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Asset allocation for a DC pension fund under a regime switching environment

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Asset allocation for a DC pension fund under a regime switching environment

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1. The Problem

- Utility maximization for a DC-fund member under regime switching and inflation
  - will be made more precise in a moment
  - Regime switching according to a Markov chain X(t) that can attain finitely many states $\{e_1, \ldots, e_N\}$, $e_1 = (1,0,\ldots,0)'$, …

2. The Ingredients

- Investment opportunities:
  - **Bank account** with nominal interest rate

(1) $$\frac{dB(t)}{B(t)} = R(t)\,dt$$

(2) $$R(t) = \langle R, X(t) \rangle := \sum_{j=1}^{N} R_j 1_{\{X(t)=e_j\}}$$

2. The Ingredients

- Investment opportunities:
  - **Index bond** with real interest rate

(3) $$\frac{dI(t)}{I(t)} = r(t)dt + \frac{dP(t)}{P(t)} = \big(r(t)+i(t)\big)dt + \sigma_I dW_1(t)$$

(4) $i(t) = \langle i, X(t) \rangle$, $r(t)$ real rate of return

$P(t)$ a price index (see K., Kruse (2004))

2. The Ingredients

- Investment opportunities:
  - **Share (index)** with BS-like dynamics

(5) $$\frac{dS(t)}{S(t)} = \mu(t)dt + \sigma_S dW_2(t)$$

(6) $$\mu(t) = \langle \mu, X(t) \rangle$$

W_1, W_2 independent Brownian motions

2. The Ingredients

- Investment opportunities:
  - **Salary (index)** with BS-like dynamics

(7) $$\frac{dY(t)}{Y(t)} = \kappa(t)dt + \sigma_1 dW_1(t) + \sigma_2 dW_2(t)$$

(8) $$\kappa(t) = R(t) + \frac{\sigma_1}{\sigma_I}\left(r(t) + i(t) - R(t)\right) + \frac{\sigma_2}{\sigma_S}\left(\mu(t) - R(t)\right)$$

„fair (mean) salary evolution condition”

Theorem

2. The Ingredients

- The problem: **Maximize expected utility**

$$(9)\quad \max_{\pi(.)\in A_Y(v)} E\left(u\left(V^{\pi}(T)\right)\right)$$

$$(10)\quad dV^{\pi}(t) = \left(V^{\pi}(t)R(t)+cY(t)\right)dt + V^{\pi}(t)\,\pi'(t)\,\sigma\left\{\theta(t)dt+dW(t)\right\}$$

$$(11)\quad \sigma = \begin{pmatrix} \sigma_I & 0 \\ 0 & \sigma_S \end{pmatrix},\ \theta(t) = \begin{pmatrix} r(t)+i(t)-R(t) \\ \mu(t)-R(t) \end{pmatrix}$$

The strategy has to be adapted to the observed prices, ***not*** to the unobservable parameters R, μ, i !

3. The Method(s)

- ***Usual approach:***
  - Set up a Hamilton-Jacobi-Bellman-Equation and solve it ….
  - Here: No good strategy as we have
    - Hidden parameters
    - Uncertain income stream

3. The Method(s)

Here: ***Multi-step approach***

- ***Estimation and control*** principle for log-utility $u(x) = \ln(x)$ (see Hinz, K. (1997))

- Solution in the „observable market“ via ***portfolio optimization with given payment streams*** (see K., Krekel (2001))

- Estimation of param. via ***non-linear filtering***

Problem

4. The Main Result

Theorem:

Let $v^* = v + cY(0)T$. Consider the process

(12) $dV^*(t) = V^*(t)\left[R(t)dt + \pi^*(t)\sigma\{\theta(t)dt + dW(t)\}\right]$

that starts with an initial value of v^* and where we have

(13) $\pi_1^*(t) = \left(r(t) + \hat{i}(t) - \hat{R}(t)\right)/\sigma_I^2$, $\pi_2^*(t) = \left(\hat{\mu}(t) - \hat{R}(t)\right)/\sigma_S^2$

Then, the final value $V^*(T)$ of this process solves the maximization problem.

4. The Main Result

Understanding the Theorem – I:

- $\hat{i}(t) := E\left(i(t) \middle| F_t\right)$, $\hat{R}(t) := E\left(R(t) \middle| F_t\right)$, $\hat{\mu}(t) := E\left(\mu(t) \middle| F_t\right)$

are the conditional mean values of the unobservable parameters given the price observations

- V*(t) is a virtual wealth process corresponding to an investment in the real market starting with v* and using the optimal portfolio process π*(t) in the market with the „hat“ parameters without contributions of the DC fund member.

4. The Main Result

Understanding the Theorem – II:

- **Where is the contribution process ?**
  - Due to the fair (mean) salary condition, the contribution process can be hedged completely
  - Its initial net present value is added to the capital available for investment
  - This is financed by a negative position in the hedging strategy
  - The negative position will be continuously offset by the incoming contribution process

4. The Main Result

Understanding the Theorem – III:

- **Where is the actual wealth process ?**
  - The optimal wealth process *V*(t) is obtained via

$$V(t) = V^*(t) - cY(t)(T - t) .$$

  - The corresponding portfolio process is given by

$$\pi_1(t) = \left(\pi_1^*(t) V^*(t) - cY(t)(T-t)\frac{\sigma_1}{\sigma_I}\right) / V(t)$$

$$\pi_2(t) = \left(\pi_2^*(t) V^*(t) - cY(t)(T-t)\frac{\sigma_2}{\sigma_S}\right) / V(t)$$

5. A Special Case

- Constant salary, deterministic interest rates, i.e.
  - $\sigma_1 = \sigma_2 = 0,\ \kappa = 0$
  - R(t) an observable deterministic function

⇓

  - Mean (fair) salary condition is not necessary for the method to go through !
  - Net present value of the payment stream equals

$$v_{add} := cY(0)\int_0^T e^{-\int_0^s R(u)du} ds$$

5. A Special Case

- Constant salary, deterministic interest rates:
  - By adding v_{add} to the initial wealth v we can proceed as in the Theorem
  - The hedging strategy for the salary payment stream is just a corresponding negative bond position which will be neutralised until time T.

6. Further Aspects

- Necessary:
  - Computation of the conditional expectations !
  - Method: Non-linear filtering (Gauge technique and EM-algorithm (no details here))
- Generalizations:
  - More general utility functions
  - Consumption
  - Death of fund member before the investment horizon
  - .....

Thank you for your attention !