Optimal construction of a fund of funds

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Problem description

How to diversify a given initial capital over a finite number of investment funds that follow different dynamic trading strategies?

- The investment funds operate in a market where a finite number of underlying assets may be traded over finite discrete time.
- The investor faces uncertain asset returns and liabilities.
- Our goal is to find a diversification over the investment funds that is optimal in terms of a convex risk measure[1].

1. R. T. Rockafellar. Coherent approaches to risk in optimization under uncertainty. Tutorials in Operations Research INFORMS 2007, pages 38–61, 2007.

Investment strategies

Several rules have been proposed for updating an investment portfolio in an uncertain dynamic environment. Popular strategies include

- Buy Hold (BH)[1]
- Fixed proportions (FP)[1]
- Constant proportion portfolio insurance (CPPI)[1]
- Target date fund (TDF)[2]

1. A. F. Perold and W. F. Sharpe. Dynamic strategies for asset allocation. Financial Analysis Journal, 51(1):149–160, 1995.
2. Z. Bodie and J. Treussard. Making investment choices as simple as possible, but not simpler. Financial Analysis Journal, 63(3):42–47, 2007.

Investment strategies

Buy Hold

- In the presence of a claim process $(c_t)_{t=1}^T$ a BH strategy can be written as

$$h_{t,j} = \begin{cases} \pi_j w_0 & t = 0, \\ R_{t,j} h_{t-1,j} - \pi_j c_t & t = 1, \ldots, T, \end{cases}$$

  where w_0 is the initial capital, $h_{t,j}$ denotes the holdings in asset $j \in J$, at time t, $R_{t,j}$ is the return on asset j over holding period $[t-1, t]$ and π_j is the proportion invested in asset j at time $t = 0$.

- If the claim process c is null the strategy requires no transactions after time $t = 0$.

Investment strategies

Fixed proportions

- In a FP strategy at each time and state the allocation is rebalanced into proportions given by a vector $\pi \in \mathbb{R}^J$ whose components sum up to one. More precisely,

$$h_t = \pi w_t,$$

where for $t = 1, \ldots, T$,

$$w_t = \sum_{j \in J} h_{t-1,j} R_{t,j} - c_t.$$

Investment strategies

Target date fund

- In a TDF, the proportion invested in risky assets $j \in J^r$ is decreased as retirement date approaches.
- In our multi-asset setting we implement TDFs a

$$\sum_{j \in J^r} \pi_{t,j} = a - bt,$$

$$h_t = \pi_t w_t,$$

  where a gives the initial proportional exposure in the risky assets and b specifies how fast the proportional exposure decreases with time.
- Within the complementary subsets J^r of “risky” and J^s of “safe” assets we apply FP allocation rules.

Investment strategies

Constant proportion portfolio insurance

- In CPPI, the proportional exposure in the risky assets follows a rule of the form

$$\pi_{t,j} = \min\left\{l; m \max\{1 - \frac{F_t}{w_t}, 0\}\right\},$$
$$h_t = \pi_t w_t,$$

  where the “floor” F_t represents the time t value of a claim that should be paid in the future and $m > 0$ gives the fraction invested in risky assets of the excess of wealth over the floor.

- The parameter l can be used to limit the maximum proportional exposure to a given upper bound.

Optimization problem

- Given an initial capital w_0 can be diversified among different strategies in order to better suit the risk preferences of the owner.
- The overall strategy obtained with diversification will cover the claims so one is free to search for an optimal diversification.
- Diversifying among parametric classes of investment strategies may produce new strategies which do not belong to the original parametric classes.[1]

1. M. Koivu and T. Pennanen. Galerkin methods in dynamic stochastic programming. Optimization, to appear.

Optimization problem

The problem of diversifying among a finite set $\{h^i \mid i \in I\}$ of funds with varying investment strategies can be written as

$$\underset{\alpha \in X}{\text{minimize}} \quad \rho\Big(\sum_{i \in I} \alpha^i w_T^i\Big),$$

where w_T^i is the terminal value of a wealth process w^i obtained by following strategy $i \in I$,

$$X = \{\alpha \in \mathbb{R}_+^I \mid \sum_{i \in I} \alpha^i = 1\}$$

and ρ is a *convex risk measure* that quantifies the preferences of the decision maker over random terminal wealth distributions.

Optimization problem

- Several choices of ρ may be considered.
- We will concentrate on the Conditional Value at Risk ($CV@R$) which is particularly convenient in the optimization context[1].
- $CV@R$ at confidence level δ is defined as the conditional expectation $CV@R_\delta(w_T) = -E[w_T \,|\, w_T \leq -V@R_\delta(w_T)]$

1. R. T. Rockafellar and S.P. Uryasev. Optimization of Conditional Value-at-Risk. Journal of Risk, 2:21–42, 2000.

Optimization problem

- The problem of optimal diversification with respect to CV@R can be written as

$$\underset{\alpha \in X, \gamma}{\text{minimize}} \quad E\left[\gamma - \frac{1}{1-\delta} \min\{\sum_{i \in I} \alpha^i w_T^i + \gamma, 0\}\right].$$

- The problem thus becomes that of minimizing a convex expectation function over a finite number of variables.
- Mathematically, it is close to the classical problem of maximizing the expected utility in a one period setting.
- The problem is easy to solve numerically.

Computational procedure

The solution procedure can be summarized as follows:

1. Generate N scenarios of asset returns R_t and claims c_t over $t = 1, \ldots, T$.
2. Evaluate each basic strategy $i \in I$ along each of the scenarios $k = 1, \ldots, N$ and record the corresponding terminal wealth $w_T^{i,k}$.
3. Solve the optimization problem

$$\underset{\alpha \in X, \gamma}{\text{minimize}} \quad \frac{1}{N} \sum_{k=1}^{N} \left[\gamma - \frac{1}{1-\delta} \min\{ \sum_{i \in I} \alpha^i w_T^{i,k} + \gamma, 0 \} \right]$$

   for the optimal diversification weights α^i.

Case study: pension fund management

Consider a closed pension fund whose aim is to cover its accrued pension liabilities with given initial capital.

- The pension claims are of the defined benefit type and they depend on the wage and consumer price indices.
- According to the "current" mortality tables, all the liabilities will be amortized in 82 years.

Case study: pension fund management

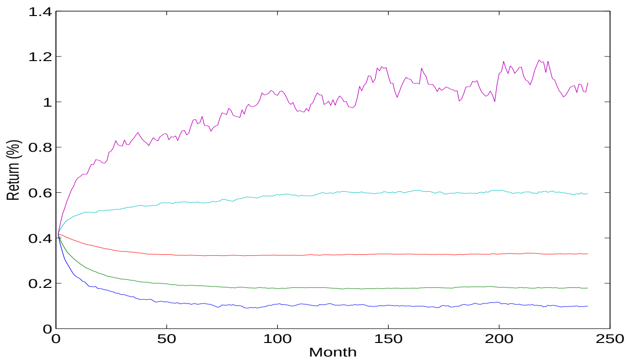
The set J of primitive assets consists of

1. Euro area money market,
2. Euro area government bonds,
3. Euro area equity,
4. US equity,
5. Euro area real estate.

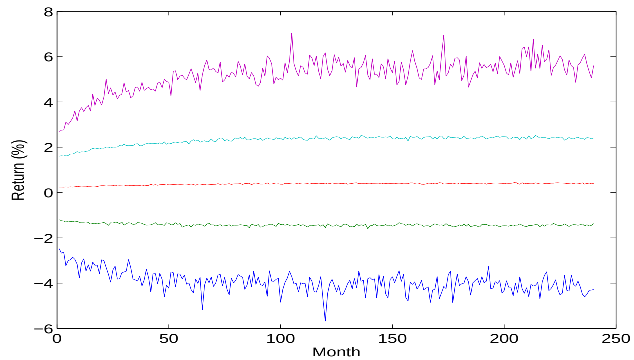
- These are the assets in which the individual funds invest.
- The assets may be viewed as investment funds themselves.
- We will model the asset returns, the wage and consumer price indices with a Vector Equilibrium Correction-model augmented with GARCH innovations.

Case study: market model

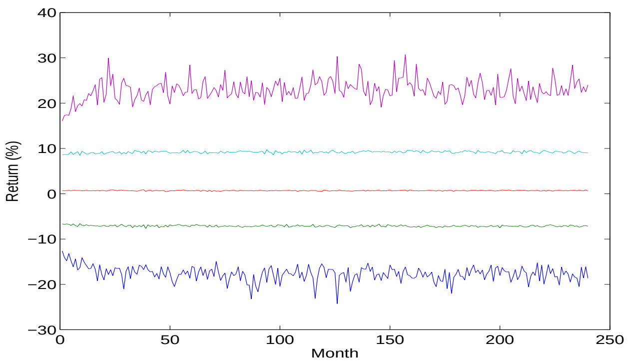
Evolution of the 0.1%, 5%, 50%, 95% and 99.9% percentiles of monthly asset return distributions over twenty years.



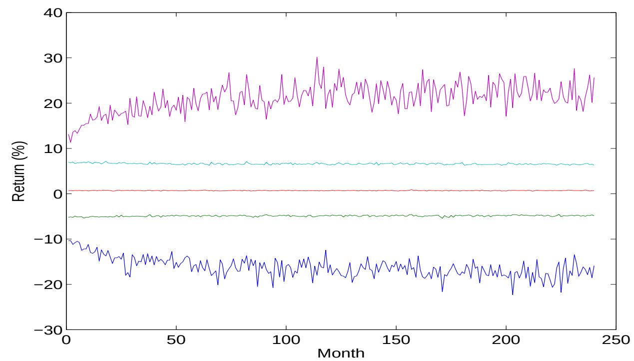
(a) Money market fund



(b) Bond fund



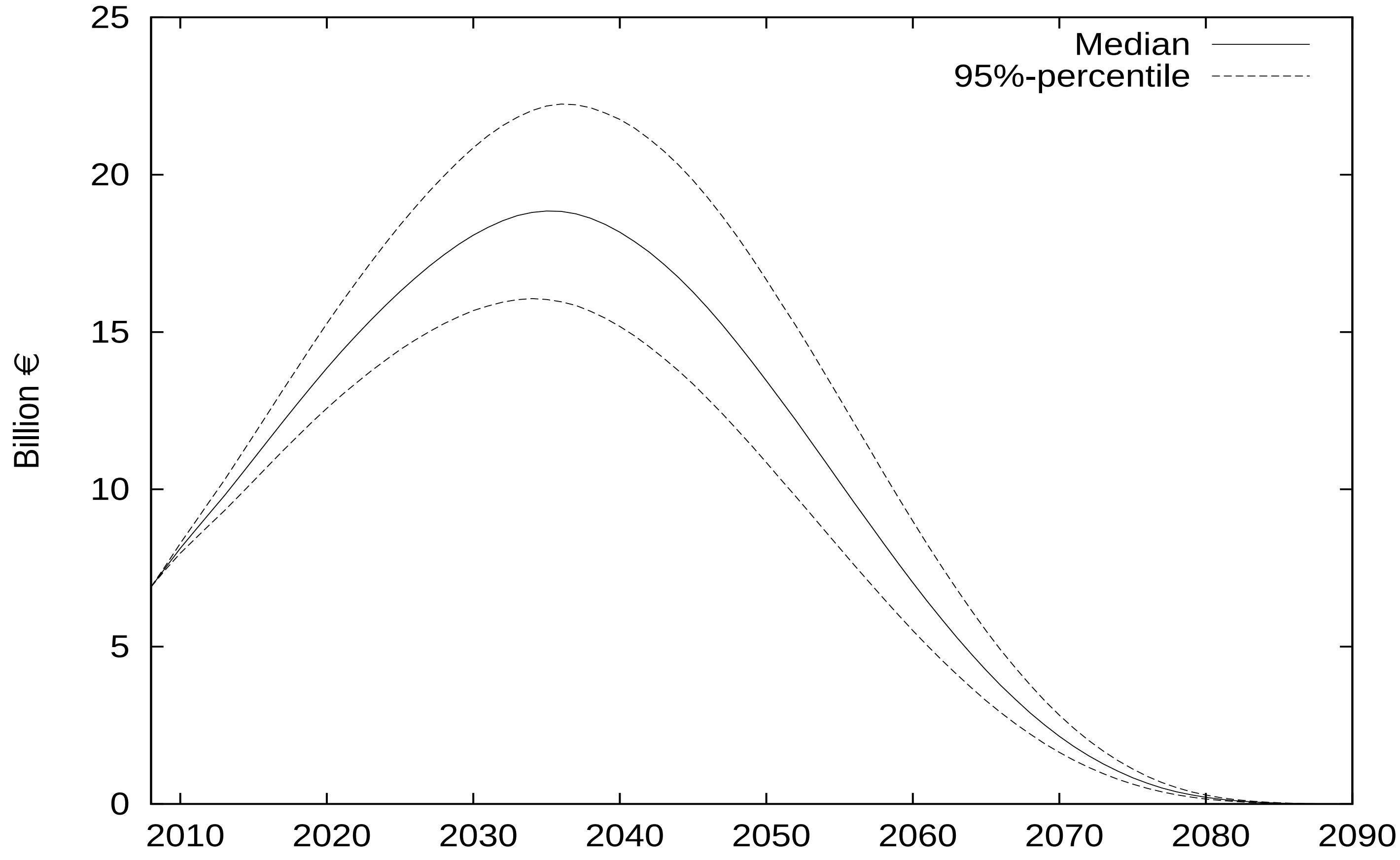
(c) Euro area equity fund



(d) US equity fund

Case study: liabilities

Median and 95% confidence interval of the projected pension expenditure



Case study: investment funds

- We use 5 BH strategies.
- We use 11 FP strategies with varying parameters π.
- We use 20 TDF strategies with varying values for a, b.
- We use 40 CPPI strategies with varying values for the multiplier m and the deterministic discount factor r in the definition of the floor

$$F_T = 0,$$
$$F_t = (1+r)F_{t-1} - \bar{c}_t \quad t = 0, \ldots, T,$$

where $\bar{c}_t$ is the median of the claim amount at time t. This corresponds to the traditional actuarial definition of "technical reserves" for an insurance portfolio.

Case study: results

- We computed an optimal diversification over the above funds given an initial capital w_0 = 225 billion euros.
- We formulated the optimization problem as a linear programming problem with 20000 scenarios, 20077 variables and 20001 constraints.
- The LP was solved with MOSEK interior point solver in 30 seconds.
- The following results (CV@R-values) were computed on an independent set of 100000 scenarios.

Case study: results

Optimally constructed fund of funds

Weight (%)	Type	Parameters	$CV@R_{97.5\%}$ (billion €)
66.5	BH	Bonds	1569
2.9	BH	Euro Equity	6567
10.4	BH	US Equity	5041
2.2	FP	$m = 0.8$	3324
3.9	CPPI	$m = 1, r = 4\%, l = 1$	1420
9.9	CPPI	$m = 2, r = 4\%, l = 1$	1907
4.2	CPPI	$m = 2.0, r = 5\%, l = 1$	2417

- CV@R of the optimally constructed fund of funds is 251
- CV@R of the best individual fund is 1020, which is 300% riskier than the optimal diversification
- This fund is not included in the optimal fund of funds

Case study: results

Optimal initial allocation in the primitive assets