Mean-Variance efficient strategies in proportional reinsurance under group correlation in a gaussian framework

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Outline

- Motivation of our research
- Gaussian world: the mean-variance efficiency approach
- Proportional reinsurance under group correlation
- The efficient set:
  - in the retention space
  - in the mean-variance space

The problem we have studied

Bruno de Finetti, 1940

Il problema dei pieni
Giornale Italiano Istituto Attuari, 9:1–88, 1940.

- a mean-variance approach for the issue of the efficient proportional reinsurance strategy
- expressive closed form formulas for the efficient retention set obtained in the case of uncorrelated risks
- original insightful procedure suggested to find the set of mean-variance efficient proportional retentions in case of correlation

Main characteristics of proportional reinsurance

Proportional reinsurance

A vector $\mathbf{x}$ of retention quotas of single policies.

- Problem of choosing the efficient vector $\mathbf{x}$
- According to the classical mean-variance approach:

$$
\begin{aligned}
\min \quad & \frac{1}{2}\mathbf{x}^T\mathbf{C}\mathbf{x} \\
& \mathbf{m}^T\mathbf{x} = E \\
& \mathbf{0} \leq \mathbf{x} \leq \mathbf{1}
\end{aligned}
$$

for all $0 \leq E \leq \sum_i m_i$, with $\mathbf{C}$, non singular covariance matrix of gains and $\mathbf{m} > 0$, vector of expected profits of each policy.

Outline of general results: KKT conditions

KKT conditions: the solution $(\mathbf{x}, \lambda)$ is efficient iff:

$$x_i = 1 \Rightarrow F_i(\mathbf{x}) \leq \lambda$$

$$0 < x_i < 1 \Rightarrow F_i(\mathbf{x}) = \lambda$$

$$x_i = 0 \Rightarrow F_i(\mathbf{x}) \geq \lambda$$

where

- λ, Lagrange multiplier of the expectation constraint in the Lagrangian of the constrained optimization problem
- the functions $F_i(\mathbf{x})$ are:

$$F_i(\mathbf{x}) = \frac{1}{2} \frac{\partial V}{\partial x_i} \Big/ \frac{\partial E}{\partial x_i} := \sum_{j=1}^{n} \frac{\sigma_{ij}}{m_i} x_j \qquad i = 1, \ldots, n \tag{1}$$

Interpretation:

- λ, unit (shadow) price in terms of quota of reinsurance
- $F_i(\mathbf{x})$ normalized marginal pseudo utility of a reinsurance of $i-$th risk. More precisely, at $\mathbf{x}$, diminution of variance over diminution of expectation for a marginal reinsurance dx_i of the reinsured quota of risk i.

Intuitive rule

Given the shadow price λ, and keeping account of the shape of marginal utility as a monotonic function of x_i, the rule is as follows:

- **Provide (buy) reinsurance**, as long as the pseudo-utility is greater than the price
- **Stop** exactly at the point where the marginal utility equals the price or, obviously, at zero

To solve this problem, you need in general a sequential procedure

A geometric dynamic idea. Optimum path.

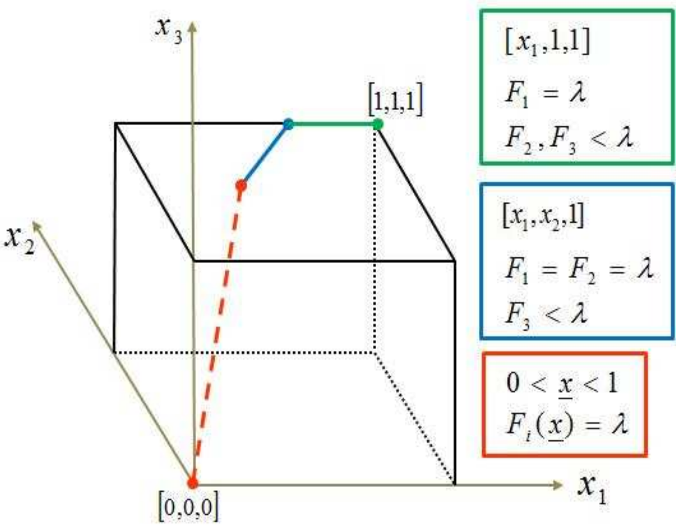
Let us transfer on the retention space:
unit n-dimension cube of feasible retentions.

Optimum path:

- Move in the unit cube along the path connecting
  Vertex **1**: full retention (largest expectation) with
  Vertex **0**: full reinsurance (minimum null variance)
  according to a maximization logic
- A piecewise linear continuous path, with corner points and specified directions

Corner point: a point in the optimum path where a risk modifies its state

An example: basic procedure in a 3-dimensional cube



From no correlation to group correlation

- No correlation case:
  - regular path (matching corners)
  - a priori order of entrance
  - closed form formulae
- Correlation case:
  - no regular path (matching and also breaking corners)
  - no a priori order of entrance
  - no closed form formulae
- Group Correlation case:
  - regular path
  - a priori order of entrance
  - **closed form formulae (NEW!!!)**

Characteristics of Group Correlation:

Portfolio of risks partitioned in g groups labeled by $q = 1, \ldots, g$

Prop. 1: constant group specific correlation **within** each group q:

$$\rho_q > 0$$

and zero correlation **between** groups

Prop. 2: $\forall i \in q$, a group specific ratio:

$$a_{i,q} = \frac{\sigma_{i,q}}{m_{i,q}} = a_q \quad i = 1, \ldots, n_q$$

whose reciprocal is the unitary loading coefficient in the framework of a safety loading coherent with the standard deviation principle $P = E + a^{-1}\sigma$

Under group correlation we have:

Pseudo utility functions:

$$\begin{aligned} F_{i,q}(\mathbf{x}) &= a_q \cdot \Big(x_{i,q}\sigma_{i,q} + \rho_q \cdot \sum_{j\neq i} x_{j,q}\sigma_{j,q}\Big) \\ &= a_q \cdot \Big[x_{i,q}\sigma_{i,q} \cdot (1-\rho_q) + \rho_q \cdot \sum_{j=1}^{n_q} x_{j,q}\sigma_{j,q}\Big] \end{aligned} \tag{2}$$

Moreover, natural ordering of risks, ⇒ i.e. order of entrance, within each group:

$$\boxed{\sigma_{1,q} > \sigma_{2,q} > \ldots > \sigma_{n_q,q}} \quad \forall q = 1, \ldots, g$$

and of groups (according to the values at $\mathbf{x} = \mathbf{1}$ (or $\mathbf{x}_q = \mathbf{1}_q$) of the utility function of the first risk of each group)

$$\boxed{F_{1,1}(\mathbf{1}_1) > F_{1,2}(\mathbf{1}_2) > \ldots > F_{1,g}(\mathbf{1}_g)}$$

Structure of mean-variance efficient retentions:

Fundamental result: Theorem 1

A necessary condition for $\mathbf{x}_q$ being the group q subset of retention of a global efficient retention $\mathbf{x}$ is that $\mathbf{x}_q$ satisfies:

$$0 \leq x_{1,q} \leq 1 \qquad (3)$$

$$x_{i,q} = \min\left[\frac{x_{1,q} \cdot \sigma_{1,q}}{\sigma_{i,q}}; 1\right] \quad \forall i = 2, \ldots, n_q \qquad (4)$$

$x_{1,q}$ is the "driver" of corresponding retention vector $\mathbf{x}_q$.

Structure of mean-variance efficient retentions:

Consider the following equation in the unknown $x_{1,q}$:

$$F_{1,1}(\mathbf{x}_1) = F_{1,q}(\mathbf{x}_q) \tag{5}$$

that is in the extended form:

$$\begin{aligned} a_1 \cdot \left[x_{1,1}\sigma_{1,1} + \rho_1 \cdot \sum_{i=2}^{n_1} \sigma_{i,1} \cdot \min\left(\frac{x_{1,1} \cdot \sigma_{1,1}}{\sigma_{i,1}}; 1 \right) \right] = \\ = a_q \cdot \left[x_{1,q}\sigma_{1,q} + \rho_q \cdot \sum_{i=2}^{n_q} \sigma_{i,q} \cdot \min\left(\frac{x_{1,q} \cdot \sigma_{1,q}}{\sigma_{i,q}}; 1 \right) \right] \end{aligned} \tag{6}$$

Fundamental result: Lemma 1

Given a feasible $x_{1,1}$, equation (6) has exactly one feasible solution

$$\widehat{x}_{1,q} \quad \textbf{iff} \quad F_{1,q}(\mathbf{1}_q) \geq F_{1,1}(\mathbf{x}_1)$$

otherwise there are no feasible solutions.

Structure of mean-variance efficient retentions:

Fundamental result: Theorem 2
Necessary and sufficient conditions on the set of drivers

$\forall q = 1, \ldots, g$, a set of drivers $\{x_{1,q}\}$ generates an efficient retention $\mathbf{x}$ (coherently with Theorem 1) **iff**:

$$0 \leq x_{1,1} \leq 1 \tag{7}$$

$$x_{1,q} = \begin{cases} \widehat{x}_{1,q} & if \quad F_{1,q}(\mathbf{1}_q) \geq F_{1,1}(\mathbf{x}_1) \\ 1 & if \quad F_{1,q}(\mathbf{1}_q) < F_{1,1}(\mathbf{x}_1) \end{cases} \tag{8}$$

A geometric look at the efficient path in the single group space:

$\mathbf{x}_q^h$: efficient retentions set of the group q generated by the driver $x_{1,q}^h = \dfrac{\sigma_{h,q}}{\sigma_{1,q}}$ (Theorem 1):

$$\begin{aligned} \mathbf{x}_q^h &= \left[x_{1,q}^h, x_{2,q}^h, \ldots, x_{n_q,q}^h\right] \\ &= \left[\frac{\sigma_{h,q}}{\sigma_{1,q}}, \frac{\sigma_{h,q}}{\sigma_{2,q}}, \frac{\sigma_{h,q}}{\sigma_{h-1,q}}, 1, \ldots, 1\right] \quad \forall \quad h = 1, \ldots, n_q \end{aligned} \tag{9}$$

$\mathbf{x}_q^h$: **corner point** at which risk h of group q enters in reinsurance.
$\mathbf{x}_q^{n_q+1} = \mathbf{0}$: last corner (the end point) of the efficient path.
Note: the set of efficient retentions is a piecewise linear path with corner points given by (9).

A geometric look at the efficient path: economic meaning of corners

At the $h-$th corner point:

- policy h_q begins to be reinsured
- its utility function **matches** the shadow price:

$$F_{h,q}(\mathbf{x}_q^h) \quad \textbf{matches} \quad \lambda(\mathbf{x})$$

- $\lambda(\mathbf{x})$: common value of utility functions of all $(h-1)_q$ risks already partially reinsured
- policies from $(h+1)_q$ to n_q have smaller values of their utility functions and are fully retained
- between two adjacent corner points: the set of the partially reinsured policies does not change. There, it is preserved the equality of their utility functions

The efficient set in the global retention space

Starting point: set of g sequences of the $\lambda_{h,q}$

$$\lambda_{h,q} = F_{h,q}(\mathbf{x}_q^h)$$

$(\lambda_{h,q}, \lambda_{h-1,q}]$: interval of shadow prices between two adjacent corner points of group q, where exactly $(h-1)_q$ policies are currently reinsured.

Let then put:

$$\lambda_q = \lambda_{1,q} = \max_{i \in q} F_{i,q}(\mathbf{1}_q) = F_{1,q}(\mathbf{1}_q) \qquad (10)$$

$$\overline{\lambda} = \max_q \lambda_q = F_{1,1}(\mathbf{1}_1) = F_{1,1}(\mathbf{1}) \qquad (11)$$

which is, keeping account of the ordering of risks and groups, the *largest value of the λ_q shadow prices* of the efficient path.

Rule providing an efficient global retention $\mathbf{x}(\lambda)$

For any choice of $0 \leq \lambda \leq \overline{\lambda}$ price:

- Choose a group, say q
- Localize the interval $(\lambda_{h,q}, \lambda_{h-1,q}]$ to which λ belongs
- $h(q,\lambda)$ is indirectly defined: there, for that λ, exactly $(h-1)_q$ policies are reinsured. If $h(q,\lambda) = 1$, then $x_{i,q} = 1$ for any i. Otherwise, the following equation must hold:

$$\lambda = a_q \cdot \left[x_{1,q}\sigma_{1,q}(1-\rho_q) + \rho_q(h-1) \cdot x_{1,q}\sigma_{1,q} + \rho_q \sum_{j=h}^{n_q} \sigma_{j,q} \right]$$
$$= a_q \cdot \left[x_{1,q}\sigma_{1,q} \cdot (1 + \rho_q \cdot (h-2)) + \rho_q \cdot \sum_{j=h}^{n_q} \sigma_{j,q} \right] \tag{12}$$

with $h = h(q,\lambda)$.

Rule providing an efficient global retention $\mathbf{x}(\lambda)$

- Solving for $x_{1,q}$, and exploiting the key relation $x_{1,q}\sigma_{1,q} = x_{i,q}\sigma_{i,q}$, in each interval $(\lambda_{h,q}, \lambda_{h-1,q}]$ we obtain for any other active policy:

$x_{i,q}$ as a function of the shadow price λ

$$x_{i,q}(\lambda) = \frac{\lambda}{a_q \cdot \sigma_{i,q} \cdot [1 + \rho_q \cdot (h-2)]} - \rho_q \cdot \frac{\sum_{j=h}^{n_q} \sigma_{j,q}}{\sigma_{i,q} \cdot [1 + \rho_q \cdot (h-2)]} \tag{13}$$

While, for $i = h, \ldots, n_q$, $x_{i,q}(\lambda) = 1$.

- Finally, repeat the procedure for any q

Remark: linear functions whose slope depends on the interval of λ.

A new set of critical subintervals

- each group gives rise to a group specific record of λ intervals
- a new set of subintervals comes from the intersection of the family of λ intervals of any group
- in each subinterval all the retentions (of the reinsured risks) are, at the same time, fully linear in λ and may be described by formulas (13)

The efficient set in the mean-variance space

The group expected return

- overall expected return of the generic group q in **each interval of** λ:

$$E_q(\lambda) = \sum_{i=1}^{h-1} x_{i,q}(\lambda) \cdot m_{i,q} + \sum_{i=h}^{n_q} m_{i,q} \quad (14)$$

$$E_q(\lambda) = \lambda \cdot \left[\frac{(h-1)}{a_q^2 \cdot [1 + \rho_q \cdot (h-2)]} \right] - \rho_q \cdot \frac{(h-1) \cdot \sum_{j=h}^{n_q} \sigma_{j,q}}{a \cdot [1 + \rho_q \cdot (h-2)]} + \sum_{i=h}^{n_q} m_{i,q} \quad (15)$$

(remind that $a_q = \sigma_{i,q}/m_{i,q} \Rightarrow m_{i,q} = \sigma_{i,q}/a_q$)

- which could be rewritten in a more expressive way:

$$E_q(\lambda) = \lambda \cdot \alpha_q + \beta_q \quad (16)$$

as a linear function of the shadow price λ.

- α_q and β_q: group and interval specific coefficients, through (a, ρ, h) parameters

The efficient set in the mean-variance space

The global expected return of the portfolio

On a specific subinterval of λ, where each group has a group specific number $(h-1)_q$ of active policies, the global expected return is given simply by the sum of the expectations of each of them:

$$E(\lambda) = \lambda \cdot \alpha + \beta \tag{17}$$

where:

$$\alpha = \sum_{q=1}^{g} \alpha_q \quad \beta = \sum_{q=1}^{g} \beta_q$$

Helpful remark:

$$\boxed{\lambda = \frac{E - \beta}{\alpha}} \tag{18}$$

Overall variance of the generic group q:

$$V_q = \mathbf{x}^\top \mathbf{V} \mathbf{x} = \sum_i \sum_j x_i x_j \rho_{ij} \sigma_i \sigma_j = \ldots = \qquad (19)$$

$$V_q(\lambda) = \alpha_q^{'} \cdot \lambda^2 + \beta_q^{'} \cdot \lambda + \gamma_q$$

where:

$$\alpha_q^{'} = \frac{(h-1)}{a_q^2 \cdot [1 + \rho_q \cdot (h-2)]} = \alpha_q \qquad (20)$$

$$\beta_q^{'} = 0 \qquad (21)$$

$$\gamma_q = -\frac{\rho_q^2 \cdot (h-1) \cdot \left(\sum_{i=h}^{n_q} \sigma_{i,q} \right)^2}{1 + \rho_q \cdot (h-2)} + 2\rho_q \cdot \sum_{i=h}^{n_q} \sigma_{i,q} \sum_{j>i} \sigma_{j,q} + \sum_{i=h}^{n_q} \sigma_{i,q}^2 \qquad (22)$$

The efficient set in the mean-variance space

The portfolio global variance

In any subinterval of λ, where each group has a group specific number $(h-1)_q$ of active policies, the global variance is given simply by the sum of the variances of each of them, owing to the null correlation between groups.

$$V(\lambda) = \lambda^2 \cdot \sum_{q=1}^{g} \alpha_q + \sum_{q=1}^{g} \gamma_q = \lambda^2 \cdot \alpha + \gamma \qquad (23)$$

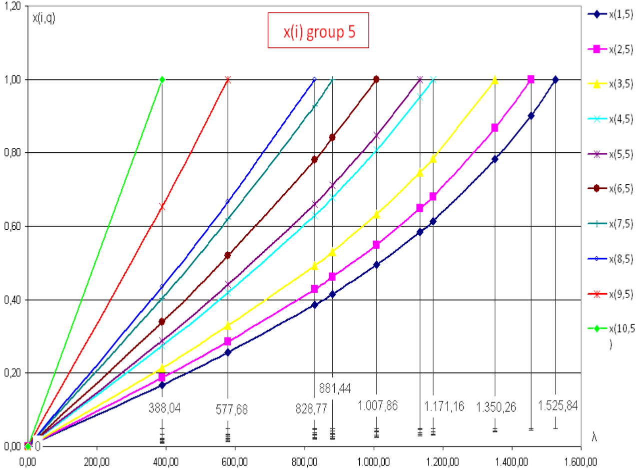
The efficient set in the mean-variance space

The portfolio global variance

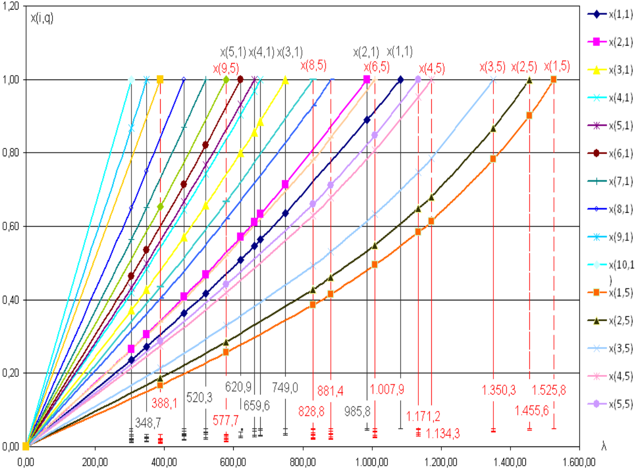
Exploiting the relation $\boxed{\lambda = \frac{E-\beta}{\alpha}}$, we have:

$$V(E) = \frac{(E-\beta)^2}{\alpha^2}\cdot\alpha + \gamma = \frac{(E-\beta)^2}{\alpha} + \gamma \qquad (24)$$

that is a set of quadratic functions in the mean-variance space.
At the connection points we have continuity and derivability (no kinks) of the graph.
Intuitively, as $\frac{\partial V}{\partial E} = 2\cdot\frac{(E-\beta)}{\alpha} = 2\lambda$ and exploiting the continuity of the shadow price also at the corner points.

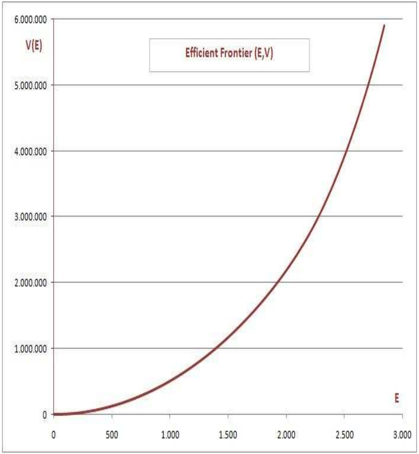


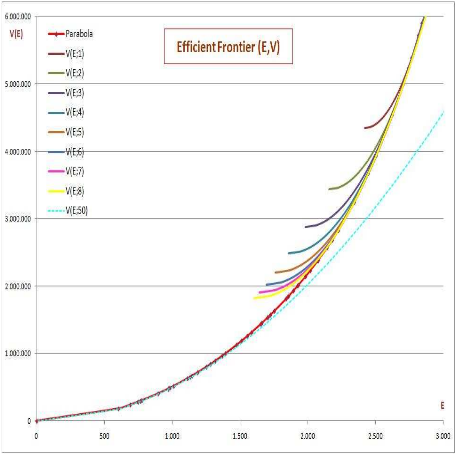
Remark: picture of $x_i(\lambda)$ is piecewise linear, from the zero retention point up to the level $x_i = 1$ (full retention point).



Intersection of λ intervals of group 1 and 5.

Graphs of $V(E)$ as an union of parabolas: the efficient frontier





Conclusions

- Problem analyzed: find a **closed form solution** of the mean-variance efficient set for proportional reinsurance under group correlation
- Main idea: mix KKT conditions and de Finetti's pseudo utility functions
- Results:
    - one-to-one correspondence between closed interval of shadow prices and the set of efficient mean-variance retentions, intuitively provide reinsurance up to the quota where marginal utility of reinsurance matches the shadow price
    - closed form formulae of these vectors as a function of the shadow price
    - the geometric picture of the efficient retention set in the retention space is piecewise linear and continuous
    - the geometric picture of the efficient retention set in a mean-variance space is a, piecewise continuous and without kinks, union of parabolas

Our results, we hope, useful for comparison between efficient retention in gaussian worlds and in other worlds, with other modern risk measures

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Thank you for your attention!!