Biffis, 2005, or Biffis & D, 2006),
  - demographic models, as in Oeppen & Vaupel (2002).
- See Pitacco et al. (2009) for an extensive presentation of the topic.

Mortality projection models

- Cairns et al. (2008a,b,c,2009) compared different mortality projection models, including
  - Lee-Carter model with and without cohort effects
  - P-splines
  - CBD model with and without cohort effects.
- Using quantitative (BIC, analysis of residuals, backtesting) and qualitative criteria, these studies evaluate the plausibility and robustness of the mortality forecasts produced by each model.
- Here, we concentrate on LC model (a similar analysis could be conducted with other models).

LC model

- Death rate at age x in year t of the form

$$m_x(t) = \exp\left(\alpha_x + \beta_x \kappa_t\right).$$

- Estimation: - In Lee & Carter (1992), OLS,
  - Binomial maximum likelihood (Cossette et al., 2007, Haberman & Renshaw, 2007),
  - Poisson maximum likelihood (Brouhns et al., 2002a,b, Renshaw & Haberman, 2003),
  - Negative Binomial maximum likelihood (Delwarde et al., 2007b),
  - Bayesian analysis (Czado et al., 2005).
- Optimal fitting period: Booth et al. (2002) or D & Goderniaux (2005).

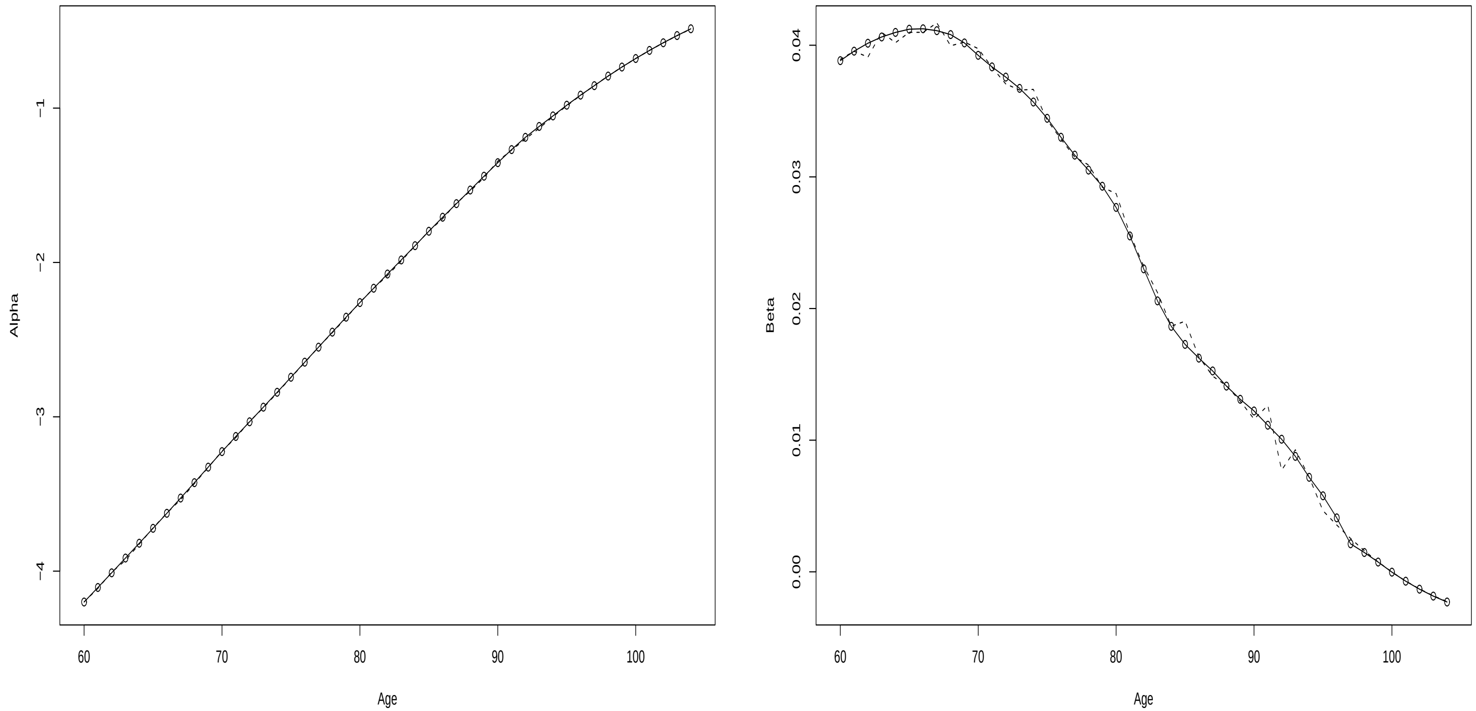
Smoothing β_x

- The estimated β_xs exhibit an irregular pattern in most cases, and this produces irregular projected life tables.
- To get smooth estimated β_xs, the OLS objective function can be modified into

$$\sum_{x,t}\Big(\ln \widehat{m}_x(t) - \alpha_x - \beta_x\kappa_t\Big)^2 + \pi_\beta \sum_x \left(\beta_{x+2} - 2\beta_{x+1} + \beta_x\right)^2$$

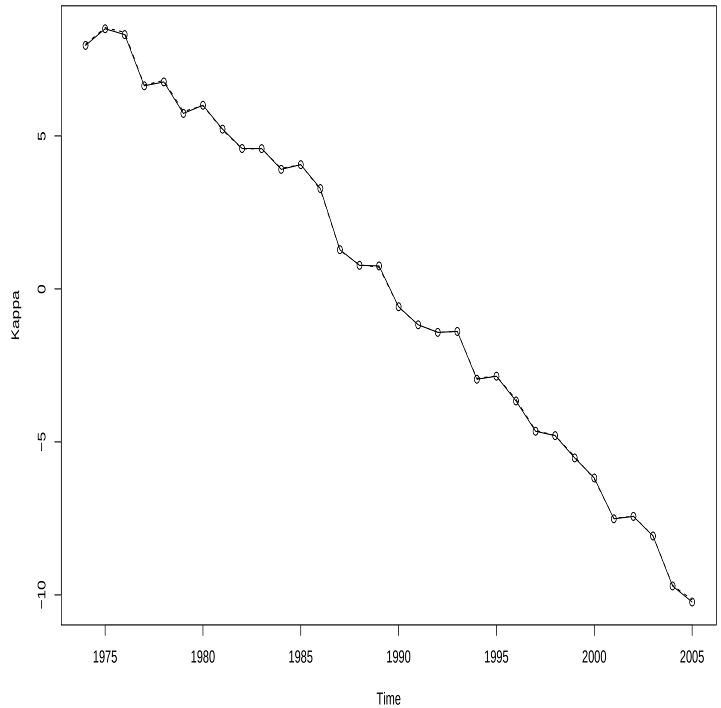
  where π_β is the smoothing parameter (penalized least-squares, Delwarde et al., 2007a).
- The optimal π_β can be selected by cross validation.
- With Poisson error structure, imposing smoothness on the estimated β_xs is achieved by penalized maximum likelihood (Delwarde et al., 2007a).

Estimated α_x and β_x



Belgian males, general population, ages 60 to 104, estimation with (circles) and without (dotted line) penalty, $\widehat{\alpha}_x$s are on the left, $\widehat{\beta}_x$s are on the right.

Estimated κ_t



Optimal starting year 1974, estimation with (circles) and without (dotted line) penalty, adjusted by refitting to the observed period life expectancy at age 60 according to Lee & Miller (2001).

Mortality reduction factors

- Time series techniques are used to forecast κ_t and

$$\widehat{m}_x(t_{\max}+k) = \exp\left(\alpha_x + \beta_x \kappa_{t_{\max}+k}\right) \text{ for } k \geq 1.$$

- Note that

$$\widehat{m}_x(t_{\max}+k) = \widehat{m}_x(t_{\max}) \times RF(x,k)$$

where

$$RF(x,k) = \exp\left(\beta_x(\kappa_{t_{\max}+k} - \kappa_{t_{\max}})\right).$$

- Hence, we could apply these reduction factors to some reference life table to avoid bias, as suggested in Lee & Miller (2001).

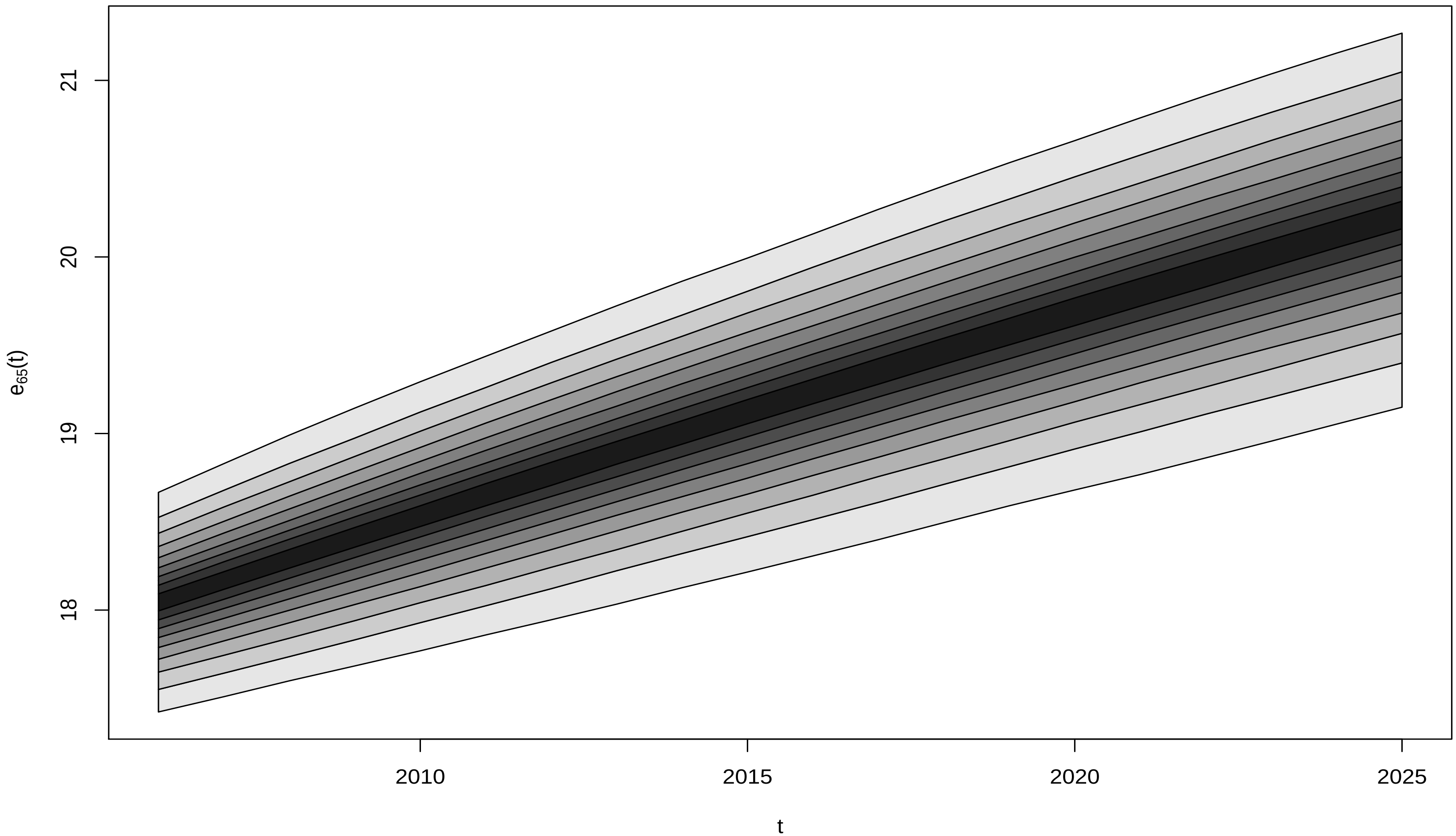
A trend-change extension

- Sweeting (2009) modelled the two factors involved in the CBD model as random fluctuations around a trend (and not as random walks), the trend changing periodically.
- This is in line with the optimal fitting period selected for the Lee-Carter model.
- Booth et al. (2002) designed procedures for selection of an optimal fitting period which identified the longest period for which the estimated mortality index parameter κ_t was linear.
- Hence, Sweeting's idea may be particularly useful for the Lee-Carter model, too.

Prediction intervals

- It is often impossible to derive prediction intervals analytically because
  - two very different sources of uncertainty have to be combined: sampling errors in the parameters of the model and forecast errors in the projected time index;
  - the measures of interest are complicated non-linear functions of the parameters α_x, β_x, and κ_t and the time series parameters.
- Bootstrap procedures help to overcome these problems as in Brouhns et al. (2002a, parametric ML bootstrap), Brouhns et al. (2005), Koissi et al. (2006, residual bootstrap), Haberman & Renshaw (2009).

Longevity fan chart for $e^{\nearrow}_{65}(t)$



Such charts have been proposed by Blake, Cairns & Dowd (2009).

Extension to several log-bilinear terms

- As pointed out by Booth et al. (2002), the original approach by Lee & Carter (1992) makes use of only the first term of the singular value decomposition of the matrix of centered log death rates.
- In principle, the second and higher order terms could be incorporated in the model:

$$\ln m_x(t) = \alpha_x + \sum_{j=1}^{r} \beta_x^{[j]} \kappa_t^{[j]},$$

  where r is the rank of the $\ln m_x(t) - \alpha_x$ matrix.
- Renshaw & Haberman (2003) successfully included the first two sets of SVD vectors.

Multivariate time series

- Lazar & D (2009) investigate the use of multivariate time series techniques for forecasting age-specific death rates: dynamic factor analysis and Johansen cointegration methodology.
- The inclusion of several time factors allows the model to capture the imperfect correlations in death rates from one year to the next.
- The benchmark Lee-Carter model appears as a special case of these approaches.
- A vector error correction model is successfully fitted to Belgian general population death rates.

Cohort effects

- The cohort effect is the observed phenomenon that people born in some specific period have experienced specific improvements in mortality compared to other generations.
- The year-of-birth or cohort year is $c = t - x$.
- Three variables may enter the model: age x, period t and cohort c.
- Renshaw & Haberman (2006) suggested to partition the force of mortality into

$$\ln m_x(t) = \alpha_x + \beta_x \kappa_t + \gamma_x \lambda_c$$

where the parameters are subject to appropriate identifiability constraints.

Adverse selection

- Mortality projection can be performed on market statistics, if available.
- Else, policyholders' specific mortality can be accounted for by different approaches.
- For instance, age shifts can be determined by Poisson maximum likelihood.
- See Yerna & D (2008) for an application to Belgian insurance mortality statistics.
- Relational models have been used by Brouhns et al. (2002a) and Delwarde et al. (2004) to account for adverse selection.

Adverse selection

- The semi-parametric relational model is specified as

$$\ln m_x^{\text{market}} = f_1(x) + f_2\big(\ln m_x^{\text{ref}}\big).$$

  where m_x^{ref} corresponds to some reference life table and where no explicit expression is postulated for the unknown functions f_1 and f_2, except that they exhibit “smoothness”.

- These functions can be estimated by OLS, Poisson, Binomial or Negative Binomial regression models.
- Plat (2009) developed a stochastic model for portfolio specific mortality experience in combination with a stochastic country population mortality process.

TYPE OF DEPENDENCE INDUCED BY PROJECTED LIFE TABLES

Conditional independence

- Consider a group of n annuitants aged x_0 in year t_0.
- Their respective remaining lifetimes are $T_1, \ldots, T_n$.
- Assume that $T_1, \ldots, T_n$ obey the Lee-Carter model.
- Given $\boldsymbol{\kappa}$, $T_1, \ldots, T_n$ are independent with

$$
{}_{\xi}p_{x_0}(\boldsymbol{\kappa}) = \begin{cases} \exp\big(-\xi m_{x_0}(t_0|\boldsymbol{\kappa})\big) \text{ if } \xi \leq 1, \\ \exp\big(-(\xi - \lfloor \xi \rfloor) m_{x_0+\lfloor \xi \rfloor}(t_0 + \lfloor \xi \rfloor | \boldsymbol{\kappa})\big) \\ \qquad \prod_{k=0}^{\lfloor \xi \rfloor - 1} \exp\big(- m_{x_0+k}(t_0 + k|\boldsymbol{\kappa})\big) \\ \qquad \text{if } \xi > 1, \end{cases}
$$

assuming that the forces of mortality are constant on each square of the Lexis diagram (but allowed to vary between squares).

Exchangeable lifetimes

- For any $t_1,\ldots,t_n\geq 0$, we have

$$\begin{aligned}\Pr[T_1\leq t_1,\ldots,T_n\leq t_n] &= \mathbb{E}\left[\prod_{i=1}^n \Pr[T_i\leq t_i|\boldsymbol{\kappa}]\right]\\ &= \mathbb{E}\left[\prod_{i=1}^n \Big(1-{}_{t_i}p_{x_0}(\boldsymbol{\kappa})\Big)\right]\\ &= \Pr[T_1\leq t_{i_1},\ldots,T_n\leq t_{i_n}]\end{aligned}$$

  for any permutation $\{i_1,\ldots,i_n\}$ of $\{1,\ldots,n\}$.
- The random variables $T_1,\ldots,T_n$ are exchangeable, that is, their joint distribution function is invariant under permutation.

Associated lifetimes

- Recall that $X_1, \ldots, X_n$ are associated if

$$\mathbb{C}\text{ov}\left[\Psi_1(X_1, \cdots, X_n), \Psi_2(X_1, \cdots, X_n)\right] \geq 0$$

  for all non-decreasing functions Ψ_1 and Ψ_2 for which the covariances exist.

- $\boldsymbol{\kappa}$ is associated $\Leftrightarrow \mathbb{C}\text{ov}[\kappa_s, \kappa_t] \geq 0$ for all s and t when $\boldsymbol{\kappa}$ is multivariate Normal.
- If $\beta_x \geq 0$ for all the ages x and $\boldsymbol{\kappa}$ is associated, then the lifetimes $T_1, \ldots, T_n$ are associated as pointed out by D & Frostig (2007).
- If $\boldsymbol{\kappa}$ obeys a random walk with drift model, then it is associated (in this case, $\boldsymbol{\kappa}$ is even MTP$_2$ and the lifetimes are WBF, see D & Frostig, 2009, D, 2009a).

Associated annuity present values

- Let

$$a_{\overline{\xi}|} = \sum_{k=1}^{\lfloor \xi \rfloor} v(0,k)$$

denote an annuity certain, where $\lfloor \xi \rfloor$ is the integer part of ξ and where $v(0,k)$ is the present value at time 0 of a unit payment made at time k, with the convention that the empty sum is zero.

- Since $T_1, \ldots, T_n$ are positively dependent, it is reasonable to expect that the present values of life annuity payments $a_{\overline{T_1}|}, \ldots, a_{\overline{T_n}|}$ also are.
- The random variables $a_{\overline{T_1}|}, \ldots, a_{\overline{T_n}|}$ are associated provided $\beta_x \geq 0$ for all x and $\boldsymbol{\kappa}$ is associated.

Value-at-Risk (VaR) and Tail-VaR

- Given a risk X and a probability level $p \in (0,1)$, the corresponding VaR is defined as

$$\text{VaR}[X;p] = F_X^{-1}(p) = \inf\{x \in \mathbb{R} | F_X(x) \geq p\}.$$

- The corresponding Tail-VaR is defined as

$$\text{TVaR}[X;p] = \frac{1}{1-p} \int_p^1 \text{VaR}[X;\xi]\, d\xi.$$

- If F_X is continuous then

$$\text{TVaR}[X;p] = \text{CTE}[X;p] = \mathbb{E}\Big[X \Big| X \geq \text{VaR}[X;p]\Big].$$

Comparison with independence

- Let $T_1^\perp, \ldots, T_n^\perp$ be independent lifetimes, each $T_i^\perp$ being distributed as T_i.
- On average, there is no effect since

$$\mathbb{E}\left[\sum_{i=1}^n a_{\overline{T_i^\perp|}}\right] = \mathbb{E}\left[\sum_{i=1}^n a_{\overline{T_i|}}\right].$$

- If $\beta_x \geq 0$ for all x and $\boldsymbol{\kappa}$ is associated then

$$\text{TVaR}\left[\sum_{i=1}^n a_{\overline{T_i^\perp|}}; p\right] \leq \text{TVaR}\left[\sum_{i=1}^n a_{\overline{T_i|}}; p\right] \text{ for any } p$$

as pointed out in D & Frostig (2007).

Present value V of life annuity payments

(1) lifetimes $T_1, \ldots, T_n$ being conditionally independent given $\boldsymbol{\kappa}$ with common ξ-year conditional survival probability ${}_\xi p_{x_0}(\boldsymbol{\kappa})$.

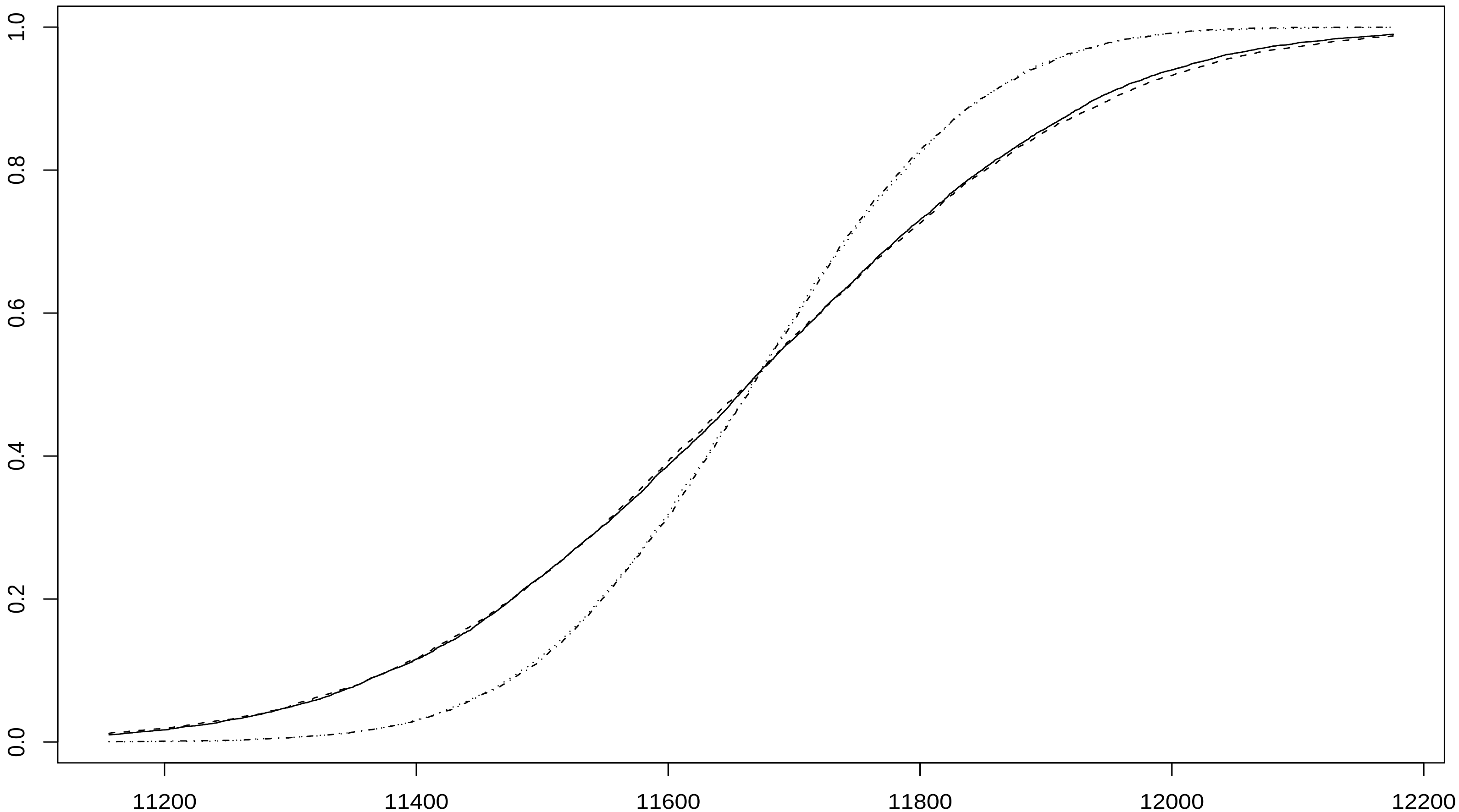
(2) lifetimes with the same copula as (1) but with deterministic death rates

$$m^{\text{det}}_{x_0+k}(t_0+k) = \exp\big(\alpha_{x_0+k} + \beta_{x_0+k}\mathbb{E}[\kappa_{t_0+k}]\big).$$

(3) lifetimes $T_1^\perp, \ldots, T_n^\perp$ being independent with common ξ-year survival probability $\mathbb{E}[{}_\xi p_{x_0}(\boldsymbol{\kappa})]$.

(4) lifetimes $T_1^\perp, \ldots, T_n^\perp$ being independent with deterministic death rates $m^{\text{det}}_{x_0+k}(t_0+k)$.

F_V with $n = 1,000$



(1) — (2) - - - (3) · · · (4) - · - ·

SYSTEMATIC RISK IN LARGE PORTFOLIOS

Large homogeneous portfolios

- For a policyholder aged x_0 in year t_0

$$a_{x_0}(t_0|\boldsymbol{\kappa}) \;=\; \mathbb{E}[a_{\overline{T_1|}}|\boldsymbol{\kappa}] = \sum_{k\geq 1} {}_kp_{x_0}(\boldsymbol{\kappa})v(0,k)$$

quantifies the systematic risk because

$$\lim_{n\to+\infty}\frac{1}{n}\sum_{i=1}^{n} a_{\overline{T_i|}} = a_{x_0}(t_0|\boldsymbol{\kappa}) = \mathbb{E}[a_{\overline{T_1|}}|\boldsymbol{\kappa}].$$

- The diversification effect is apparent from

$$\text{TVaR}\left[\sum_{i=1}^{n+1} a_{\overline{T_i|}};p\right] - \text{TVaR}\left[\sum_{i=1}^{n} a_{\overline{T_i|}};p\right] \leq \frac{1}{n}\text{TVaR}\left[\sum_{i=1}^{n} a_{\overline{T_i|}};p\right].$$

Survival probability ${}_kp_{x_0}(\boldsymbol{\kappa})$

- We have ${}_kp_{x_0}(\boldsymbol{\kappa}) = \exp(-S_k)$ with

$$S_k = \sum_{j=0}^{k-1} \exp\Big(\alpha_{x_0+j} + \beta_{x_0+j}\kappa_{t_0+j}\Big) = \sum_{j=0}^{k-1} \delta_j \exp(X_j),$$

  where $\delta_j = \exp(\alpha_{x_0+j}) > 0$ and $X_j = \beta_{x_0+j}\kappa_{t_0+j}$.

- Note that S_k is a sum of correlated LogNormal random variables when $\boldsymbol{\kappa}$ is multivariate Normal.
- In this case,

$$X_j \sim \mathcal{N}or(\mu_j, \sigma_j^2)$$

  with $\mu_j = \beta_{x_0+j}\mathbb{E}[\kappa_{t_0+j}]$ and $\sigma_j^2 = (\beta_{x_0+j})^2\mathbb{V}[\kappa_{t_0+j}]$.

Comonotonic approximations for S_k

From Dhaene et al. (2002a,b), we know that

$$S_k \approx S_k^u = \sum_{j=0}^{k-1} \delta_j \exp(\mu_j + \sigma_j Z) \text{ where } Z \sim \mathcal{N}or(0,1)$$

and

$$S_k \approx S_k^l = \sum_{j=0}^{k-1} \delta_j \exp\left(\mu_j + r_j(k)\sigma_j Z + \frac{1}{2}(1-(r_j(k))^2)\sigma_j^2\right)$$

where $r_i(d)$, $i = 0, \ldots, d-1$, is given by

$$\frac{\sum_{j=0}^{d-1} \delta_j \exp(\mu_j)\beta_{x+i}\beta_{x+j}\mathbb{C}\text{ov}[\kappa_{t_0+i}, \kappa_{t_0+j}]}{\sigma_i \sqrt{\sum_{j=0}^{d-1}\sum_{k=0}^{d-1} \delta_j\delta_k \exp(\mu_j+\mu_k)\beta_{x+j}\beta_{x+k}\mathbb{C}\text{ov}[\kappa_{t_0+j}, \kappa_{t_0+k}]}}.$$

Approximation for ${}_kp_{x_0}(\boldsymbol{\kappa})$ and $a_{x_0}(t_0|\boldsymbol{\kappa})$

- D & Dhaene (2007) suggested

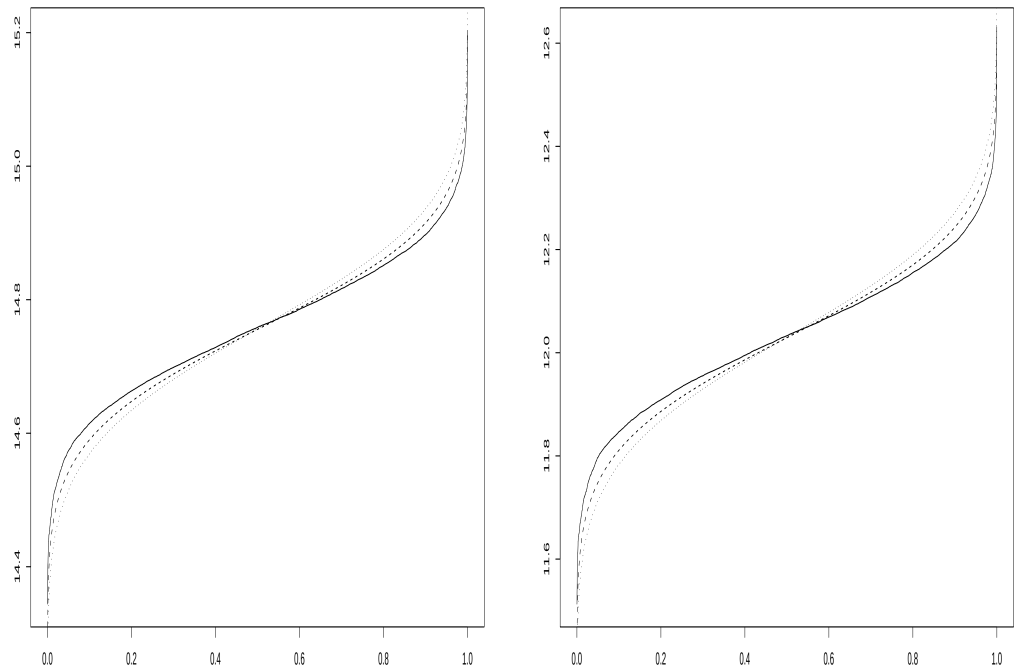
$$
\begin{aligned}
{}_kp_{x_0}(\boldsymbol{\kappa}) &\approx \exp(-S_k^u) \\
&\approx \exp(-S_k^l).
\end{aligned}
$$

- Then, D (2007,2008) proposed to use

$$
\begin{aligned}
a_{x_0}(t_0|\boldsymbol{\kappa}) &\approx \sum_{k\geq 1} \exp(-S_k^u) v(0,k) \\
&\approx \sum_{k\geq 1} \exp(-S_k^l) v(0,k).
\end{aligned}
$$

- Quantile functions of these approximations are easily calculated and provide accurate credible intervals, as demonstrated by D, Haberman & Renshaw (2008).

Quantile function of $a_{65}(2005|\boldsymbol{\kappa})$



Individual life insurance market (left) and group life insurance market (right): the results obtained from 10,000 simulations correspond to the continuous line (—), the results obtained from the S_k^l-approximation correspond to the broken line (- - -), and the results obtained from the S_k^u-approximation correspond to the dotted line ($\cdots$).

Longevity indexed life annuities

- Reinsurance treaties covering longevity risk, if available, are usually expensive.
- Securitization offers an interesting alternative to reinsurance, as well as “natural hedging".
- Considering the cost of these risk management tools, a viable alternative might well be to leave the systematic part of the mortality risk with the annuitants.
- D, Haberman & Renshaw (2009) suggest to let the annuity payments depend on the actual mortality improvements.
- If the actual longevity exceeds that of a reference forecast then the payments are reduced accordingly.

Longevity indexed life annuities

- This proposal is related to Group Self-Annuitization (GSA), studied by Piggott et al. (2005) and Valdez et al. (2006).
- With GSA, retirees pool together and form a fund to provide for protection against longevity.
- Compared with GSA, the type of annuities discussed by D, Haberman & Renshaw (2009) offers a superior protection to policyholders.
- The annuitants only bear the systematic part of the longevity risk, whereas the insurer covers the random fluctuation of mortality as well as the expected future mortality improvements and possible departures from the guaranteed interest rates.

Indexed life annuity

- Let us consider an individual buying an indexed life annuity contract at age x_0 in year t_0.
- Let $p^{\text{ref}}_{x_0+k}(t_0+k)$ be a forecast for the survival of some reference population to which the individual belongs.
- The annual payment of 1 due at time k is adjusted by

$$i_{t_0+k} = \prod_{j=1}^{k-1} \left(\frac{p^{\text{ref}}_{x_0+j}(t_0+j)}{p^{\text{obs}}_{x_0+j}(t_0+j)} \right).$$

- As the annuitant is not in a position to absorb all of the longevity risk, it seems reasonable to limit the impact of the index on the annuity payments, using

$$i_{t_0+k}(i_{\min}, i_{\max}) = \max\{\min\{i_{t_0+k}, i_{\max}\}, i_{\min}\}$$

Indexed life annuity

- In the Lee-Carter model,

$$I_{t_0+k}(i_{\min}, i_{\max}|\boldsymbol{\kappa}) = \max\left\{\min\left\{{}_kp_{x_0}^{\text{ref}}\exp\left(S_k\right), i_{\max}\right\}, i_{\min}\right\}.$$

- Hence,

$$a_{x_0}(t_0, i_{\min}, i_{\max}|\boldsymbol{\kappa}) = \sum_{k\geq 1} I_{t_0+k}(i_{\min}, i_{\max}|\boldsymbol{\kappa})\exp\left(-S_k\right)v(0,k)$$

$$\approx \sum_{k\geq 1}\max\left\{\min\left\{{}_kp_{x_0}^{\text{ref}}\exp\left(S_k^u\right), i_{\max}\right\}, i_{\min}\right\}\exp\left(-S_k^u\right)v(0,k)$$

$$\approx \sum_{k\geq 1}\max\left\{\min\left\{{}_kp_{x_0}^{\text{ref}}\exp\left(S_k^l\right), i_{\max}\right\}, i_{\min}\right\}\exp\left(-S_k^l\right)v(0,k)$$

ALDA

- The advanced-life delayed annuities (ALDA) proposed by Milevsky (2005) are deferred (inflation-adjusted) life annuity contracts.
- The deferment period can be seen as a deductible: the policyholder finances his consumption until some advanced age, 80, 85 or even 90, say, and the insurer starts paying the annuity at this age provided the annuitant is still alive.
- Hence, the ALDA transforms the consumer choice and asset-allocation problem from a stochastic date of death to a deterministic one in which the terminal horizon becomes the annuity payment commencement date.
- The longevity risk involved in the ALDA is quite substantial for the insurance company.

ALDA

- As the premium amount reflects this high risk, the product can become quite expensive unless some indexing applies.
- As the index is publicly available, the annuitant is able to adjust his consumption level during the deferred period.
- Note that we could also think of alternative indexing mechanisms for ALDA.
- Considering a deferred life annuity bought at age 65 with payments starting at age 80, say, we could let the starting age vary according to actual longevity improvements: if longevity increases more than expected, then payments may start at age 82 instead of 80, for instance.

Longevity index

- Other indices than I_{t_0+k} could be used to adapt the amount of periodic payments.
- The period life expectancy at some preset age (the retirement age for instance) appears as a natural candidate for being used as a longevity index.
- In the Lee-Carter framework, it is essentially equivalent to κ_t under Lee & Miller (2001) adjustment; see D (2009b) for an extensive study.
- It possesses an intuitive meaning and somewhat "smooths" I_{t_0+k}.
- It is published yearly by National Institutes of Statistics as well as by international bodies.

Distribution in the Lee-Carter framework

- The period life expectancy at age x in year t_0+k, given $\boldsymbol{\kappa}$, is given by

$$e_x^{\uparrow}(t_0+k|\boldsymbol{\kappa}) \;=\; \frac{1}{2}+\sum_{d\geq 1}\exp\left(-\sum_{j=0}^{d-1}m_{x+j}(t_0+k|\boldsymbol{\kappa})\right)$$

- If all the β_{x+j}s are positive, then $e_x^{\uparrow}(t_0+k|\boldsymbol{\kappa})$ is a decreasing function of the time index κ_{t_0+k}.
- The quantile function of $e_x^{\uparrow}(t_0+k|\boldsymbol{\kappa})$ can be explicitly computed in the LC case.
- The joint distribution of $(e_x^{\uparrow}(t_0+1|\boldsymbol{\kappa}),\ldots,e_x^{\uparrow}(t_0+k|\boldsymbol{\kappa}))^T$ can then easily be recovered from the multivariate Normal distribution of $\boldsymbol{\kappa}$.

MODEL RISK
(work in progress with Andrew Cairns... only preliminary ideas at this stage)

Model risk

- Let $a_{x_0}(t_0|\boldsymbol{m})$ be the price of an immediate annuity sold to an x_0-aged individual in calendar year t_0, viewed as a function of the life table applying to the annuitant (represented by means of a vector $\boldsymbol{m}$ of death rates).
- To predict the cost of this annuity, we have at our disposal a set of K mortality projection models $\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_K$, say.
- The available data set is denoted as $\mathcal{D}$.
- Henceforth, we assume that

$$\Pr[\mathcal{M}_k] = \frac{1}{K} \text{ for all } k$$

so that one of the models considered is the true one.

Model risk

- The mean of $a_{x_0}(t_0|\boldsymbol{m})$ given the available observations is

$$\mathbb{E}[a_{x_0}(t_0|\boldsymbol{m})|\mathcal{D}] = \sum_{k=1}^{K} \mathbb{E}[a_{x_0}(t_0|\boldsymbol{m})|\mathcal{D},\mathcal{M}_k]\,\mathrm{Pr}[\mathcal{M}_k|\mathcal{D}]$$

  where $\mathbb{E}[a_{x_0}(t_0|\boldsymbol{m})|\mathcal{D},\mathcal{M}_k]$ is the prediction of the annuity price using model k.

- Similarly,

$$\mathbb{V}\text{ar}[a_{x_0}(t_0|\boldsymbol{m})|\mathcal{D}] = \sum_{k=1}^{K} \mathbb{E}[(a_{x_0}(t_0|\boldsymbol{m}))^2|\mathcal{D},\mathcal{M}_k]\,\mathrm{Pr}[\mathcal{M}_k|\mathcal{D}]$$

$$-\Big(\mathbb{E}[a_{x_0}(t_0|\boldsymbol{m})|\mathcal{D}]\Big)^2.$$

Model risk

- The weights $\Pr[\mathcal{M}_k|\mathcal{D}]$, $k = 1, 2, \ldots, K$, assigned to each model should reflect their appropriateness given the data so that model selection criteria are good candidates in that respect.
- Consider information criteria of the form

  $$I = -2 \ln L + \pi$$

  where L is the likelihood function, evaluated by substituting the maximum likelihood estimates of the parameters and π is a penalty that is a function of the number of parameters p and/or the number of observations n.
- Standard penalties are $\pi = 2p$ (AIC) and $\pi = p \ln n$ (BIC).

Model risk

- Let $I_k = -2 \ln L_k + \pi_k$ be the value of the information criterion for model $\mathcal{M}_k$.
- The comparison of model l with model k can be based on
$$\frac{L_l \exp(-\pi_l/2)}{L_k \exp(-\pi_k/2)} = \frac{\exp(-I_l/2)}{\exp(-I_k/2)}.$$
- If the penalties are equal for the two models, that is, $\pi_l = \pi_k$, then this is just the ratio of the respective likelihoods, also called Bayes factor.
- A plausible choice for defining the weight to be assigned to model k is

$$\Pr[\mathcal{M}_k|\mathcal{D}] = \frac{\exp(-I_k/2)}{\sum_{j=1}^{K} \exp(-I_j/2)}.$$

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