On the Pricing of Inflation-Indexed Caplets

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Outline

Basic Principles on Inflation

Inflation can result from

- an inequilibrium between the macroeconomic supply and demand,
- a rise in prices in foreign countries,
- rising costs,
- increasing government debt.

$\Rightarrow$ The monetary value falls leading to a loss of purchase power in the nominal currency.

Inflation is measured as the percentage increment of a consumer or retail price index which is the value of a basket of goods and services.

We denote the index' value at time t by $I(t)$ and therefore we can express the overall inflation rate over some time interval $[t, T]$ as

$$i(t, T) = \frac{I(T) - I(t)}{I(t)}.$$

Basic Principles on Inflation

The **Fisher Equation**

$$r_N(t,T) = r_R(t,T) + \mathbb{E}[i(t,T)]$$

explains the relation between the nominal interest rates, the real interest rates and the expected inflation rate over a certain time period from t to T.

Here $r_N(t,T)$ and $r_R(t,T)$ denote the cumulative interest rates over the time interval $[t,T]$.

Hence for annual continuous interest and inflation rates the Fisher equation rewrites

$$e^{(r_N - r_R)(T-t)} = \mathbb{E}\left[\frac{I(T)}{I(t)}\right].$$

Inflation-linked Bonds and Inflation Swaps

Inflation-linked bonds offer cash flows that are either

- directly linked to the development of the inflation index from its issue date t_0 up to the payment dates t_i (so-called "Canadian model"):

$$\textit{Nominal Cash Flow} = \textit{Real Cash Flow} \cdot \frac{I(t_i)}{I(t_0)}$$

  Some inflation-linked government bonds include a deflation protection of the invested nominal such that the redemption value is floored by the face value of the bond.

  Issuing countries are the U.K., U.S., Canada, Sweden, France, Greece, Italy, Germany, Japan, Australia, South Africa,...

- directly linked to the future inflation rate over a time interval $[t_{i-1}, t_i]$

$$\textit{Nominal Cash Flow} = \textit{Real Cash Flow} \cdot (1 + i(t_{i-1}, t_i))$$

Inflation Swaps

- **Zero coupon inflation swaps** are swaps where the two counterparties exchange a fixed nominal coupon against the overall inflation rate up to maturity.

  $\Rightarrow$ with our without deflation protection

- **Inflation Rate Swaps** are swaps where the two counterparties exchange a fixed nominal coupon against the future inflation rate payed over fixed time intervalls similar to a plain vanilla interest rate swap.

  $\Rightarrow$ with our without deflation protection

Inflation Options

- **Options on the overall inflation rate** securitize the right of a compensation payment, if the inflation rate over a fixed time interval from t_0 to T exceeds a pre-specified level, where t_0 is some time before or equal to the issue date of the option. Payoff at maturity:

$$\max(i(t_0, T) - K; 0) \cdot N \quad or \quad \max(K - i(t_0, T); 0) \cdot N$$

- **Options on the future inflation rate** securitize the right of a compensation payment, if the future inflation rate over a future time interval from t^* to T exceeds a pre-specified level, where t^* is some time in the future. Payoff at maturity:

$$\max(i(t^*, T) - K; 0) \cdot N \quad or \quad \max(K - i(t^*, T); 0) \cdot N$$

Inflation Caps and Inflation Floors usually occur as implicit options and equal these two types of inflation options.

Analogy to Forward Starting Options

We suppose we have an asset A with price $A(t)$ at time t.

Forward starting options are path-dependent options on a certain change of the underlying A over a future time interval:

- on the absolute performance of A with payoff structure

$$C_{FWS}(T) = (A(T) - k \cdot A(t^*))^+ \quad \text{for some } 0 < \mathrm{t}^* < \mathrm{T}$$

- on the return of A with payoff structure

$$C_{RFWS}(T) = \left(\frac{A(T)}{A(t^*)} - K\right)^+ = \left(\frac{A(T) - A(t^*)}{A(t^*)} - (K-1)\right)^+ \quad \text{for some } 0 < \mathrm{t}^* < \mathrm{T}$$

The prices of forward starting options are influenced by the future price behavior and thereby by today's expectations on the future volatility.

Outline

FWS on the Return of an Underlying

Rubinstein (1991)

Theorem

We consider a Black-Scholes framework where the dynamics of the asset price $A(t)$ under the equivalent martingale measure is given by

$$dA(t) = (r - d)A(t)dt + \sigma A(t)dW(t)$$

with constant volatility σ, interest rate r and dividend yield d while $W(t)$ denotes a Brownian motion.
The price of a forward starting option with payoff $(A(T)/A(t^*) - K)^+$ at time t prior to strike determination at t^* is then given by

$$C_{RFWS}(t) = e^{-d(T-t^*)}e^{-r(t^*-t)}N(d(t^*)) - Ke^{-r(T-t)}N(d(t^*) - \sigma\sqrt{T-t^*})$$

with

$$d(t^*) = \frac{-\ln(K) + (r - d + \frac{1}{2}\sigma^2)(T - t^*)}{\sigma\sqrt{T-t^*}}.$$

Outline

Heston's Stochastic Volatility Model

Heston (1993)

Let $(\Omega, \mathcal{F}, Q)$ be a probability space with the natural filtration of market information $(\mathcal{F}_t)_{t\geq 0}$. We assume the following stochastic differential equations for the asset price $A(t)$

$$dA(t) = (r-d)A(t)dt + \sqrt{\nu(t)}A(t)dW_A(t)$$

and the mean-reverting variance process $\nu(t)$

$$d\nu(t) = \kappa\left(\theta - \nu(t)\right)dt + \sigma\sqrt{\nu(t)}dW_\nu(t)$$

where r denotes the constant, risk free interest rate, d is the constant dividend yield, $\kappa \geq 0$ is the mean reversion speed, θ the mean reversion level, $\sigma\sqrt{\nu(t)}$ the volatility of volatility and W_S, W_ν denote Brownian motions with correlation ρ.
To ensure that the variance process $\nu(t)$ remains positive, one assumes that the stability condition $2\kappa\theta > \sigma^2$ holds.

European Option Prices in the Heston Model

Heston (1993)

Theorem

We assume a Heston framework as before. The price of a plain vanilla call at time t with maturity T, $t < T$, and payoff $(A(T) - K)^+$ can be calculated as

$$\begin{aligned} C(t; A(t); \nu(t)) =& e^{-d(T-t)} A(t) P_1 \left(t; e^{-d(T-t)} A(t); \nu(t); K\right) \\ & - K e^{-r(T-t)} P_2 \left(t; e^{-d(T-t)} A(t); \nu(t); K\right) \end{aligned}$$

with

$$P_j(t; e^{-d(T-t)} A(t); \nu(t); K) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty Re \left(\frac{e^{i\phi(\ln(A(t)/K) - d(T-t)) + C_j(\phi) + D_j(\phi)\nu(t)}}{i\phi} \right) d\phi$$

for $j = 1, 2$, where the functions C_j and D_j are depending on time to maturity $T - t$, the mean reversion speed κ as well as the mean reversion level θ, the vol of vol parameter σ and the correlation ρ between the two driving Brownian motions W_S and W_ν.

Forward Starting Options in the Heston Model

Kruse, Nögel (2005)

Theorem

Let $0 \leq t < t^* < T$. Under the same assumptions as in Heston's model the price of a forward starting option with payoff $(A(T)/A(t^*) - K)^+$ is given by

$$C_{RFWS}(t; A(t); \nu(t)) = e^{-d(T-t^*)} e^{-r(t^*-t)} \hat{P}_1(e^{-d(T-t^*)}; t; \nu(t)) - k e^{-r(T-t)} \hat{P}_2(e^{-d(T-t^*)}; t; \nu(t))$$

with

$$\hat{P}_j(e^{-d(T-t^*)}; t; \nu(t)) := \int_0^\infty P_j(t^*; e^{-d(T-t^*)}; v; k) f(v|\nu(t)) dv$$

where the P_j, $j = 1, 2$, equal the Heston probabilities and $f(v|\nu(t))$ is the transition density from state $\nu(t)$ with

$$f(v|\nu(t)) = B f_{\chi^2(\Lambda, R)}(Bv)$$

for $B = \frac{4\kappa}{\sigma^2}(1 - e^{-\kappa(t^*-t)})^{-1}$. $f_{\chi^2(\Lambda,R)}(v)$ is the density of a non-central Chi-squared random variable with non-centrality parameter $\Lambda = Be^{-\kappa(t^*-t)}\nu(t)$ and $R = \frac{4\kappa\theta}{\sigma^2}$ degrees of freedom.

Outline

Modeling Inflation

Korn, Kruse (2004)

Consider a Black-Scholes-type framework where the dynamics of the inflation index I is given by

$$dI(t) = (r_N - r_R)I(t)dt + \sigma_I I(t)dW_I(t)$$

with constant volatility σ_I and constant nominal interest rate r_N as well as constant real interest rate r_R. $W_I(t)$ denotes a Brownian motion.

The Fisher equation still holds in this model since

$$\mathbb{E}\left[\frac{I(T)}{I(t)}\middle|\mathcal{F}_t\right] = e^{(r_N - r_R)(T-t)}$$

Caplet Prices

Korn, Kruse (2004)

Theorem

Under the assumption of a Black-Scholes-type framework for the inflation index the price of an inflation option on the inflation index with payoff

$$C(T, I(T)) = (I(T) - K)^+$$

at maturity T is at time $t < T$ given by

$$C(t, I(t)) = I(t)e^{-r_R(T-t)}N(d) - Ke^{-r_N(T-t)}N(d - \sigma_I\sqrt{T-t})$$

with

$$d = \frac{\ln\left(\frac{I(t)}{K}\right) + (r_N - r_R + \frac{1}{2}\sigma_I^2)(T-t)}{\sigma_I\sqrt{T-t}}.$$

Caplet Prices

Kruse (2009)

Corollary

Under the assumption of a Black-Scholes-type framework for the inflation index the price of an inflation caplet over the time period from t_0 up to T with payoff

$$C_I(T; I(T)) = \left(\frac{I(T) - I(t_0)}{I(t_0)} - k \right)^+$$

at maturity T is at time t with $t_0 < t < T$ given by

$$C_I(t, I(t)) = \frac{I(t)}{I(t_0)} e^{-r_R(T-t)} N(d) - (1+k) e^{-r_N(T-t)} N(d - \sigma_I \sqrt{T-t})$$

with

$$d = \frac{\ln\left(\frac{I(t)}{I(t_0)(1+k)}\right) + (r_N - r_R + \frac{1}{2}\sigma_I^2)(T-t)}{\sigma_I \sqrt{T-t}}.$$

Caplet Prices

Kruse (2009)

Theorem

Under the assumption of a Black-Scholes-type framework for the inflation index the price of an inflation caplet over the future time period from T_{i-1} up to T_i with payoff

$$C_I(T) = (i(T_{i-1}, T_i) - K)^+ = \left(\frac{I(T_i) - I(T_{i-1})}{I(T_{i-1})} - K \right)^+$$

at maturity T_i is at time $t < T_{i-1}$ given by

$$C(t, I(t)) = e^{-r_R(T_i - T_{i-1})} e^{-r_N(T_{i-1} - t)} N(d) - (1 + K) e^{-r_N(T_i - t)} N(d - \sigma_I \sqrt{T_i - T_{i-1}})$$

with

$$d = \frac{-\ln(1 + K) + \left(r_N - r_R + \frac{1}{2}\sigma_I^2 \right) \cdot (T_i - T_{i-1})}{\sigma_I \sqrt{T_i - T_{i-1}}}.$$

Inflation Options under the Assumption of Constant Volatility

Different approaches to modelling inflation and the pricing of inflation options under the assumption of a constant volatility are e.g.:

- Huston (1998)
- Jarrow and Yildirim (2003)
- Mercurio (2005)

These models use a foreign currency analogy which is based on the idea that the inflation rate is the exchange rate between the ”nominal” and the ”real currency”.

Implied Volatilities in the Black-Scholes Framework

We consider an example of calibration to U.S. market data as of November 3, 2004.

- Caplet prices for different strikes (1%, 1,50%, 2%, 2,50%, 3% and 3,5%) and different maturies (1, 2, 3, ..., 10 years)

T_i/Strike	1%	1,50%	2%	2,50%	3%	3,50%
1	178,10	134,40	95,10	62,40	37,80	21,20
2	360,40	277,40	202,90	140,30	92,00	58,00
3	539,90	419,90	312,10	221,10	150,20	99,30
4	714,50	558,90	418,90	300,40	207,40	140,00
5	886,40	696,10	524,30	378,30	263,10	179,00
6	1041,70	819,50	619,00	448,30	313,40	214,60
7	1196,80	944,70	717,10	523,00	368,80	255,10
8	1338,80	1059,50	807,50	592,30	420,90	293,80
9	1477,40	1172,60	897,60	662,40	474,60	334,60
10	1610,50	1281,90	985,40	731,60	528,30	376,00

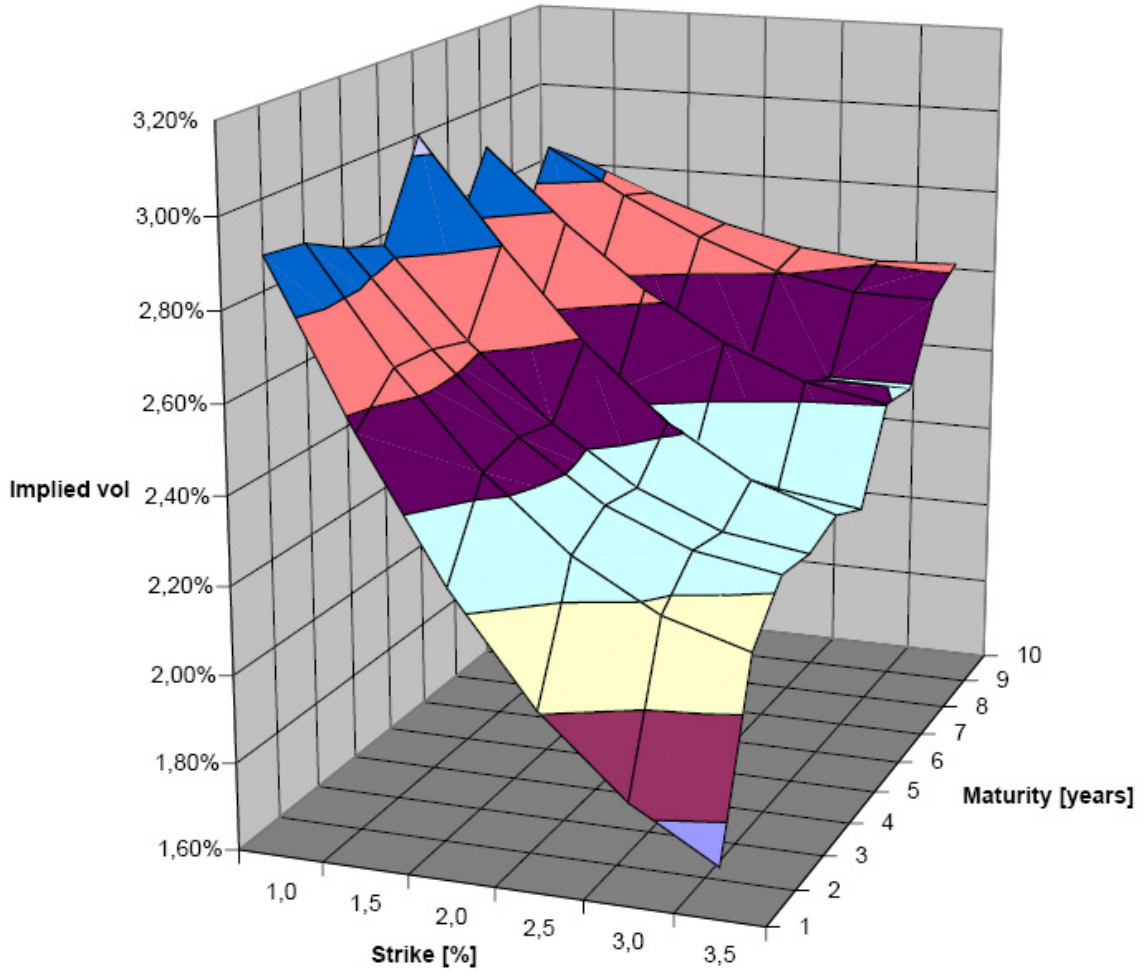
Implied Volatilities in the Black-Scholes Framework

- USD nominal and real discount factors, ZC swap rates

T_i	$D_{r_N}(0, T_i)$	ZC Rates	$D_{r_R}(0, T_i)$
1	0,97701	2,111%	0,99763
2	0,94982	2,188%	0,99184
3	0,91835	2,240%	0,98146
4	0,88433	2,278%	0,96771
5	0,84862	2,293%	0,95048
6	0,81179	2,300%	0,93046
7	0,77460	2,310%	0,90887
8	0,73785	2,320%	0,88645
9	0,70218	2,325%	0,86354
10	0,66773	2,335%	0,84109

The zero coupon inflation swap rates are used to strip the current real discount factors for the relevant maturities.

Implied Volatilities in the Black-Scholes Framework



Outline

Modeling Inflation

Kruse (2009)

We now consider a Heston framework where the inflation index I follows the diffusion process

$$dI(t) = (r_N - r_R)I(t)dt + \sqrt{\nu(t)}I(t)dW_I(t)$$

and the mean-reverting variance process $\nu(t)$

$$d\nu(t) = \kappa\left(\theta - \nu(t)\right)dt + \sigma\sqrt{\nu(t)}dW_\nu(t)$$

where r_N and r_R denote the constant, risk free nominal and real interest rate, $\kappa \geq 0$ is the mean reversion speed, θ the mean reversion level, $\sigma\sqrt{\nu(t)}$ the volatility of volatility and W_I and W_ν denote Brownian motions with correlation ρ. To ensure that the inflation variance process $\nu(t)$ remains positive, we assume again that the stability condition holds

$$2\kappa\theta > \sigma^2.$$

The Fisher equation still holds in this setting, since

$$\mathbb{E}\left[\frac{I(T)}{I(t)}\middle|\mathcal{F}_t\right] = \mathbb{E}\left[e^{(r_N-r_R)(T-t)-\frac{1}{2}\int_t^T \nu(s)ds+\int_t^T \sqrt{\nu(s)}dW_I(s)}\middle|\mathcal{F}_t\right] = e^{(r_N-r_R)(T-t)}$$

Caplet Prices

Kruse (2009)

Theorem

Assuming the above Heston framework for the inflation index the price of an inflation caplet over the time period from t_0 up to T with payoff

$$C_I(T; I(T)) = \left(\frac{I(T) - I(t_0)}{I(t_0)} - k \right)^+$$

at maturity T is at time t with $t_0 < t < T$ given by

$$\begin{aligned} C_I(t, I(t), \nu(0)) = & \frac{I(t)}{I(t_0)} e^{-r_R(T-t)} P_1 \left(t, e^{-r_R(T-t)} I(t), \nu(t), I(t_0)(1+k) \right) \\ & - (1+k) e^{-r_N(T-t)} P_2 \left(t, e^{-r_R(T-t)} I(t), \nu(t), I(t_0)(1+k) \right) \end{aligned}$$

where P_j, $j = 1, 2$ equal the Heston probabilities.

Caplet Prices

Kruse (2009)

Theorem

Assuming the above Heston framework the price of an inflation caplet over the future time period from T_{i-1} up to T_i with payoff

$$C_I(T) = (i(T_{i-1}, T_i) - K)^+ = \left(\frac{I(T_i) - I(T_{i-1})}{I(T_{i-1})} - K \right)^+$$

at maturity T_i is at time $t < T_{i-1}$ given by

$$\begin{aligned} C(t, I(t), \nu(t)) &= e^{-r_R(T_i - T_{i-1})} e^{-r_N(T_{i-1} - t)} \hat{P}_1 \left(t, e^{-r_R(T_i - T_{i-1})}, \nu(t), 1 + K \right) \\ &\quad - (1 + K) e^{-r_N(T_i - t)} \hat{P}_2 \left(t, e^{-r_R(T_i - T_{i-1})}, \nu(t), 1 + K \right) \end{aligned}$$

where the probabilities $\hat{P}_j$, $j = 1, 2$ are given by

$$\hat{P}_j(t, e^{-r_R(T_i - T_{i-1})}, \nu(t), 1 + K) := \int_0^\infty P_j(T_{i-1}, e^{-r_R(T_i - T_{i-1})}, v, 1 + K) f(v|\nu(t)) dv$$

where the P_j, $j = 1, 2$, equal the Heston probabilities and $f(v|\nu(t))$ is the transition density of the volatility process from state $\nu(t)$ as before.

Inflation Options under the Assumption of Stochastic Volatility

A different approach under stochastic volatility has been introduced by

- Mercurio and Moreni (2006)

again based on the foreign currency-analogy.

Conclusions

- We have recalled some facts on inflation and introduced important financial products on inflation.
- A brief recollection of the pricing of forward starting options in the Black-Scholes model as well as in the Heston model has been given.
- We constructed closed-form solutions to the pricing of inflation caplets in a Black-Scholes framework and had a look at implied volatilities to motivate a stochastic volatility model for the inflation index.
- We have introduced stochastic volatility and have received a useful closed-form formula for inflation caplets under the assumption of stochastic volatility as in Heston's model.

References

For a detailed discussion on the pricing of forward starting options see

Rubinstein, M. (1991): Pay Now, Choose Later. Risk, 4 (2), 13.

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The main topic of the talk can be found in

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A description of a simple Black-Scholes type framework for inflation can be found in

Korn, R.; Kruse, S. (2004): ”A simple model to value inflation-linked financial products.” (in German), Blätter der DGVFM, XXVI (3), 351-367.

Further Readings

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