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QUASI-EXACT NUMERICAL EVALUATION OF SYNTHETIC CDO PRICES

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AGENDA

- **Sub-prime mortgage crisis**
  - **Facts & aftermath / Wikipedia / New York Times**
  - **Lessons / nature & magnitude of credit risk transfer**
- **Pricing model**
  - **synthetic CDO tranche / fair spread / one-factor copula model / one-factor Gaussian copula model**
- **Evaluation**
  - **conditional tranche loss / pseudo compound Poisson approximation / Panjer recursive algorithm / Examples**

SUB-PRIME MORTGAGE CRISIS

- **Facts & aftermath**
  - Wikipedia
    - Merrill Lynch's 2008 large losses partly due to drop in value of un-hedged portfolio of CDO's after AIG ceased offering CDS's on Merrill's CDO's
  - New York Times (May 14, 2009)
    - "New Regulations Sought on Derivatives": swaps and other derivatives should be traded on exchanges and backed by capital reserves (T.F. Geithner from Obama administration)

SUB-PRIME MORTGAGE CRISIS

- **Lessons learnt from the crisis**
  - Nature of credit risk transfer
    - Risk characteristics of CDO‘s (Gibson(2004))
  - Magnitude of credit risk transfer
    - Loss distributions of synthetic CDO‘s
    - Evaluation methods: analytical, semi-analytical, Monte-Carlo simulation, exact calculation
    - New quasi-exact numerical method: recursive algorithm for pseudo compound Poisson approximation (Hipp, Hürlimann, Michel)

Pricing model / synthetic CDO tranche

- synthetic collateralized debt obligation (CDO) transfers the credit risk on a portfolio of credit default swaps (CDS).
- synthetic CDO pool made up of m *names* with *notional values* $N(k)$ and *recovery rate* $R(k)$, $k=1,\ldots,m$. The *loss-given-default* of name k is $LGD(k)=N(k)\cdot\{1-R(k)\}$.
- *premium dates*: $0=t(0)<t(1)<\ldots<t(n)=T$ (=*maturity date*)
- *discount factors*: $D(0), D(1),\ldots, D(n)$
- synthetic CDO tranche of *size* S and *attachment point* ℓ has *payoff function* at time $t(i)$: $L(i)=\min\{\max(Lp(i)-\ell,0),S\}$, with $Lp(i)$ the *pool's cumulative losses* up to time $t(i)$ (similar to *limited stop-loss reinsurance* in actuarial science)

Pricing model / fair spread

- Let s be the *fair spread* of CDO tranche per annum
- *default leg* (=losses to maturity): $DL = \sum D(i) \cdot \{L(i) - L(i-1)\}$
- *contingent* (=PV of default leg): $PV(DL) = E[DL]$
- *premium leg* (=premiums to maturity)

  $PL = s \cdot \sum D(i) \cdot \Delta(i) \cdot \{S - L(i)\}$, $\Delta(i) = t(i) - t(i-1)$
- *fee* (=PV of premium leg): $PV(PL) = E[PL]$
- *market-to-market value* CDO tranche: MV = fee - contingent
- Pricing equation fair spread is $MV = 0$ or $E[PL - DL] = 0$
- Valuation problem reduced to computation

  of $E[L(i)]$ ⇔ specification of default process for each name

  & correlation structure of default events

Pricing model / one-factor model

- T(k) : *random default time* of name k
- q(k,i)=P(T(k)<t(i)) : *risk-neutral default probabilities*
- dependence structure of default times determined by *creditworthiness indices* $Y(k) = \sqrt{\rho(k)}\cdot X + \sqrt{1-\rho(k)}\cdot Z(k)$, with *systematic risk factor* X, mutually independent *idiosyncratic factors* Z(k), all independent of X, and *correlation factors* ρ(k), through one-factor copula model:

  $q(k,i) = P(Y(k) < H(k,t(i)))$ with

  H(k,t(i)) : *default threshold* of name k at time t(i)
- references: Vasicek(1987) (loss distribution of pool of loans) Li(2000), Gordy & Jones(2003), Hull & White(2004), etc.

Pricing model / one-factor Gaussian

- Assume *standard normal* distributions for X and Z(k):

  (i) $H(k,t(i)) = \Phi^{-1}(q(k,i))$

  (ii) $Cov[Y(k),Y(j)] = \sqrt{\rho(k)\rho(j)}$

  (iii) $q(k,i) = P(Y(k)<H(k,t(i))|X=x)$ (*conditional default*

  $= \Phi\{[\Phi^{-1}(q(k,i)) - x\sqrt{\rho(k)}]/\sqrt{1-\rho(k)}\}$ *probabilities*)

- Model extensions:
- X random vector => multi-factor copula model
- X , Z(k) Student-t => double-t copula model of Hull & White
- X , Z(k) NIG => Kalemanova et al.(2005) & Lüscher(2005)
- Further generalizations: Burtschell et al.(2005), Bee(2007) and Albrecher et al.(2007).

Evaluation / conditional tranche loss

- In one-factor Gaussian model one has

  $E[\,L(i)\,] = \int E[\,L(i) \mid X=x\,] \cdot d\Phi(x)$ with

  $E[\,L(i) \mid X=x\,] = E[\,L(i)=\min\{\max(Lp(i)-\ell,0),S\} \mid X=x\,]$ and

  $Lp(i) = \sum LGD(k) \cdot I\{\,Y(k) < \Phi^{-1}(q(k,i))\,\}$, where the random indicators $I\{\cdot\}$ are mutually independent conditional on X

  => reduction to computation of conditional expected cumulative tranche losses $E[\,L(i) \mid X=x\,]$, $i=1,\ldots,n$
- Random sum $Lp(i)$ similar to "individual model of risk theory" in actuarial science studied by Kornya(1983), Hipp(1986), De Pril(1986/89), Dhaene and De Pril(1994), Hürlimann(1989/90/2004), Sundt and Vernic(2009), etc.

Evaluation / pseudo comp. Poisson appr.

- conditional on $X=x$ the characteristic function of $Lp(i)$ is
  $\varphi(t) = \prod \varphi(k,t)$, $\varphi(k,t) = \exp\{ \ln[1+c(k)\cdot(e^{it\cdot LGD(k)}-1)] \}$
- J-th order appr. $\varphi(J,t)$: truncate logarithmic expansion
  $\ln(1+x) = \sum (-1)^{j+1}\cdot x^j/j$ at J-th term to get finite sums
  $\ln \varphi(J,t) = \sum \sum (-1)^{j+1}\cdot[c(k)\cdot(e^{it\cdot LGD(k)}-1)]^j/j$
- **J=1: compound Poisson approximation (λ(1), h(1,y))**
  $\ln \varphi(1,t) = \lambda(1)\cdot[\psi(1,t)-1]$, $\psi(1,t) = 1/\lambda(1)\cdot\sum c(k)\cdot e^{it\cdot LGD(k)}$
  $\lambda(1) = \sum c(k)$, $h(1,y) = 1/\lambda(1)\cdot \sum c(k)\cdot I\{LGD(k)=y\}$, $y=1,2,\ldots$
- **J>1: pseudo compound Poisson approximation**
  $\ln \varphi(J,t) = \lambda(J)\cdot[\psi(J,t)-1]$, Poisson parameter $\lambda(J)$ and
  pseudo severity distribution $h(J,y)$ determined as follows:

Evaluation / pseudo comp. Poisson appr.

pseudo compound Poisson approximations of low order:

J=2: $\lambda(2) = \sum c(k)\cdot(1+\tfrac{1}{2}\cdot c(k))$

$$h(2,y) = 1/\lambda(2)\cdot \left[\sum c(k)(1+c(k))\cdot I\{LGD(k)=y\} - \tfrac{1}{2} \sum c(k)^2\cdot I\{2\cdot LGD(k)=y\} \right],\quad y=1,2,\ldots$$

J=3: $\lambda(3) = \sum c(k)\cdot(1+\tfrac{1}{2}\cdot c(k)+\tfrac{1}{3}\cdot c(k)^2)$

$$h(3,y) = 1/\lambda(3)\cdot \left[\sum c(k)(1+c(k)+c(k)^2)\cdot I\{LGD(k)=y\} - \sum c(k)^2(\tfrac{1}{2}+c(k))\cdot I\{2\cdot LGD(k)=y\} + \tfrac{1}{3} \sum c(k)^3\cdot I\{3\cdot LGD(k)=y\} \right],$$

J=4: $\lambda(4) = \sum c(k)\cdot(1+\tfrac{1}{2}\cdot c(k)+\tfrac{1}{3}\cdot c(k)^2+\tfrac{1}{4}\cdot c(k)^3)$

$$h(4,y) = 1/\lambda(4)\cdot \left[\sum c(k)(1+c(k)+c(k)^2+c(k)^3)\cdot I\{LGD(k)=y\} - \sum c(k)^2(\tfrac{1}{2}+c(k)+\tfrac{1}{2}\cdot 3\cdot c(k)^2)\cdot I\{2\cdot LGD(k)=y\} + \sum c(k)^3(\tfrac{1}{3}+c(k))\cdot I\{3\cdot LGD(k)=y\} - \tfrac{1}{4} \sum c(k)^4\cdot I\{4\cdot LGD(k)=y\} \right],$$

Evaluation / pseudo comp. Poisson appr.

- **some mathematical properties:**
- **h(J,y) for J>1** is not a true probability measure but **only a signed measure**
- **conditions for** a pseudo compound Poisson distribution

  (λ, h(y)) to be **a true probability distribution** identified by Lévy(1937) (see Lukacs(1970), Johnson et al.(1992)):

  a negative value h(y)<0 is preceded by a positive value and followed by at least two positive values

  CDO examples: criterion is fulfilled for J=3 but not for J=2,4
- **solution to moment problem:** J-th order approximation preserves first J moments (Dhaene et al.(1996))

Evaluation / Panjer recursive algorithm

- probability function $f(z)$, $z=0,1,2,\ldots$ of a pseudo compound Poisson distribution $(\lambda, h(y))$ satisfies **Panjer recursion formula** (Hürlimann(1990)):

  $f(0) = \exp(-\lambda)$, $z \cdot f(z) = \sum y \cdot h(y) \cdot f(z-y)$, $z=1,2,\ldots$
- numerical stable algorithm (Panjer & Wang(1993))
- convergence of J-th order approximation in distribution obtained from the error bound (Hipp & Michel(1990)):

  $|F(z) - F^{(J)}(z)| \le \exp(\varepsilon) - 1$, $\varepsilon = \sum \varepsilon(k)$, with

  $\varepsilon(k) = [1/(J-1)] \cdot [2c(k)]^{(J-1)} / [1-2c(k)]$, $c(k)<1/2$

Evaluation / Numerical examples

- **Example 1**: **completely homogenous pool**

  m=100 names, t(i)=i=1,2,3,4,5 dates, LGD(k)=1 all k, risk-neutral default probabilities $q(k,i)=q(i)=1-\exp(-0.01\cdot i)$, correlation factors $\rho(k)=\rho=0.3$, flat interest rates of 5%

  The cumulative losses Lp(i) are binomially distributed and converge to the Vasicek limiting distribution as $m \to \infty$

- **Table 1**: **par spreads** for completely homogeneous pool

	par spread for different distributions					
CDO tranches	J=1	J=2	J=3	J=4	exact	Vasicek
equity	21.794%	21.875%	21.876%	21.876%	21.876%	26.095%
mezzanine	6.004%	6.024%	6.024%	6.024%	6.024%	6.488%
senior	0.271%	0.269%	0.269%	0.269%	0.269%	0.201%

Evaluation / Numerical examples

- **Example 2**: **inhomogenous pool**

  5 sub-pools with 20 names each with LGD(k)=k=1,2,3,4,5, t(i)=i=1,2,3,4,5 dates, $q(k,i)=1-\exp\{-(0.005+0.005\cdot k)\cdot i\}$, correlation factors $\rho(k)=0.25+0.05\cdot k$, flat interest rates of 5%

- **Table 2**: **par spreads** for inhomogeneous pool

	par spread for different distributions			
CDO tranches	J=1	J=2	J=3	J=4
equity	25.954%	26.087%	26.091%	26.091%
mezzanine	11.002%	11.078%	11.080%	11.080%
senior	3.002%	3.060%	3.076%	3.082%

- **Conclusions**:

  J=3,4 => **quasi-exact spreads** for synthetic CDO tranches

  J=2 => **almost accurate spreads**

  J=1 => compound Poisson approximation **not reliable**