Breakout Session Topic 5:

Portfolio and risk management

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Estimating Portfolio Value-at-Risk with Multivariate Mixture Time Series Models

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Outline

- Introduction
- The Dataset
- Multivariate Mixture Autoregressive Models
- A Small Comparison Study
- Future Works

Introduction

- **The important role of stochastic equity return in measuring the obligations embedded in insurance and/or investment products are now well-recognized**
- **CIA Task Force on Segregated Fund Investment Guarantees (March 2002)**
  - Provide useful guidance for applying stochastic techniques to value segregated fund guarantees in a Canadian GAAP valuation environment
- **AAA Life Capital Adequacy Subcommittee (2002, 2005)**
  - Stochastic scenario analysis is recommended to determine minimum capital requirements for variable products with guarantees

Introduction

- **Recent actuarial works in the univariate setting**
  - Hardy (2001): A Regime-switching Model for Long-Term Stock Returns
  - Wong and Chan (2005): Mixture Gaussian Time Series Modeling of Long-Term Market Returns
  - Hardy, Freeland and Till (2006): Validation of Long-Term Equity Return Models for Equity-Linked Guarantees
- **Recent actuarial works in the multivariate setting**
  - Boundreault and Panneton (2009): Multivariate Models of Equity Returns for Investment Guarantees Valuation

The Dataset

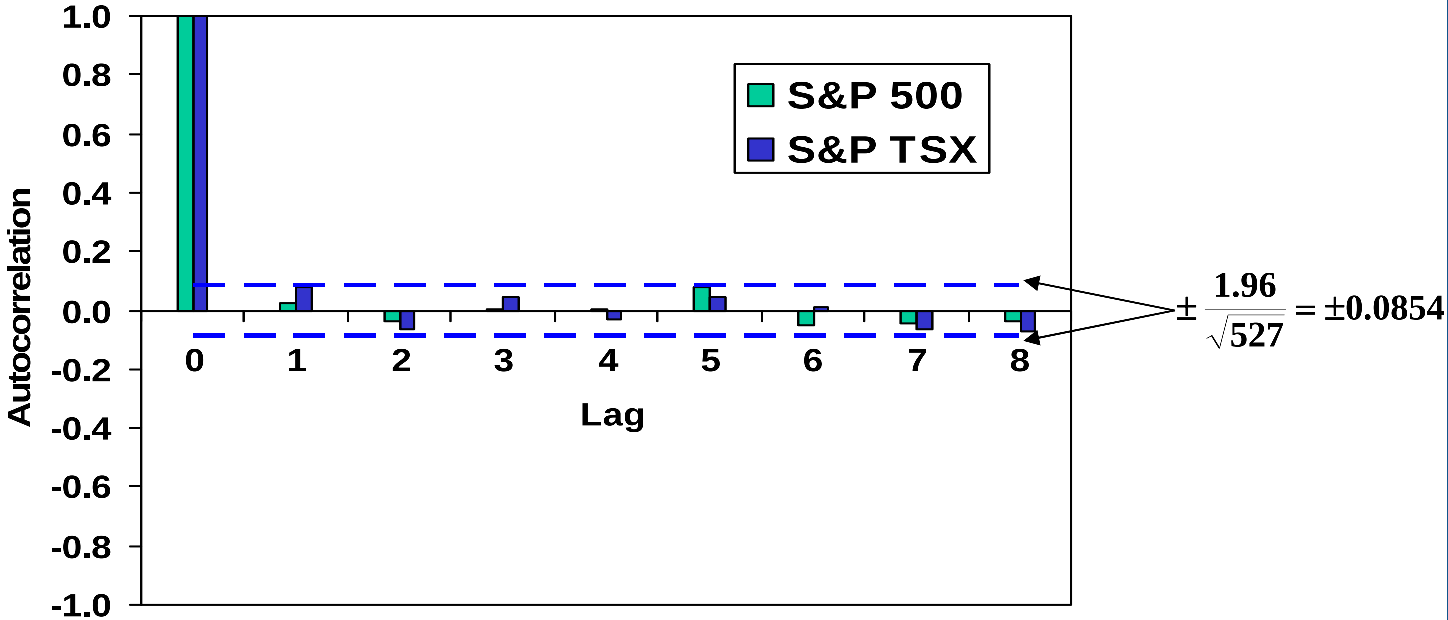
- **Monthly index data from January 1956 to December 1999 for the following markets**
  - United States (S&P 500 total return index)
  - Canada (S&P TSX total return index)
- **Summary statistics for log-index returns:**

Index	Mean	S.D.	Skewness	Excess Kurtosis
S&P 500	0.009630	0.0416	–0.6399	3.0139
S&P TSX	0.008137	0.0451	–0.9099	3.9136

- **The marginal distributions of returns exhibit**
  - Negative skewness
  - Heavy-tailedness

The Dataset

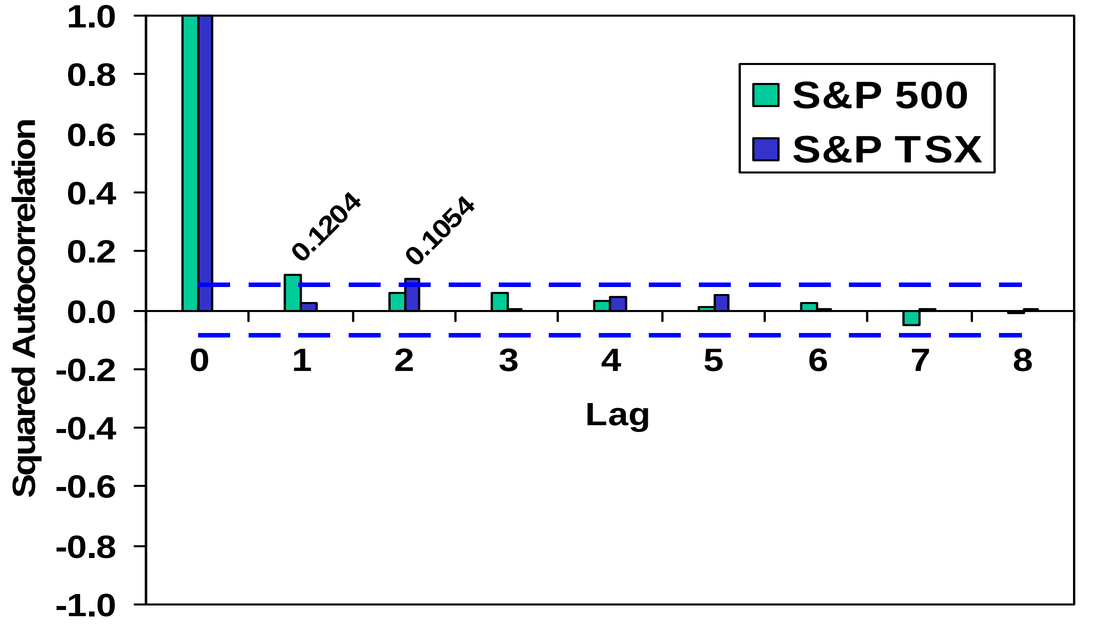
- **The autocorrelation function (ACF)**



- **None of the autocorrelation is significant different from zero**
  - No mean structure for *linear* time series models

The Dataset

- The squared autocorrelation function (SQACF)



- **The squared autocorrelations are significant at lag 1 and lag 2 for S&P 500 and S&P TSX returns respectively**
  - Models with non-constant variances should be used

The Dataset

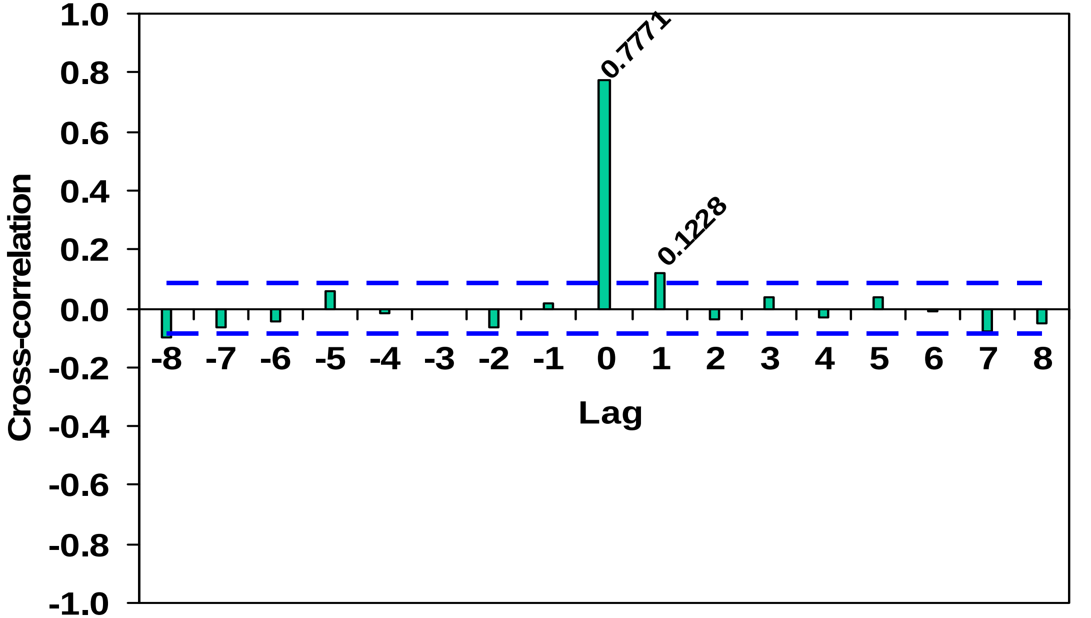
- **For multivariate time series, the *cross-correlation* function (CCF) should be considered**
- **The formula for sample lag k cross-correlation between two time series X_t and Y_t is defined as**

$$\rho_{XY}(k)=\frac{\sum_{t=1}^{T-k}\left(x_t-\bar{x}\right)\left(y_{t+k}-\bar{y}\right)}{\sqrt{\sum_{t=1}^{T}\left(x_t-\bar{x}\right)^2}\sqrt{\sum_{t=1}^{T}\left(y_t-\bar{y}\right)^2}}$$

  - It measures the strength of relationship between X_t and Y_{t+k}
  - If there exists strong relationship, it may be possible to make use of the information for the variable X at time t to predict the value of Y at time $t+k$

The Dataset

- **The cross-correlation function**



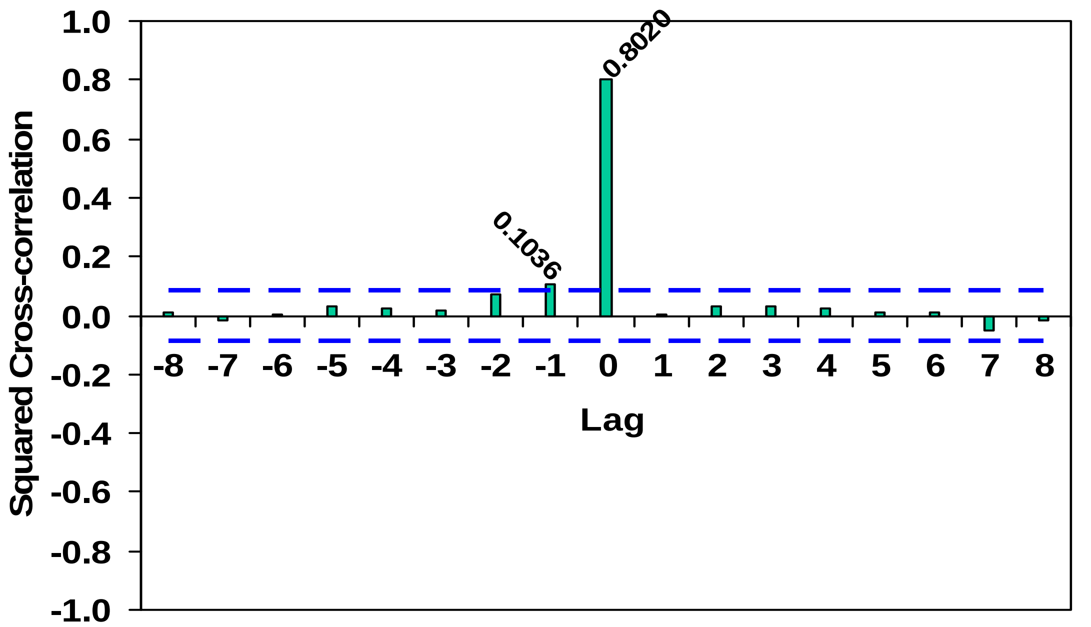
- Note that $\rho_{XY}(k) \neq \rho_{XY}(-k)$

The Dataset

- **The correlation between U.S. and Canadian market returns is quite high**
- **The lag 1 cross-correlation between S&P 500 and S&P TSX index returns is significantly different from zero**
  - It may be possible to improve the prediction of S&P TSX index returns by utilizing the S&P 500 index returns in the previous month
- **In contrast to univariate setting, mean structure should be included in the multivariate setting**
  - Models such as vector autoregressive models may be useful

The Dataset

- The cross-correlation function of the squared index returns (SQCCF)



- The volatility in Canadian market might have some effect of the volatility in U.S. market next month

Mixture Autoregressive Models

- **Wong and Li (2000) introduce the class of mixture autoregressive (MAR) models for univariate time series**

$$F\left(y_t \middle| \mathcal{F}_{t-1}\right) = \sum_{k=1}^{K} \alpha_k N\left(\frac{y_t - \phi_{k0} - \phi_{k1} y_{t-1} - \cdots \phi_{kp_k} y_{t-p_k}}{\sigma_k}\right)$$

- **Properties:**
  - Conditional means and variances change over time
  - Conditional distributions can change from short-tailed to fat-tailed, or from unimodal to multimodal
  - The autocorrelation function for a MAR model is similar to that of an autoregressive model
- **For actuarial applications, see Wong and Chan (2005)**

Multivariate Mixture Autoregressive Models

- **Fong, Li, Yau and Wong (2007) introduce the mixture vector autoregressive model for an *n*-dimensional vector Y_t**

$$F\left(Y_t \middle| \mathcal{F}_{t-1}\right) = \sum_{k=1}^{K} \alpha_k N\left\{\Sigma_k^{-1/2}\left(Y_t - \Phi_{k0} - \Phi_{k1}Y_{t-1} - \cdots - \Phi_{kp_k}Y_{t-p_k}\right)\right\}$$

  - This model is denoted by MVAR(n, K; p_1, …, p_K)
  - $N(\cdot)$ is the cumulative distribution function of the n-dimensional standard normal distribution
  - α_ks are the mixing proportion sum to one
  - Φ_{k0}s are an n-dimensional vectors
  - $\Phi_{k1}, \ldots, \Phi_{kp_k}$ are $n \times n$ coefficient matrices
  - Σ_ks are the $n \times n$ variance-covariance matrices

Multivariate Mixture Autoregressive Models

- **It can be rewritten as**

$$Y_t = \Phi_{k0} + \Phi_{k1} Y_{t-1} + \cdots + \Phi_{kp_k} Y_{t-p_k} + \Sigma_k^{1/2} \varepsilon_t \quad \textbf{with prob. } \alpha_k$$

  - Here, ε_t is a random variate from a n-dimensional standard normal distribution, i.e. $N(0, I_n)$ with I_n the identity matrix

- **The conditional means are changing over time**

$$E\left(Y_t \middle| \mathcal{F}_{t-1}\right) = \sum_{k=1}^{K} \alpha_k \left(\Phi_{k0} + \Phi_{k1} Y_{t-1} + \cdots + \Phi_{kp_k} Y_{t-p_k}\right) = \sum_{k=1}^{K} \alpha_k \mu_{kt}$$

- **The conditional variances are changing over time**

$$\mathrm{var}\left(Y_t \middle| \mathcal{F}_{t-1}\right) = \sum_{k=1}^{K} \alpha_k \Sigma_k + \sum_{k=1}^{K} \alpha_k \mu_{kt} \mu_{kt}' - \left(\sum_{k=1}^{K} \alpha_k \mu_{kt}\right)\left(\sum_{k=1}^{K} \alpha_k \mu_{kt}\right)'$$

- **The autocorrelation functions are similar to those of a vector autoregressive process**

Multivariate Mixture Autoregressive Models

- **Parameter estimation can be performed by maximizing the log-likelihood with EM algorithm (Dempster, Laird and Rubin, 1977)**
- **Standard errors of the parameter estimates can be computed by the Missing Information Principle (Louis, 1982)**
  - Note that the parameter estimates are not necessarily normal distributed
- **Unnecessary parameters in the autoregressive coefficient matrices can be dropped**
  - Estimation of the remaining parameters can be done via constrained estimation

Multivariate Mixture Autoregressive Models

- **Model selection for the number of components, *K*, is a complicated problem**
  - It corresponds to the testing problem with nuisance parameters that do not exist under the null hypothesis (Davies 1977, 1987)
- **Bayesian information criterion (BIC) can provide a rough guideline in selecting *K***
  - In most applications, two- and three-component models should be sufficient
- **Likelihood ratio test can be performed to test if some parameters are not necessary after *K* is fixed**
  - To obtain a parsimonious model

Multivariate Mixture Autoregressive Models

- **The empirical conditional or predictive distribution can be generated by simulating Y_t, given the information up to time t – 1, a large number of times, says 50,000**
- **The VaR can be obtained by determining the empirical percentiles from the simulated data**

Multivariate Mixture Autoregressive Models

- **Let $Y_t = (Y_{1t}, Y_{2t})'$ where Y_{1t} and Y_{2t} are log-index returns of S&P 500 and S&P TSX respectively**
- **The fitted two-component model for S&P 500 and S&P TSX index returns is a MVAR(2, 2; 1, 0) model**

$$F\left(Y_t \middle| \mathcal{F}_{t-1}\right) =$$

$$0.2398\, N\left[\begin{pmatrix} 0.0037 & 0.0033 \\ 0.0033 & 0.0044 \end{pmatrix}^{-\frac{1}{2}} \left\{ \begin{pmatrix} -0.0022 \\ -0.0083 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0.4323 & 0 \end{pmatrix} Y_{t-1} \right\}\right]$$

$$+\, 0.7602\, N\left[\begin{pmatrix} 0.0010 & 0.0008 \\ 0.0008 & 0.0011 \end{pmatrix}^{-\frac{1}{2}} \begin{pmatrix} 0.0133 \\ 0.0127 \end{pmatrix}\right]$$

Multivariate Mixture Autoregressive Models

- The fitted three-component model for S&P 500 and S&P TSX index returns is a MVAR(2, 3; 1, 1, 0) model

$$F\left(Y_t \middle| \mathcal{F}_{t-1}\right) =$$

$$0.7422\, N\left[\begin{pmatrix} 0.0013 & \boxed{0.0010} \\ 0.0010 & 0.0015 \end{pmatrix}^{-\frac{1}{2}} \left\{\begin{pmatrix} 0.0142 \\ 0.0087 \end{pmatrix} + \begin{pmatrix} -0.1377 & 0 \\ 0 & 0.0909 \end{pmatrix} Y_{t-1}\right\}\right]$$

$$+\, 0.2215\, N\left[\begin{pmatrix} 0.0012 & \boxed{0.0012} \\ 0.0012 & 0.0013 \end{pmatrix}^{-\frac{1}{2}} \left\{\begin{pmatrix} 0.0012 \\ 0.0098 \end{pmatrix} + \begin{pmatrix} 0.5544 & -0.1684 \\ 0.6848 & -0.5083 \end{pmatrix} Y_{t-1}\right\}\right]$$

$$+\, 0.0363\, N\left[\begin{pmatrix} 0.0074 & \boxed{0.0077} \\ 0.0077 & 0.0103 \end{pmatrix}^{-\frac{1}{2}} \begin{pmatrix} -0.0342 \\ -0.0547 \end{pmatrix}\right]$$

$\rho = 0.7172$

$\rho = 0.9353$

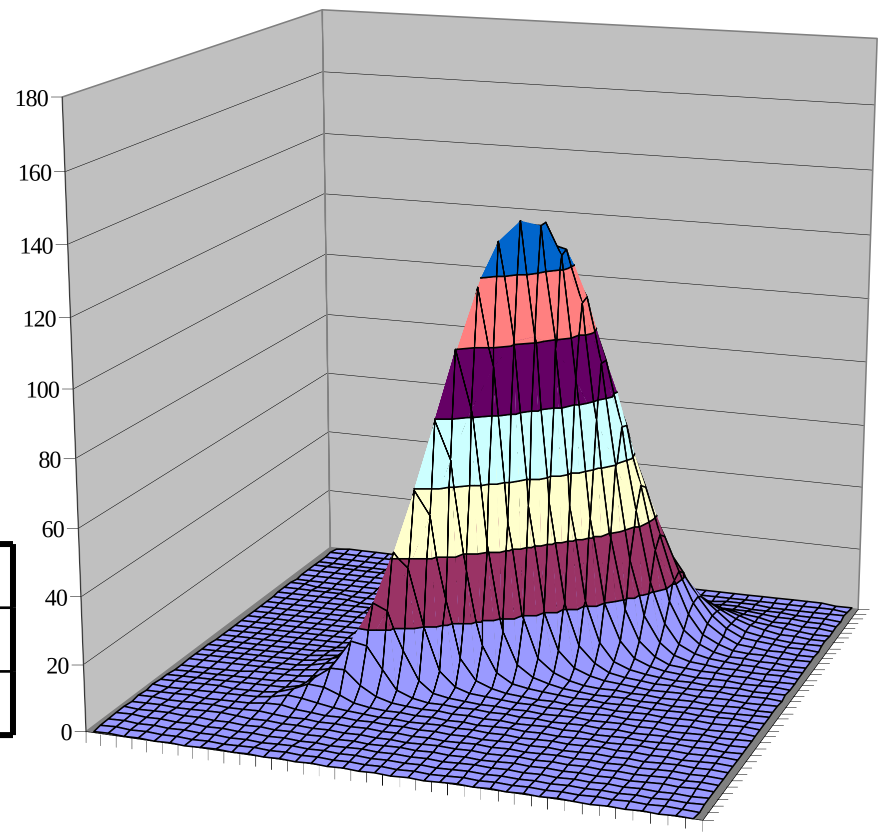
$\rho = 0.8852$

$$Y_{2t} = 0.0098 + 0.6848 Y_{1t-1} - 0.5083 Y_{2t-1} + \text{error}$$

MVAR(2, 3; 1, 1, 0) Model

- **Shape of conditional distribution for September 1981**

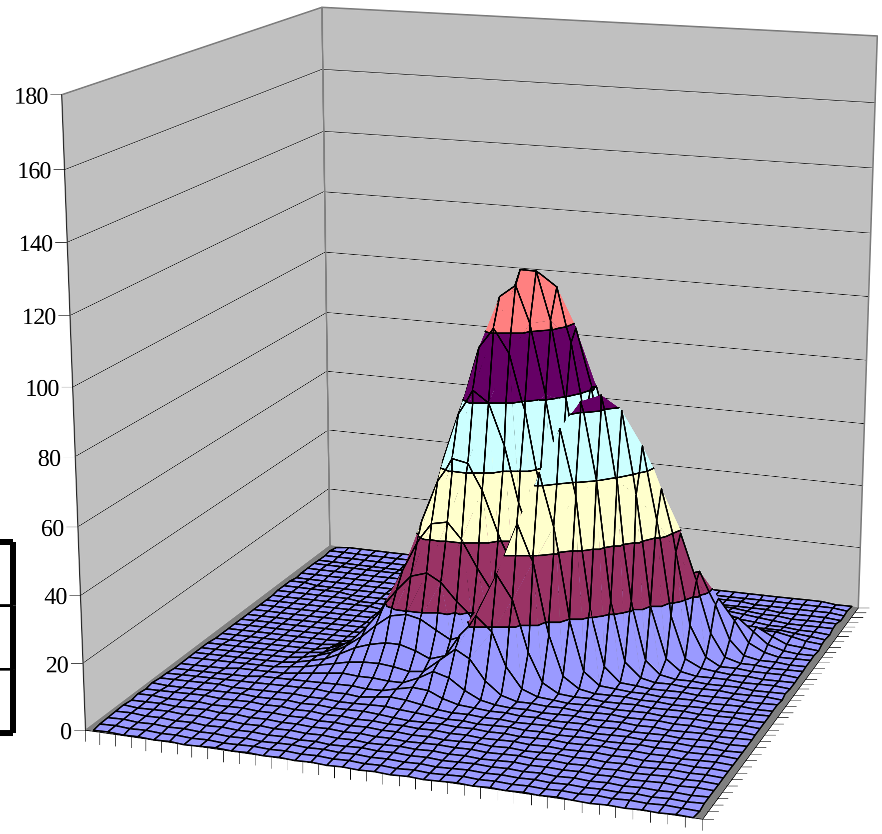
Returns for August 1981	
S&P 500	**–0.0595**
S&P TSX	**–0.0316**



MVAR(2, 3; 1, 1, 0) Model

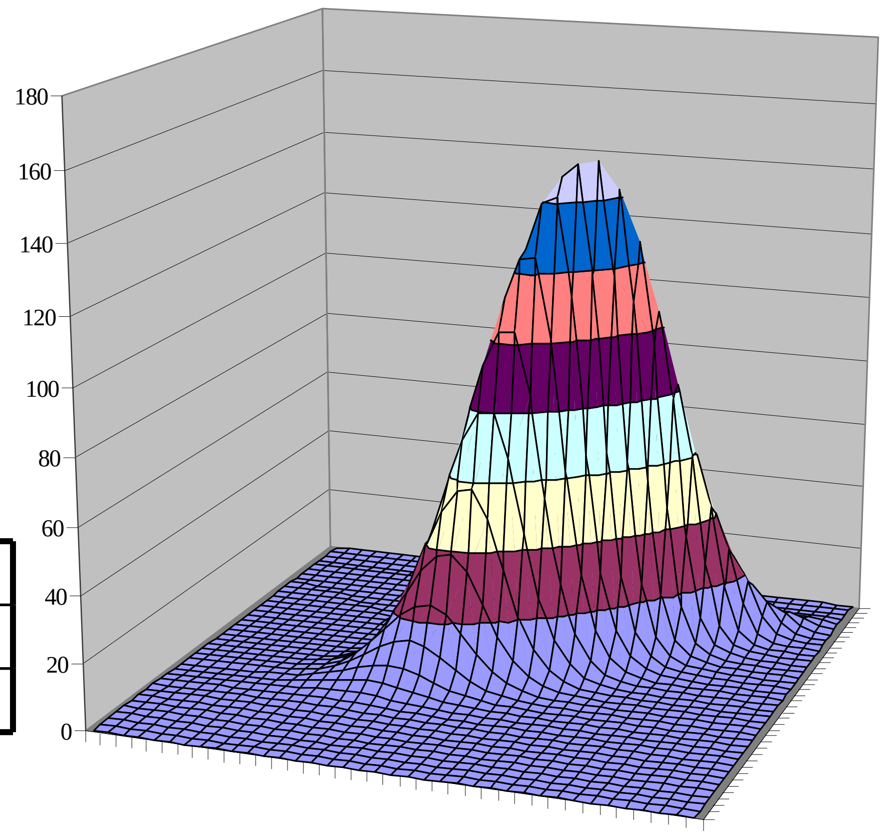
- Shape of conditional distribution for October 1981

Returns for September 1981	
S&P 500	–0.0505
S&P TSX	–0.1410



MVAR(2, 3; 1, 1, 0) Model

- Shape of conditional distribution for November 1981

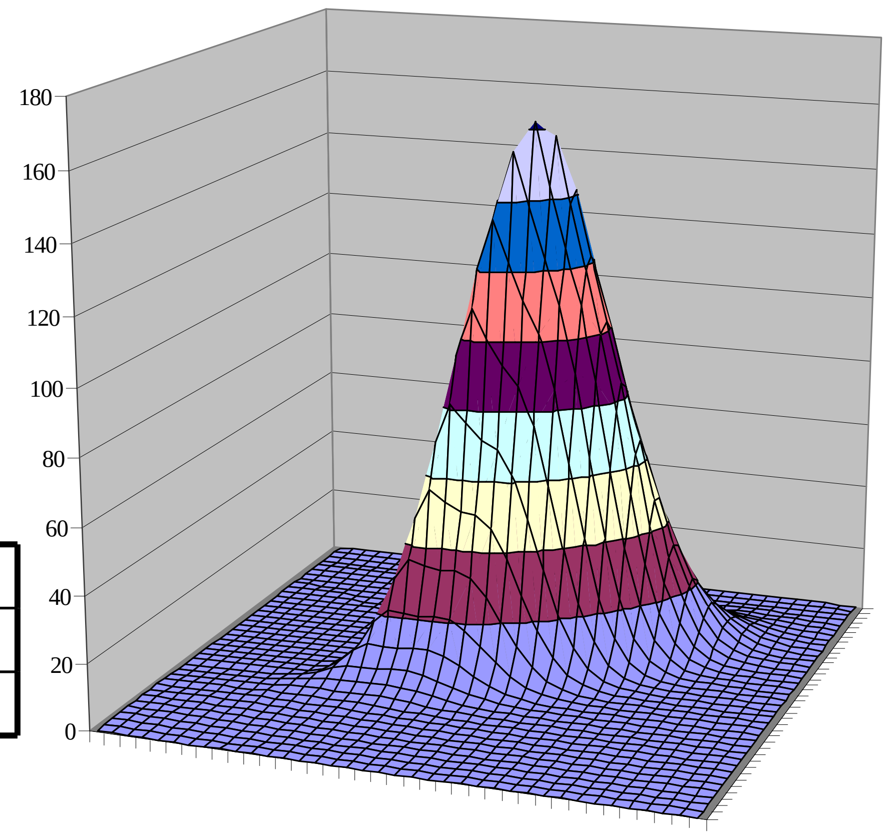


Returns for October 1981	
S&P 500	0.0526
S&P TSX	–0.0199

MVAR(2, 3; 1, 1, 0) Model

- **Shape of conditional distribution for December 1981**

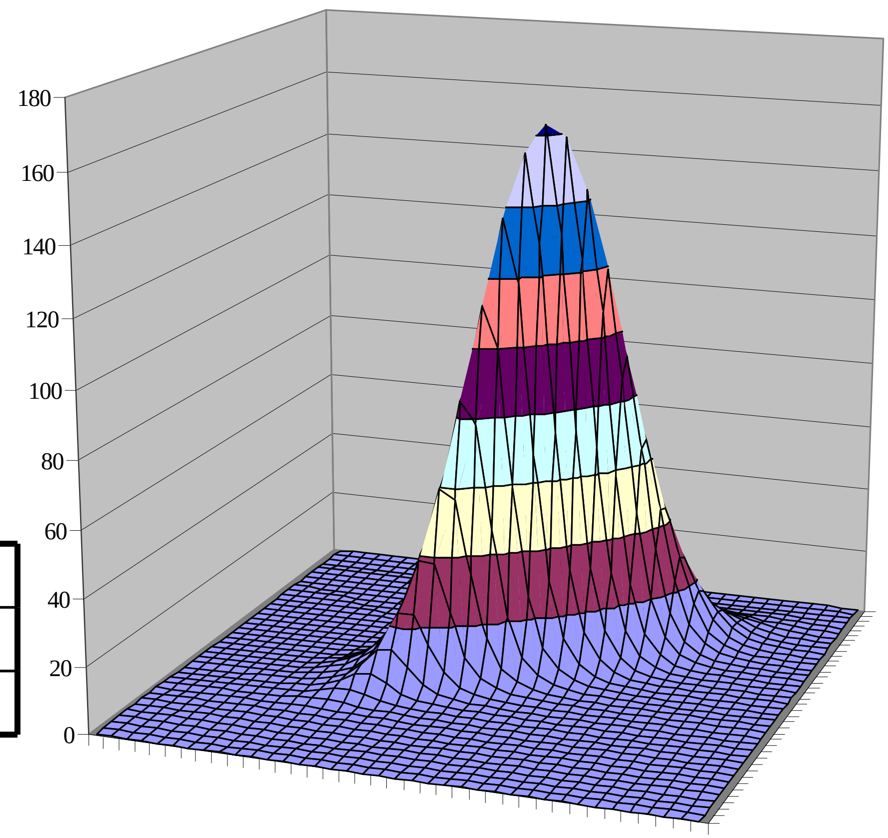
Returns for November 1981	
S&P 500	0.0404
S&P TSX	0.0928



MVAR(2, 3; 1, 1, 0) Model

- **Shape of conditional distribution for January 1982**

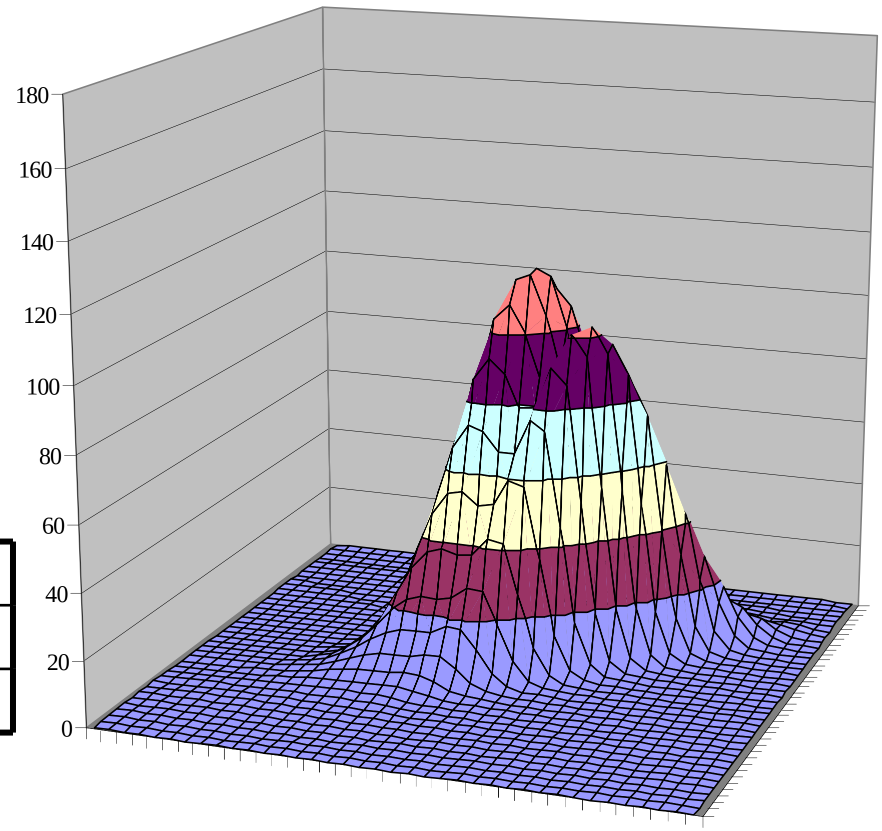
Returns for December 1981	
S&P 500	**–0.0260**
S&P TSX	**–0.0236**



MVAR(2, 3; 1, 1, 0) Model

- Shape of conditional distribution for February 1982

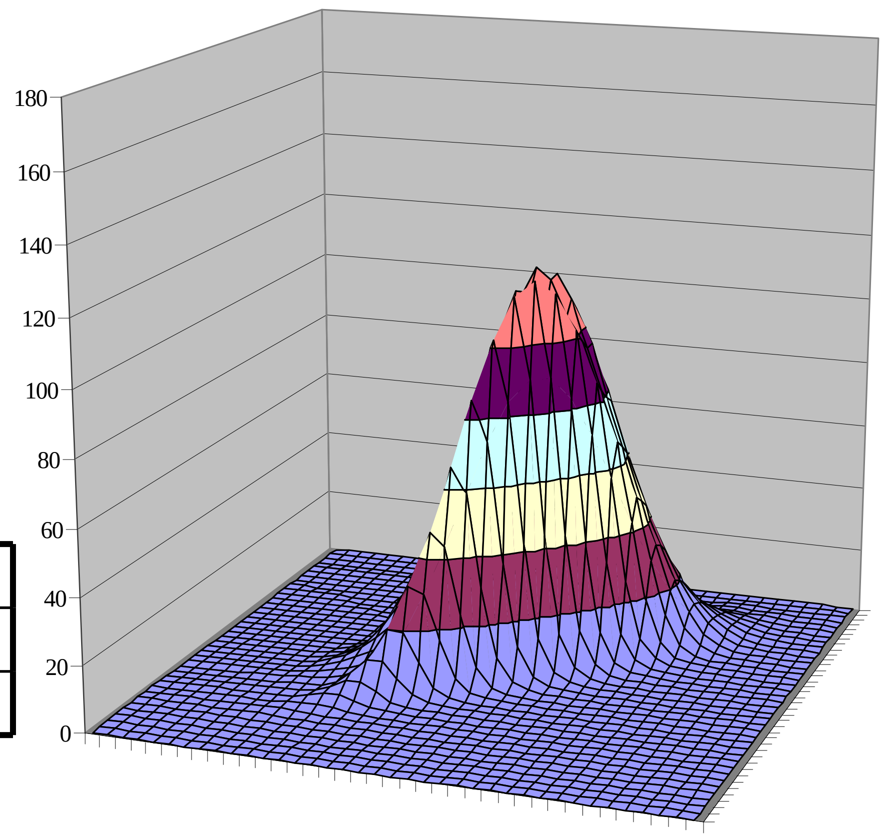
Returns for January 1982	
S&P 500	–0.0131
S&P TSX	–0.0878



MVAR(2, 3; 1, 1, 0) Model

- Shape of conditional distribution for March 1982

Returns for February 1982	
S&P 500	–0.0575
S&P TSX	–0.0633



A Small Comparison Study

- **Independent normal model, ILN**

$$F\left(Y_t\right) = N\left[\begin{pmatrix} 0.0017 & 0.0015 \\ 0.0015 & 0.0020 \end{pmatrix}^{-\frac{1}{2}} \begin{pmatrix} 0.0096 \\ 0.0081 \end{pmatrix}\right]$$

- **Vector autoregressive model of order 1, VAR(1)**

$$F\left(Y_t \middle| \mathcal{F}_{t-1}\right) =$$

$$N\left[\begin{pmatrix} 0.0017 & 0.0014 \\ 0.0014 & 0.0020 \end{pmatrix}^{-\frac{1}{2}} \left\{\begin{pmatrix} 0.0093 \\ 0.0068 \end{pmatrix} + \begin{pmatrix} 0.0313 & -0.0069 \\ 0.1631 & -0.0351 \end{pmatrix} Y_{t-1}\right\}\right]$$

A Small Comparison Study

- **Two-component mixture of independent normal model, MIND(2)**

$$F\left(Y_t\right) = 0.1819\, N\left[\begin{pmatrix} 0.0043 & 0.0038 \\ 0.0038 & 0.0052 \end{pmatrix}^{-\frac{1}{2}} \begin{pmatrix} -0.0063 \\ -0.0147 \end{pmatrix}\right]$$

$$+\, 0.8181\, N\left[\begin{pmatrix} 0.0011 & 0.0008 \\ 0.0008 & 0.0012 \end{pmatrix}^{-\frac{1}{2}} \begin{pmatrix} 0.0132 \\ 0.0132 \end{pmatrix}\right]$$

A Small Comparison Study

- **Three-component mixture of independent normal model, MIND(3)**

$$F\left(Y_t\right) = 0.5044\, N\left[\begin{pmatrix} 0.0008 & 0.0006 \\ 0.0006 & 0.0008 \end{pmatrix}^{-\frac{1}{2}} \begin{pmatrix} 0.0172 \\ 0.0171 \end{pmatrix}\right]$$

$$+\, 0.4900\, N\left[\begin{pmatrix} 0.0022 & 0.0018 \\ 0.0018 & 0.0025 \end{pmatrix}^{-\frac{1}{2}} \begin{pmatrix} 0.0039 \\ 0.0016 \end{pmatrix}\right]$$

$$+\, 0.0056\, N\left[\begin{pmatrix} 0.0033 & 0.0014 \\ 0.0014 & 0.0006 \end{pmatrix}^{-\frac{1}{2}} \begin{pmatrix} -0.1672 \\ -0.2247 \end{pmatrix}\right]$$

A Small Comparison Study

- **Sample characteristics of simulated time series of length 527 over 10,000 replications are compared**
- **For S&P 500 return series**

Model	Mean	S.D.	Skewness	Excess Kurtosis
Data	0.009630	0.0416	–0.6399	3.0139
ILN	0.009593	0.0415	–0.0002	–0.0098
VAR(1)	0.009579	0.0415	–0.0039	–0.0102
MIND(2)	0.009641	0.0415	–0.3941	1.7204
MIND(3)	0.009658	0.0415	–0.6821	2.7396
MVAR(2,2;1,0)	0.009646	0.0415	–0.3133	1.3470
MVAR(2,3;1,1,0)	0.009524	0.0413	–0.4382	1.9829

A Small Comparison Study

- For S&P TSX return series

Model	Mean	S.D.	Skewness	Excess Kurtosis
Data	0.008137	0.0451	–0.9099	3.9136
ILN	0.008091	0.0451	0.0002	–0.0104
VAR(1)	0.008109	0.0450	–0.0030	–0.0121
MIND(2)	0.008105	0.0450	–0.5532	2.0665
MIND(3)	0.008152	0.0450	–0.9058	3.6985
MVAR(2,2;1,0)	0.008731	0.0448	–0.3670	1.8153
MVAR(2,3;1,1,0)	0.007779	0.0450	–0.7066	3.4580

A Small Comparison Study

- Sample time series properties are compared

Model	S&P 500 SQACF(1)	S&P TSX SQACF(2)	CCF(0)	CCF(1)	SQ-CCF(0)	SQ-CCF(–1)
Data	0.1204	0.1054	0.7771	0.1228	0.8020	0.1036
ILN	–0.0024	–0.0023	0.7771	–0.0008	0.6019	–0.0018
VAR(1)	–0.0009	–0.0029	0.7766	0.1218	0.6015	–0.0007
MIND(2)	–0.0019	–0.0011	0.7769	–0.0005	0.6946	–0.0015
MIND(3)	–0.0017	–0.0024	0.7758	–0.0005	0.7780	–0.0011
MVAR(2,2;1,0)	–0.0009	–0.0021	0.7748	0.0939	0.6783	–0.0007
MVAR(2,3;1,1,0)	0.0476	0.0022	0.7745	0.1023	0.7561	0.0312

A Small Comparison Study

- **In-sample validation of Value-at-Risk for S&P 500 return series**

Model	Level			
	1%	2%	5%	10%
ILN	1.71%	3.04%	5.32%	9.13%
VAR(1)	1.90%	3.04%	5.13%	8.94%
MIND(2)	0.76%	1.71%	5.32%	10.84%
MIND(3)	0.76%	2.28%	4.56%	10.27%
MVAR(2,2;1,0)	0.76%	1.71%	4.18%	10.27%
MVAR(2,3;1,1,0)	0.95%	2.47%	5.89%	10.46%

A Small Comparison Study

- **In-sample validation of Value-at-Risk for S&P TSX return series**

Model	Level			
	1%	2%	5%	10%
ILN	2.09%	3.04%	4.75%	8.56%
VAR(1)	1.52%	2.85%	4.56%	8.56%
MIND(2)	0.76%	1.52%	4.56%	10.27%
MIND(3)	0.76%	2.47%	4.75%	10.08%
MVAR(2,2;1,0)	0.57%	1.71%	3.99%	9.89%
MVAR(2,3;1,1,0)	0.76%	2.47%	5.32%	10.27%

A Small Comparison Study

- **Assuming a 50% S&P 500 – 50% S&P TSX portfolio**
- **No adjustments are made for currency effect**
- **In-sample validation of Value-at-Risk**

Model	Level			
	1%	2%	5%	10%
ILN	1.52%	2.85%	4.37%	10.08%
VAR(1)	1.52%	2.47%	4.37%	10.08%
MIND(2)	0.76%	1.52%	3.99%	11.22%
MIND(3)	0.95%	1.90%	4.56%	11.03%
MVAR(2,2;1,0)	0.76%	1.14%	3.80%	10.84%
MVAR(2,3;1,1,0)	0.95%	2.09%	5.51%	10.84%

A Small Comparison Study

- **The mean VaR values for the equal-weighted portfolio**

Model	Level			
	1%	2%	5%	10%
ILN	–0.0861	–0.0750	–0.0583	–0.0435
VAR(1)	–0.0859	–0.0748	–0.0581	–0.0433
MIND(2)	–0.1150	–0.0914	–0.0593	–0.0389
MIND(3)	–0.1052	–0.0836	–0.0588	–0.0395
MVAR(2,2;1,0)	–0.1082	–0.0879	–0.0594	–0.0393
MVAR(2,3;1,1,0)	–0.1042	–0.0772	–0.0546	–0.0389

A Small Comparison Study

- **Conclusion**
  - ILN and VAR(1) fail to produce reliable VaR estimates
  - MIND(2) and MIND(3) fail to capture the cross-correlation between the two return series
  - MVAR models produce marginal distributions with some biases in the mean for the S&P TSX return series
  - All models here fail to capture the dependence in squared return series
  - MIND(3) and MVAR(2, 3; 1, 1, 0) models provide the best description of the conditional distribution at the left tail
    - But MVAR(2, 3; 1, 1, 0) model provide better description of the dependence between the two return series

Future Works

- **Comparison will be made with other multivariate models**
  - Regime-switching models
  - GARCH models
  - See Boundreault and Panneton (2009)
- **CTE or Tail VaR computation will be considered**

Future Works

- **In univariate time series literature, mixture time series models with changing conditional variance have been introduced (Wong and Li, 2001)**

$$F\left(y_t \middle| \mathcal{F}_{t-1}\right) = \sum_{k=1}^{K} \alpha_k N\left(\frac{e_{k,t}}{\sqrt{h_{k,t}}}\right)$$

$$e_{k,t} = y_t - \phi_{k0} - \phi_{k1} y_{t-1} - \cdots \phi_{kp_k} y_{t-p_k}$$

$$h_{k,t} = \beta_{k0} + \beta_{k1} e_{k,t-1}^2 + \cdots + \beta_{kq_k} e_{k,t-q_k}^2$$

- **Multivariate generalization can be done which can capture the cross-correlation effect in both raw and squared series**
  - Better description of the data and more reliable VaR estimates may be obtained

The End