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Asset allocation for a DC pension fund under a regime switching environment

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Asset allocation for a DC pension fund under a regime switching environment

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1. The Problem

- Utility maximization for a DC-fund member under regime switching and inflation
 - will be made more precise in a moment
 - Regime switching according to a Markov chain $X(t)$ that can attain finitely many states $\{e_1, \dots, e_N\}$, $e_1 = (1, 0, \dots, 0)'$, ...

2. The Ingredients

- Investment opportunities:
 - **Bank account** with nominal interest rate

$$(1) \quad \frac{dB(t)}{B(t)} = R(t) dt$$

$$(2) \quad R(t) = \langle R, X(t) \rangle := \sum_{j=1}^N R_j 1_{\{X(t)=e_j\}}$$

2. The Ingredients

- Investment opportunities:
 - **Index bond** with real interest rate

$$(3) \quad \frac{dI(t)}{I(t)} = r(t)dt + \frac{dP(t)}{P(t)} = (r(t) + i(t))dt + \sigma_I dW_1(t)$$

$$(4) \quad i(t) = \langle i, X(t) \rangle \quad , \quad r(t) \text{ real rate of return}$$

$P(t)$ a price index (see K., Kruse (2004))

2. The Ingredients

- Investment opportunities:
 - **Share (index)** with BS-like dynamics

$$(5) \quad \frac{dS(t)}{S(t)} = \mu(t) dt + \sigma_S dW_2(t)$$

$$(6) \quad \mu(t) = \langle \mu, X(t) \rangle$$

W_1, W_2 independent Brownian motions

2. The Ingredients

- Investment opportunities:
 - **Salary (index)** with BS-like dynamics

$$(7) \quad \frac{dY(t)}{Y(t)} = \kappa(t) dt + \sigma_1 dW_1(t) + \sigma_2 dW_2(t)$$

$$(8) \quad \kappa(t) = R(t) + \frac{\sigma_1}{\sigma_I} (r(t) + i(t) - R(t)) + \frac{\sigma_2}{\sigma_S} (\mu(t) - R(t))$$

„fair (mean) salary evolution condition”

2. The Ingredients

- The problem: **Maximize expected utility**

$$(9) \quad \max_{\pi(\cdot) \in A_Y(V)} E\left(u\left(V^\pi(T)\right)\right)$$

$$(10) \quad dV^\pi(t) = \left(V^\pi(t)R(t) + cY(t)\right) dt \\ + V^\pi(t) \pi'(t) \sigma \left\{ \theta(t) dt + dW(t) \right\}$$

$$(11) \quad \sigma = \begin{pmatrix} \sigma_I & 0 \\ 0 & \sigma_S \end{pmatrix}, \quad \theta(t) = \begin{pmatrix} r(t) + i(t) - R(t) \\ \mu(t) - R(t) \end{pmatrix}$$

The strategy has to be adapted to the observed prices, **not** to the unobservable parameters R, μ, i !

3. The Method(s)

- *Usual approach:*
 - Set up a **Hamilton-Jacobi-Bellman-Equation** and solve it
 - Here: No good strategy as we have
 - **Hidden parameters**
 - **Uncertain income stream**

3. The Method(s)

Here: Multi-step approach

- ***Estimation and control*** principle for log-utility $u(x) = \ln(x)$ (see Hinz, K. (1997))
- Solution in the „observable market“ via ***portfolio optimization with given payment streams*** (see K., Krekel (2001))
- Estimation of param. via ***non-linear filtering***

4. The Main Result

Theorem:

Let $v^* = v + cY(0)T$. Consider the process

$$(12) \quad dV^*(t) = V^*(t) \left[R(t) dt + \pi^*(t) \sigma \{ \theta(t) dt + dW(t) \} \right]$$

that starts with an initial value of v^* and where we have

$$(13) \quad \pi_1^*(t) = (r(t) + \hat{i}(t) - \hat{R}(t)) / \sigma_I^2, \quad \pi_2^*(t) = (\hat{\mu}(t) - \hat{R}(t)) / \sigma_S^2$$

Then, the final value $V^*(T)$ of this process solves the maximization problem.

4. The Main Result

Understanding the Theorem – I:

- $\hat{i}(t) := E(i(t)|F_t)$, $\hat{R}(t) := E(R(t)|F_t)$, $\hat{\mu}(t) := E(\mu(t)|F_t)$

are the conditional mean values of the unobservable parameters given the price observations

- $V^*(t)$ is a virtual wealth process corresponding to an investment in the real market starting with v^* and using the optimal portfolio process $\pi^*(t)$ in the market with the „hat“ parameters without contributions of the DC fund member.

4. The Main Result

Understanding the Theorem – II:

- **Where is the contribution process ?**
 - Due to the fair (mean) salary condition, the contribution process can be hedged completely
 - Its initial net present value is added to the capital available for investment
 - This is financed by a negative position in the hedging strategy
 - The negative position will be continuously offset by the incoming contribution process

4. The Main Result

Understanding the Theorem – III:

- **Where is the actual wealth process ?**

- The optimal wealth process $V(t)$ is obtained via

$$V(t) = V^*(t) - cY(t)(T - t) .$$

- The corresponding portfolio process is given by

$$\pi_1(t) = \left(\pi_1^*(t)V^*(t) - cY(t)(T - t)\frac{\sigma_1}{\sigma_I} \right) / V(t)$$

$$\pi_2(t) = \left(\pi_2^*(t)V^*(t) - cY(t)(T - t)\frac{\sigma_2}{\sigma_S} \right) / V(t)$$

5. A Special Case

- Constant salary, deterministic interest rates, i.e.
 - $\sigma_1 = \sigma_2 = 0, \kappa = 0$
 - $R(t)$ an observable deterministic function
 - ⤵
 - Mean (fair) salary condition is not necessary for the method to go through !
 - Net present value of the payment stream equals

$$V_{add} := cY(0) \int_0^T e^{-\int_0^s R(u) du} ds$$

5. A Special Case

- Constant salary, deterministic interest rates:
 - By adding v_{add} to the initial wealth v we can proceed as in the Theorem
 - The hedging strategy for the salary payment stream is just a corresponding negative bond position which will be neutralised until time T .

6. Further Aspects

- Necessary:
 - Computation of the conditional expectations !
 - Method: Non-linear filtering (Gauge technique and EM-algorithm (no details here))
- Generalizations:
 - More general utility functions
 - Consumption
 - Death of fund member before the investment horizon
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Thank you for your attention !