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Interest Rate Guarantee in Defined Benefit Pension

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Background

- The Norwegian pension system
  - defined benefit
  - interest rate guarantee
- Motivation for pricing the interest rate guarantee
  - The new legislation requires pricing for all risk-components
    - Actuarial risk
    - Financial risk
    - Administration risk
  - The new legislation requires up front paying of the interest rate guarantee $\longrightarrow$ put option

Notation

V_t	- Premium reserve
$\pi((X-1)T, XT)$	- Premium
$\beta((X-1)T, XT)$	- Benefit payments
PF_t	- Premium fund
AR_t	- Additional reserve
S_t^c	- Client assets
S_t^b	- Buffer assets
σ_c	- Volatility of client assets
σ_b	- Volatility of buffer assets
τ	- Correlation between (log-) assets
ρ	- Risk free interest rate
r	- Guaranteed interest rate

Model for the assets

Client assets and buffer assets are given by

$$dS_t^c = \mu^c S_t^c \, dt + \sigma^c S_t^c \, dW_t^c, \quad \begin{cases} S_0^c = V_0 + PF_0 + AR_0 + E[\pi(0,T)] \\ S_T^c = S_{T_-}^c - \beta(0,T)(1+\rho)^{T/2}, \end{cases}$$

$$dS_t^b = \mu^b S_t^b \, dt + \sigma^b S_t^b \, dW_t^b, \quad S_0^b \text{ is from last year balance sheet EOY.}$$

Where W_t^c and W_t^b are correlated Brownian motions with correlation factor τ.

Model for the guarantee

The return on client assets need to cover at least the following amount:

$$K = (V_0 + PF_0)((1+r)^T - 1) + \int_0^T ((1+r)^{T-t} - 1)\, d(V_t + PF_t).$$

This may be approximated by a discrete Thiele on the form

$$\begin{aligned} K \approx\ & (V_0 + PF_0)((1+r)^T - 1) \\ & + (\pi(0,T) + PF_T - PF_0)((1+r)^T - 1) \\ & - \beta(0,T)((1+r)^{\frac{T}{2}} - 1). \end{aligned}$$

Structure of the option

Hence we end up with an option payoff of the form

$$((K - AR_T - \alpha S_T^b) - (S_{T-}^c - S_0^c - \beta(0,T)((1+\rho)^{T/2} - 1)))^+$$
$$= (K^*(\pi(0,T), \beta(0,T)) - (S_{T-}^c + \alpha S_T^b))^+$$

This leads to a basket option with stochastic strike:

$$\Pi_t^T = e^{-\rho(T-t)} E_Q[(K^*(\pi(0,T), \beta(0,T)) - (S_T^c + \alpha S_T^b))^+ | \mathcal{F}_t],$$

where the (B&S) dynamics of the assets under Q are given by

$$dS_t^c = \rho S_t^c \, dt + \sigma^c S_t^c \, dB_t^c,$$
$$dS_t^b = \rho S_t^b \, dt + \sigma^b S_t^b \, dB_t^b.$$

One underlying asset and stochastic strike

Proposition

Let $S_T = S_0 \exp((\rho - \frac{1}{2}\sigma_S^2)T + \sigma_s\sqrt{T}X)$ *and* $K = \mu_K T + \sigma_K\sqrt{T}Y$. *Then price of an European put option at time* $t = 0$ *with normally distributed strike,* K*, and maturity* T *is given by*

$$\begin{aligned}\Pi_0^T =& e^{-\rho T}E_Q[(K - S_T)^+] \\ =& e^{-\rho T}\mu_K T\Phi(X \le d(Y), Y \ge -a) + e^{-\rho T}\frac{\sigma_K\sqrt{T}}{\sqrt{2\pi}}\int_{-a}^{\infty} Y\Phi(d(Y))e^{-\frac{1}{2}Y^2}\,dY \\ &- S_0\Phi(X \le d(Y) + \sigma_S\sqrt{T}, Y \ge -a)\end{aligned}$$

where

$$d(Y) = \frac{1}{\sigma_S\sqrt{T}}\Big(\ln\big(\frac{\mu_K T + \sigma_K\sqrt{T}Y}{S_0}\big) - (\rho - \frac{1}{2}\sigma_S^2)T\Big), \qquad a = \frac{\mu_K}{\sigma_K}\sqrt{T}.$$

X *and* Y *are independent standard normal variables and* Φ *is the cumulative bivariate normal distribution.*

Two underlying assets and fixed strike

Proposition

Assume that the strike, K, is a given constant and that the two underlying assets, S_t^c and S_t^b, are given by the B&S risk neutral measure Q. Then the unique price of the option at time $t = 0$ with maturity T can be expressed as:

$$\begin{aligned}\Pi_0^T =& Ke^{-\rho T}\Phi(Y_1 \leq a, Y_2 \leq d(Y_1)) - S_0^c\Phi(Y_1 \leq a - \sigma_c\sqrt{T}, Y_2 \leq d(Y_1 + \sigma_c\sqrt{T})) \\ &- \alpha S_0^b\Phi(Y_1 \leq a - \sigma_b\tau\sqrt{T}, Y_2 \leq d(Y_1 + \sigma_b\tau\sqrt{T}) - \sigma_b\sqrt{1-\tau^2}\sqrt{T})\end{aligned}$$

where Y_1 and Y_2 are independent standard normal variables, Φ is the cumulative bivariate normal distribution and

$$d(Y_1) = \frac{\left(\ln\left(K - S_0^c e^{\sigma_c\sqrt{T}Y_1 + T(\rho - \frac{1}{2}\sigma_c^2)}\right) - \ln(\alpha S_0^b) - T(\rho - \frac{1}{2}\sigma_b^2) - \sigma_b\tau\sqrt{T}Y_1\right)}{\sigma_b\sqrt{1-\tau^2}\sqrt{T}},$$

$$a = \frac{1}{\sigma_c\sqrt{T}}\left(\ln\left(\frac{K}{S_0^c}\right) - (\rho - \frac{1}{2}\sigma_c^2)T\right).$$

Initial parameter set

$$V_0 = 100, \quad PF_i = 10 \text{ for all } i, \quad AR_i = 5 \text{ for all } i,$$

$$S^c_{0-} = V_0 + PF_0 + TA, \quad S^b_0 = 10, \quad \sigma_c = \frac{0.1}{\sqrt{T}}, \quad \sigma_b = \frac{0.1}{\sqrt{T}}, \quad \tau = 0.5,$$

$$T = 252, \quad N = 1, \quad \frac{\sigma_\pi}{\sqrt{n}} = 0.1, \quad \frac{\sigma_\beta}{\sqrt{n}} = 0.05, \quad r = 0.03, \quad \rho = 0.03,$$

$$E[\pi((X-1)T, XT)] = 10 \text{ for all } X, \quad E[\beta((X-1)T, XT)] = 5 \text{ for all } X.$$

Numerical examples

Parameters	Option prices
$PF_i = \{-10, -5, 0, 5, 10\}$	**0.33, 0.40, 0.48, 0.57, 0.66**
$TA = \{0, 5, 10\}$	1.36, 0.66, 0.29
$S_0^b = \{5, 10\}$	1.43, 0.66
$\rho = \{0.02, 0.03, 0.04\}$	0.81, 0.66, 0.54
$\sigma_c = \{\frac{0.05}{\sqrt{T}}, \frac{0.10}{\sqrt{T}}, \frac{0.15}{\sqrt{T}}\}$	**0.02, 0.66, 2.08**
$\sigma_b = \{\frac{0.05}{\sqrt{T}}, \frac{0.10}{\sqrt{T}}, \frac{0.15}{\sqrt{T}}\}$	0.61, 0.66, 0.71
$\tau = \{-0.5, 0, 0.5, 1\}$	0.47, 0.57, 0.66, 0.75
$N = \{1, 3, 5, 10\}$	**0.66, 3.26, 5.78, 11.52**
$E[\pi((X-1)T, XT)] = \{10, 15\}$	0.66, 0.76
$E[\beta((X-1)T, XT)] = \{5, 10\}$	0.66, 0.66
$\frac{\sigma_\pi}{\sqrt{n}} = \{0.1, 0.15\}$	0.66, 0.66
$\frac{\sigma_\beta}{n} = \{0.05, 0.10\}$	0.66, 0.66

Asset model in incomplete market

The assets in an incomplete market are given by

$$\begin{aligned} dS_t^c &= S_{t-}^c \Big\{ \mu^c(t)dt + \sigma^c(t)\tau_1 dW_t^{(1)} + \sigma^c(t)\sqrt{1-\tau_1^2}dW_t^{(2)} \\ &\quad + \int_{\mathbb{R}_0} \gamma_1^c(t,z)\tau_2 \widetilde{N}_1(dt,dz) + \int_{\mathbb{R}_0} \gamma_2^c(t,z)\sqrt{1-\tau_2^2}\widetilde{N}_2(dt,dz) \Big\} \end{aligned}$$

$$dS_t^b = S_{t-}^b \left\{ \mu^b(t)dt + \sigma^b(t)dW_t^{(1)} + \int_{\mathbb{R}_0} \gamma_1^b(t,z)\widetilde{N}_1(dt,dz) \right\},$$

where $W_t^{(i)}$, $i = 1,2$ are independent standard Brownian motions and $\widetilde{N}_i(dt,dz) = N_i(dt,dz) - \nu_i(dz)dt$, $i = 1,2$ independent compensated Poisson random measures with Lévy measures ν_i on $\mathbb{R}_0 := \mathbb{R} \setminus \{0\}$, $i = 1,2$. Further $\tau_i \in [0,1]$, $i = 1,2$ are correlation parameters and $\mu^c(t), \mu^b(t), \sigma_t^c(t), \sigma_t^b(t), \gamma_1^b(t,z), \gamma_i^c(t,z), i = 1,2$ are predictable processes.

Principle of risk indifference price

The insurance company is risk indifferent to entering the market on its own or entering the market after having given the interest rate guarantee:

$$\Phi_G(x+p) = \Phi_0(x).$$

Here p is the price of the guarantee. Further

$$\Phi_0(x) = \inf_{\pi \in \mathcal{P}} \rho(X_x^{(\pi)}(T))$$

and

$$\Phi_G(x+p) = \inf_{\pi \in \mathcal{P}} \rho(X_x^{(\pi)}(T) - G). \tag{1}$$

where $\rho(\cdot)$ is a convex risk measure, $G = g(S_t^c, S_t^b)$ is the interest guarantee claim and $X_x^{(\pi)}(t)$ is a self financing portfolio of a risk free asset and the assets S_t^c, S_t^b. Here, $\rho(\cdot)$ is given by a "worst scenario" risk measure which gives the largest risk neutral price.

Risk indifference price

The price based on Equation (1) can be found by a maximum principle. As a special case this price turns out to be

$$p = p_{risk}^{seller} = \sup_{Q \in \mathcal{L}} E_Q[G], \tag{2}$$

where $\mathcal{L}$ is the set of all equivalent martingale measures.

Numerical example I

In our numerical examples we have chosen to estimate p in (2) by using a constant parametric form on the Radon Nikodym derivative when all parameters are constant. I.e. choose the admissible controls $\theta(t,z) = (\theta_0, \theta_1, \theta_2, \theta_3) \in \mathbb{R}^4$ such that

$$\begin{aligned} dK_\theta(t) &= K_\theta(t^-)\Big[\theta_0(t)dW_t^{(1)} + \theta_1(t)dW_t^{(2)} \\ &\quad + \int_{\mathbb{R}_0} \theta_2(t,z)\widetilde{N}_1(dt,dz) + \int_{\mathbb{R}_0} \theta_3(t,z)\widetilde{N}_2(dt,dz)\Big], \\ K_\theta(0) &= k > 0 \end{aligned}$$

is a martingale, and find the price of the interest rate guarantee by

$$\hat{p} = \max_{\theta \in \mathbb{R}^4} E[K_\theta(1)G].$$

Numerical example II

In our examples we let the jumps be a Poisson process with intensity λ_i and jump size γ_i^c and γ_1^b, $i = 1, 2$. Further we let the base parameters be given by

$$K = 103, \quad S_0^c = 100, \quad S_0^b = 10, \quad \mu_c = 0.06, \quad \mu_b = 0.07,$$
$$\sigma_c = 0.10, \quad \sigma_b = 0.15, \quad \rho = 0, \quad r = 0.03,$$
$$\tau_1 = 0.5, \quad \tau_2 = 0.3, \quad \lambda_1 = 0.5, \quad \lambda_2 = 0.3,$$
$$T = 1, \quad \gamma_1^c = 0.04, \quad \gamma_2^c = 0.04, \quad \gamma_1^b = 0.06.$$

In addition we will put a constraint on θ to be positive.

Numerical example III

Parameters	Risk indifference price	B & S price
$\tau_2 = \{-0.3, 0, 0.3, 0.6\}$	$\{1.63, 1.64, 1.67, 1.68\}$	1.58
$\lambda_1 = \{0.5, 2.0\}$	$\{1.67, 1.72\}$	1.58
$\lambda_2 = \{0.3, 1.2\}$	$\{1.67, 1.81\}$	1.58
$\gamma_1^c = \gamma_2^c = \{0.01, 0.04, 0.08\}$	$\{1.60, 1.67, 1.84\}$	1.58
$\gamma_1^b = \{0.01, 0.06, 0.10\}$	$\{1.65, 1.67, 1.68\}$	1.58

Summary

- We have introduced a model for interest rate guarantees and corresponding assets well adapted to the Norwegian defined benefit system.
- Methods have been presented to price the interest rate guarantee in a complete asset market.
- A corresponding problem has been solved using risk indifferent pricing in incomplete markets.
- We have looked at numerical examples both in complete and incomplete asset markets.

Further details

Further details on the following can be found in the paper

- Analysis of the sensitivity of prices in the complete market case based on different parameters.
- Explanation on the numerical methods applied.
- Discussion on interest rate guarantees over multi periods.
- Derivation of the risk neutral price.
- Description a convex risk measure.
- Introduction of a Hamiltonian and optimum solutions through backward stochastic differential equations.