Pension Fund Management Based on Constrained Consumption-Investment

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Based on work with Holger Kraft

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- Classical and modern asset management and LDI
- Terminal wealth problems with VaR constraints
- Intermediate payments
- Intermediate constraints

1 Classical and modern asset management and LDI

- Classical asset management: mean-variance-optimization, one-period-model

$$\max_{\pi: Var[A] < k} E[A] \text{ versus } \min_{\pi: E[A] > k} Var[A]$$

Figure 1: Harry Markowitz (1927-)

- Modern asset management: utility optimization, dynamic strategies

$$\max_{\pi} E\left[u\left(A\right)\right]$$

Figure 2: Robert C. Merton (1944-   )

- LDI (Liability-Driven Investment) is essentially involving L in the objective and/or the constraint

$$\max_{\pi : A > L} E\left[u\left(A\right)\right]$$
$$\max_{\pi} E\left[u\left(A - L\right)\right]$$
$$\max_{\pi : P(A > L) \geq 1 - \varepsilon} E\left[u\left(A\right)\right]$$
$$\max_{\pi : E^{Q}\left[(L - A)^{+}\right] \leq \varepsilon} E\left[u\left(A\right)\right]$$

- What is the time horizon of the objectives and/or the constraint? And what about intermediate payments and/or constraints?

2 Terminal wealth problems with VaR constraints

$$\max_{\pi : P(A(T) > L(T)) \geq 1-\varepsilon} E\left[u\left(A\left(T\right)\right)\right]$$

- Invest first $A'(0)$ in the optimal portfolio solving

$$\max_{\pi} E\left[u\left(A'(T)\right)\right]$$

- Buy, for the residual amount, $A(0) - A'(0)$ an option with payoff

$$\left(L(T) - A'(T)\right) 1\left[l < A'(T) < L(T)\right]$$

  The two positions sum up to the claim (with value $A(0)$)

$$\max\left(A'(T), L(T) 1\left[l < A'(T)\right]\right)$$

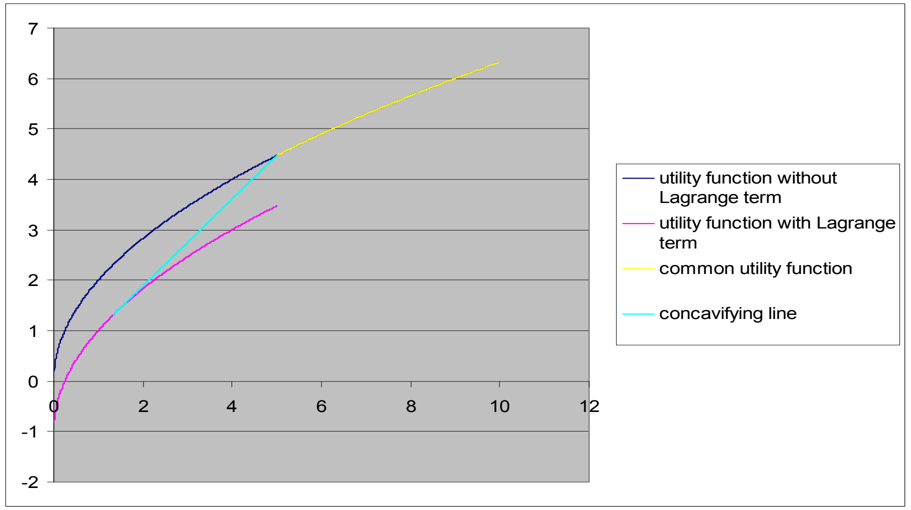

Figure 3: Utility function, auxiliary utility function with Lagrange term, and concavifying line

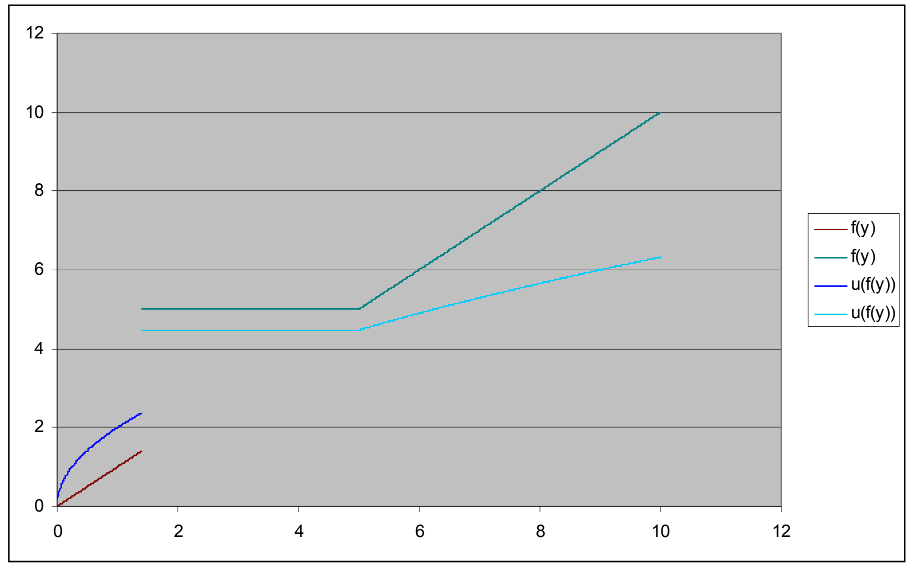

Figure 4: Optimal wealth and utility of optimal wealth as a function of A'

3 Intermediate payments

$A(T)$ finances also a payment of C at time $T_1 < T_2$ (generalizes to n periods):

$$\max_{\substack{\pi : P(C(T_1) > K(T_1)) \geq 1 - \varepsilon_1 \\ P(A(T_2) > L(T_2)) \geq 1 - \varepsilon_2}} E\left[u\left(C\left(T_1\right)\right) + u\left(A\left(T_2\right)\right)\right]$$

- Allocate the initial capital A to the two 'projects': $A^1(0) + A^2(0)$
- Solve each of the two terminal value problems separately

- Calculate the allocation $A^1(0)$ such that

$$\begin{aligned} & \frac{\partial}{\partial A^1(0)} \max_{\pi : P\left(A^1(T_1) > K(T_1)\right) \geq 1-\varepsilon_1} E\left[u\left(A^1(T_1)\right)\right] \\ = \; & \frac{\partial}{\partial A^2(0)} \max_{\pi : P\left(A^2(T_2) > L(T_2)\right) \geq 1-\varepsilon_2} E\left[u\left(A^2(T_2)\right)\right] \end{aligned}$$

4 Intermediate constraints

A exceeds L at time T_1 and time $T_2 > T_1$ (generalizes to n periods):

$$\max_{\substack{\pi : P(A(T_1) > L(T_1)) > 1-\varepsilon_1 \\ P(A(T_2) > L(T_2)) > 1-\varepsilon_2}} E\left[u\left(A\left(T_2\right)\right)\right]$$

- Invest first $A''(0)$ in the optimal portfolio solving

$$\max_{\pi} E\left[u\left(A''\left(T_2\right)\right)\right]$$

- Buy, for the amount $A'(0) - A''(0)$ a put option leading to the total time T_2 payoff

$$\max\left(A''\left(T_2\right), L\left(T_2\right) 1\left[l_2 < A''\left(T_2\right)\right]\right)$$

with value process $A'(t)$

- Buy, for the residual amount $A(0) - A'(0)$ a put option leading to the total time T_1 payoff

$$\max\left(A'(T_1), L(T_1) 1\left[l_1 < A''(T_2)\right]\right)$$

with value $A(0)$