Risk-minimization with mortality derivatives: mixed dynamic and static hedging

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Introduction

Focus on

- **Mixed dynamic and static hedging:**
  Dahl, Glar, Møller (2009). Mixed dynamic and static risk-minimization with an application to survivor swaps.
  Preprint, 2009

- **Valuation and hedging with traditional bonds:**
  Dahl, Møller (2006): Valuation and hedging of life insurance liabilities with systematic mortality risk.
  Insurance: Mathematics and Economics 39:193–217

- **Dynamic hedging with mortality derivatives:**
  Dahl, Melchior and Møller (2008). On systematic mortality risk and risk-minimization with survivor swaps.
  Scandinavian Actuarial Journal 2008(2-3):114–146

Financial market and discounted price processes

Savings account $B^* = 1$

Usual risky assets $X = (X^1, \dots, X^d)$. Traded **dynamically**

Illiquid asset Y. Traded at **fixed discrete times**
Example: mortality derivative (survivor swaps)

Martingale measure Q for assets (X, Y)

Goals:

- Derive optimal mixed dynamic and static risk-minimizing strategies
- Compare results with usual dynamic hedging
- Assess efficiency

Application to survivor swaps

Consider **portfolio of insured lives**

Survivor swap payment rate =
current number of survivors − "expected" number of survivors

("expected" number fixed at time 0. May include market price of risk)

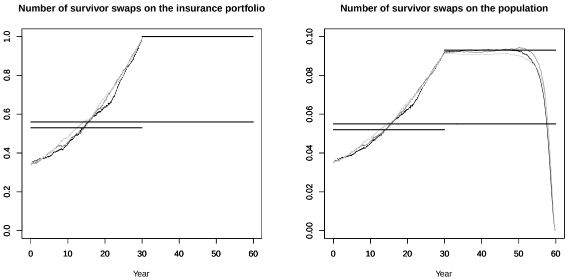
Market price of survivor swap at future times involves
– conditional expected number of survivors (new information)
– current zero coupon bond prices

How should we use the survivor swap for hedging?
Illiquid asset. Not realistic with dynamic trading
Maybe trade one or two times

From continuous to discrete trading

Liability with payment process $A(t)$, $t \in [0, T]$
Determine full dynamic strategy $(\xi(t), \vartheta(t), \eta(t))$, $t \in [0, T]$

What if Y only traded at times $0 = t_0 < \cdots < t_n = T$?
Naive guess: use $\vartheta(t) = \vartheta(t_i)$, $t \in (t_i, t_{i+1}]$, and $\xi(t)$ unchanged



Not optimal if trend in $\vartheta(t)$ or if X and Y are correlated...

Risk-minimization - full dynamic hedging

(Föllmer/Sondermann, Schweizer, Møller)
Financial market: $X(t)$, $Y(t)$, $B^*(t) = 1$

Trading strategy $\varphi = (\xi, \vartheta, \eta)$

Value process $V(t,\varphi) = \xi(t)X(t) + \vartheta(t)Y(t) + \eta(t)$

Payment process $A = (A(t))_{0 \le t \le T}$

Cost process
$C(t,\varphi) = V(t,\varphi) - \int_0^t \xi(u)\, dX(u) - \int_0^t \vartheta(u)\, dY(u) + A^*(t)$

Risk process $R(t,\varphi) = E^{Q}\left[(C(T,\varphi) - C(t,\varphi))^2 \middle| \mathcal{F}(t) \right]$

Criterion of risk-minimization
Minimize $R(t,\varphi)$ over φ for all t

Market value decomposition

$$\begin{aligned} V^*(t) &:= E^Q[A^*(T) \mid \mathcal{F}(t)] \\ &= V^*(0) + \int_0^t \xi^A(u)\, dX(u) + \int_0^t \vartheta^A(u)\, dY(u) + L^A(t) \end{aligned}$$

where

- L^A is a Q-martingale
- (X, Y) and L^A are orthogonal

Theorem. Risk-minimizing strategy $\varphi = (\xi, \vartheta, \eta)$, $V(T, \varphi) = 0$:

$\xi(t) = \xi^A(t)$ **dynamic strategy for X**

$\vartheta(t) = \vartheta^A(t)$ $\leftarrow$ **not possible with illiquid asset!**

$\eta(t) = V^*(t) - A^*(t) - \xi^A(t)X(t) - \vartheta^A(t)Y(t)$

Minimal risk process

$$R(t, \varphi) = E^Q\left[(L^A(T) - L^A(t))^2 \middle| \mathcal{F}(t) \right]$$

Obtaining a decomposition with an illiquid asset

"Project" (decompose) illiquid asset on liquid assets:

$$dY(t) = \xi^Y(t)dX(t) + dL^Y(t)$$

First term: Financial risk, e.g. interest rate risk
Second term: Non-hedgeable insurance risk

Insert in market value decomposition

$$\begin{aligned} V^{*,Q}(t_i) &= V^{*,Q}(t_0) + \int_0^{t_i} \Big(\xi^A(u) + \vartheta^A(u)\xi^Y(u)\Big) dX(u) \\ &\quad + \sum_{j=1}^{i} \Big(\int_{t_{j-1}}^{t_j} \vartheta^A(u) dL^Y(u) + \Delta L^A(t_j) \Big) \end{aligned}$$

with $\Delta L^A(t_j) = L^A(t_j) - L^A(t_{j-1})$.
Leads to decomposition with orthogonal terms

The mixed discrete and continuous strategy

Minimizing the risk process

$$R(t,\varphi) = \mathrm{E}^Q\left[(C(T,\varphi) - C(t,\varphi))^2 \,\middle|\, \mathcal{F}(t)\right]$$

leads to minimization of

$$\mathrm{E}^Q\left[\left(\int_{t_i}^{t_{i+1}} \vartheta^A(u)dL^Y(u) - \vartheta(t_i)\Delta L^Y(t_{i+1})\right)^2 \middle| \mathcal{F}(t_i)\right]$$

Usual quadratic problem with solution

$$\widehat{\vartheta}(t_i) = \frac{\mathrm{E}^Q\left[\int_{t_i}^{t_{i+1}} \vartheta^A(u)dL^Y(u)\Delta L^Y(t_{i+1}) \middle| \mathcal{F}(t_i)\right]}{\mathrm{E}^Q\left[\left(\Delta L^Y(t_{i+1})\right)^2 \middle| \mathcal{F}(t_i)\right]}$$

Interpretation: Optimal position $\widehat{\vartheta}(t_i)$ is a risk-adjusted average of the future dynamic strategy $\vartheta^A(t)$ on interval $(t_i, t_{i+1}]$

Mixed dynamic and static hedging

Theorem
The unique mixed discrete- and continuous-time risk-minimizing strategy associated with A^* is

$$\begin{aligned}
\widehat{\vartheta}^*(t) &= \widehat{\vartheta}^A(t_{j-1}) = \frac{\mathrm{E}^Q\left[\int_{t_{j-1}}^{t_j} \vartheta^A(u)dL^Y(u)\Delta L^Y(t_j)\middle|\mathcal{F}(t_{j-1})\right]}{\mathrm{E}^Q\left[(\Delta L^Y(t_j))^2\middle|\mathcal{F}(t_{j-1})\right]} \\
\widehat{\xi}^*(t) &= \widehat{\xi}^A(t) = \xi^A(t) + \xi^Y(t)(\vartheta^A(t) - \widehat{\vartheta}^A(t_{j-1})) \\
\eta(t) &= V^{*,Q}(t) - A^*(t) - \widehat{\xi}^A(t)X(t) - \widehat{\vartheta}^A(t_{j-1})Y(t)
\end{aligned}$$

$t \in (t_{j-1}, t_j]$.

Note: Correction term for $\xi(t)$ arises to adjust for modified risk from illiquid asset

Modeling the mortality intensity:

Known at time 0:
$\mu^\circ(x+t)$ is mortality intensity "today" at all ages $x+t$

Unknown at time 0:
$\zeta(x,t)$ is relative change in the mortality from 0 to t, age x

Stochastic mortality intensity:
$\mu(x,t) = \mu^\circ(x+t)\zeta(x,t)$

True survival probability from t to T given information $\mathcal{I}(t)$:

$$\mathcal{S}(x,t,T) = E^P\left[e^{-\int_t^T \mu(x,\tau)d\tau} \middle| \mathcal{I}(t) \right]$$

The model:

2-dim. time-inhomogeneous CIR model known from finance:

$$d\zeta(x,t) = (\gamma(x,t) - \delta(x,t)\zeta(x,t))dt + \sigma(x,t)\sqrt{\zeta(x,t)}dW^{\mu}(t)$$

Survival probability in affine model

$$\mathcal{S}(x,t,T) = e^{A^{\mu}(x,t,T) - B^{\mu}(x,t,T)\mu(x,t)}$$

Forward mortality intensity

$$f^{\mu}(x,t,T) = -\frac{\partial}{\partial T}\log \mathcal{S}(x,t,T)$$

Financial market

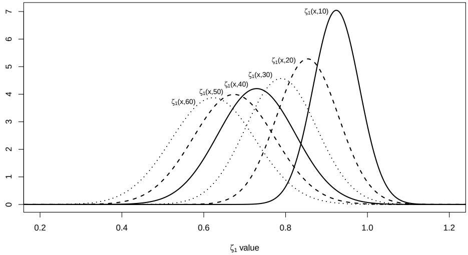
Vasiček model for short rate:

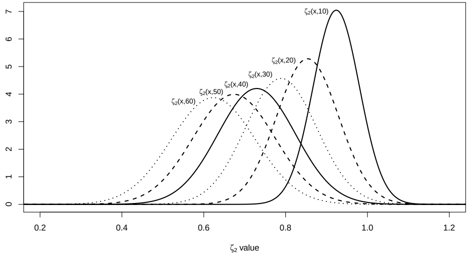
$$dr(t) = (\gamma^{r,\alpha} - \delta^{r,\alpha} r(t))\,dt + \sqrt{\gamma^{r,\sigma}}dW^{r}(t)$$

Zero coupon bond prices

$$P(t,T) = e^{A^{r}(t,T) - B^{r}(t,T)r(t)}$$

Mortality improvement distribution





Parameters:

Pf.	1	2
γ_j	0.0001800	0.0001805
δ_j	0.0080	0.0081
$\sigma_{j,1}$	0.006	0.000
$\sigma_{j,2}$	0.018	0.019

Initial values:

$\zeta_j(x,0) = 1$

Mean reversion levels:

γ_j/δ_j

Portfolio 1: 0.0225

Portfolio 2: 0.0223

Two portfolios of insured lives

Counting processes and martingales

$$N_j(x,t) = \sum_{i=1}^{n_j} 1_{(T_{j,i} \le t)}$$

$$M_j(x,t) = N_j(x,t) - \int_0^t (n_j - N_j(x,u-))\mu_j(x,u)du$$

Insurance payment process (Benefits − premiums on pf 1)

$$dA(t) = (n_1 - N_1(x,\overline{T}))\Delta A_0(\overline{T})d1_{(t \ge \overline{T})} + a_0(t)(n_1 - N_1(x,t))dt + a_1(t)dN_1(x,t)$$

(a_i, A_0 deterministic functions)

Market value process

$$V^{*,Q}(t) = E^Q\left[\int_{[0,T]} e^{-\int_0^\tau r(u)du} dA(\tau) \middle| \mathcal{F}(t)\right]$$
$$= \int_{[0,t]} e^{-\int_0^\tau r(u)du} dA(\tau)$$
$$+ e^{-\int_0^t r(u)du}(n - N_1(x,t))V^Q(t, r(t), \mu_1(x,t))$$

where

$$V^Q(t, r(t), \mu_1(x,t))$$
$$= \int_t^T P(t,\tau)\mathcal{S}_1^Q(x,t,\tau)\left(a_0(\tau) + a_1(\tau)f^{\mu_1,Q}(x,t,\tau)\right)d\tau$$
$$+ P(t,\overline{T})\mathcal{S}_1^Q(x,t,\overline{T})\Delta A_0(\overline{T})1_{(t<\overline{T})}$$

Survivor swap price process

$$\widetilde{Z}_j^*(x,t) = (n_j - N_j(x,t))\int_t^T P^*(t,u)\mathcal{S}_j^Q(x,t,u)du$$
$$-n_j\, {}_t\widetilde{p}_x \int_t^T P^*(t,u)\, {}_{u-t}\widetilde{p}_{x+t}\, du$$

Price and market value dynamics

Zero coupon bond price dynamics:

$$dP^*(t,T) = -\sigma^r B^r(t,T) P^*(t,T) dW^{r,Q}(t)$$

Survivor swap price dynamics:

$$dZ_j^{*,Q}(x,t) = \nu_j^{Z,Q}(t) dM_j^Q(x,t) + \eta_j^{Z,Q}(t) dW^{r,Q}(t) + \rho_j^{Z,Q}(t) dW^{\mu,Q}(t)$$

Market risk: $\eta_j^{Z,Q}(t) dW^{r,Q}(t)$
Remaining risk: Unsystematic and systematic mortality risk

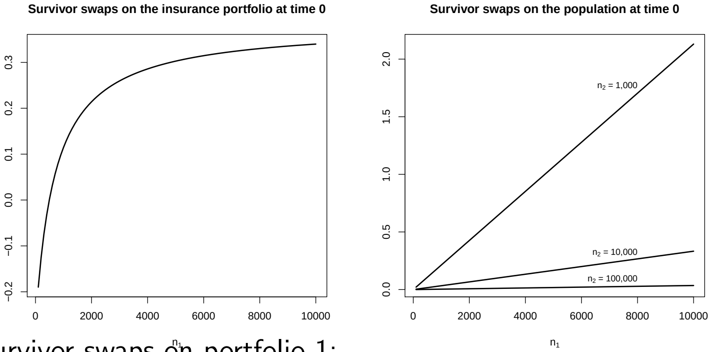
Market value dynamics:

$$dV^{*,Q}(t) = \nu^{V,Q}(t) dM_1^Q(x,t) + \eta^{V,Q}(t) dW^{r,Q}(t) + \rho^{V,Q}(t) dW^{\mu,Q}(t)$$

Market value decomposition:

$$dV^{*,Q}(t) = \xi_j^A(t) dX(t) + \vartheta_j^A(t) dZ_j^*(x,t) + dL_j^A(t)$$

Dependence on the number of policy-holders n_1

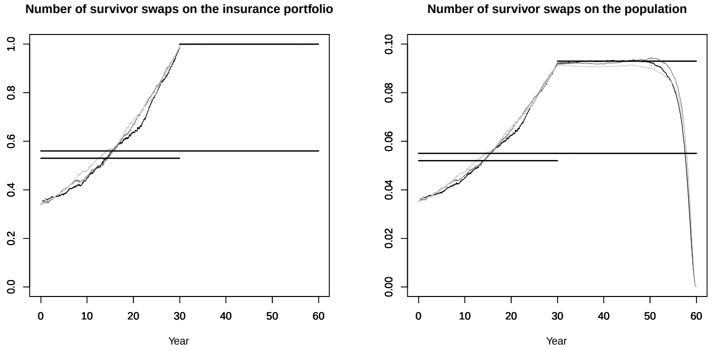


Survivor swaps on portfolio 1:

$$\vartheta_1^A(t) = \frac{\nu^{V,Q}(t) + \rho_1^{V,Q}(t)(\kappa_{1,1}^Q(t))^{-1} + \rho_2^{V,Q}(t)(\kappa_{1,2}^Q(t))^{-1}}{\nu_1^{Z,Q}(t) + \rho_{1,1}^{Z,Q}(t)(\kappa_{1,1}^Q(t))^{-1} + \rho_{1,2}^{Z,Q}(t)(\kappa_{1,2}^Q(t))^{-1}}$$

Here: $\rho_j^{V,Q}(t)$ and $\rho_{1,j}^{Z,Q}(t)$ are proportional to n_1

Comparison of discrete and continuous strategy



Discrete strategy:

$$\widehat{\vartheta}_j^A(0) = \frac{\mathrm{E}^Q\left[\int_0^T \vartheta_j^A(u)dL^{Z_j^*,Q}(u)\Delta L^{Z_j^*,Q}(T)\middle|\mathcal{F}(0)\right]}{\mathrm{E}^Q\left[\left(\Delta L^{Z_j^*,Q}(T)\right)^2\middle|\mathcal{F}(0)\right]}$$

Intrinsic risk - continuous hedging

n_1	n_2	$\frac{\sqrt{R(0,\varphi_V^*)}}{n_1}$	$\frac{\sqrt{R(0,\varphi_B^*)}}{n_1}$	$\frac{\sqrt{R(0,\varphi_1^*)}}{n_1}$	$\frac{\sqrt{R(0,\varphi_2^*)}}{n_1}$
100	1,000	0.634	0.111	0.048	0.103
100	10,000	0.634	0.111	0.048	0.098
1,000	10,000	0.627	0.062	0.032	0.037
1,000	100,000	0.627	0.062	0.032	0.035
10,000	100,000	0.626	0.055	0.013	0.020

Notation

φ_V^*: No hedging (only savings account)
φ_B^*: Bond market
φ_1^*: Bond market + survivor swap on portfolio 1
φ_2^*: Bond market + survivor swap on portfolio 2

Intrinsic risk - mixed hedging

n_1	n_2	$\frac{\sqrt{R(0,\varphi^*_{D^1_1})}}{n_1}$	$\frac{\sqrt{R(0,\varphi^*_{D^1_2})}}{n_1}$	$\frac{\sqrt{R(0,\varphi^*_{D^2_1})}}{n_1}$	$\frac{\sqrt{R(0,\varphi^*_{D^2_2})}}{n_1}$
100	1,000	0.105	0.104	0.073	0.104
100	10,000	0.105	0.100	0.073	0.100
1,000	10,000	0.045	0.040	0.038	0.039
1,000	100,000	0.045	0.039	0.038	0.037
10,000	100,000	0.022	0.026	0.019	0.024

Notation

D^1_j: Constant swap for portfolio j on $(0, 60]$

D^2_j: Trading swap for portfolio j on $(0, 30]$ and $(30, 60]$