

# Risk-minimization with mortality derivatives: mixed dynamic and static hedging

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# Introduction

Focus on

- ▶ **Mixed dynamic and static hedging:**  
Dahl, Glar, Møller (2009). Mixed dynamic and static risk-minimization with an application to survivor swaps.  
*Preprint, 2009*
- ▶ **Valuation and hedging with traditional bonds:**  
Dahl, Møller (2006): Valuation and hedging of life insurance liabilities with systematic mortality risk.  
*Insurance: Mathematics and Economics* 39:193–217
- ▶ **Dynamic hedging with mortality derivatives:**  
Dahl, Melchior and Møller (2008). On systematic mortality risk and risk-minimization with survivor swaps.  
*Scandinavian Actuarial Journal* 2008(2-3):114–146

# Financial market and discounted price processes

**Savings account**  $B^* = 1$

**Usual risky assets**  $X = (X^1, \dots, X^d)$ . Traded **dynamically**

**Illiquid asset**  $Y$ . Traded at **fixed discrete times**

Example: mortality derivative (survivor swaps)

**Martingale measure**  $Q$  for assets  $(X, Y)$

## Goals:

- ▶ Derive optimal mixed dynamic and static risk-minimizing strategies
- ▶ Compare results with usual dynamic hedging
- ▶ Assess efficiency

# Application to survivor swaps

Consider **portfolio of insured lives**

**Survivor swap payment rate** =

current number of survivors – “expected” number of survivors

(“expected” number fixed at time 0. May include market price of risk)

**Market price of survivor swap** at future times involves

- conditional expected number of survivors (new information)
- current zero coupon bond prices

**How should we use the survivor swap for hedging?**

**Illiquid asset. Not realistic with dynamic trading**

**Maybe trade one or two times**

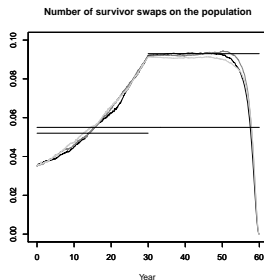
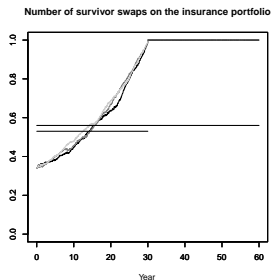
# From continuous to discrete trading

Liability with payment process  $A(t)$ ,  $t \in [0, T]$

Determine full dynamic strategy  $(\xi(t), \vartheta(t), \eta(t))$ ,  $t \in [0, T]$

What if  $Y$  only traded at times  $0 = t_0 < \dots < t_n = T$ ?

**Naive guess:** use  $\vartheta(t) = \vartheta(t_i)$ ,  $t \in (t_i, t_{i+1}]$ , and  $\xi(t)$  unchanged



**Not optimal** if trend in  $\vartheta(t)$  or if  $X$  and  $Y$  are correlated...

# Risk-minimization - full dynamic hedging

(Föllmer/Sondermann, Schweizer, Møller)

Financial market:  $X(t)$ ,  $Y(t)$ ,  $B^*(t) = 1$

**Trading strategy**  $\varphi = (\xi, \vartheta, \eta)$

**Value process**  $V(t, \varphi) = \xi(t)X(t) + \vartheta(t)Y(t) + \eta(t)$

**Payment process**  $A = (A(t))_{0 \leq t \leq T}$

**Cost process**

$$C(t, \varphi) = V(t, \varphi) - \int_0^t \xi(u) dX(u) - \int_0^t \vartheta(u) dY(u) + A^*(t)$$

**Risk process**  $R(t, \varphi) = E^Q [(C(T, \varphi) - C(t, \varphi))^2 | \mathcal{F}(t)]$

## Criterion of risk-minimization

Minimize  $R(t, \varphi)$  over  $\varphi$  for all  $t$

# Market value decomposition

$$\begin{aligned}V^*(t) &:= E^Q[A^*(T) \mid \mathcal{F}(t)] \\ &= V^*(0) + \int_0^t \xi^A(u) dX(u) + \int_0^t \vartheta^A(u) dY(u) + L^A(t)\end{aligned}$$

where

- ▶  $L^A$  is a  $Q$ -martingale
- ▶  $(X, Y)$  and  $L^A$  are orthogonal

**Theorem.** Risk-minimizing strategy  $\varphi = (\xi, \vartheta, \eta)$ ,  $V(T, \varphi) = 0$ :

$$\xi(t) = \xi^A(t) \quad \text{dynamic strategy for } X$$

$$\vartheta(t) = \vartheta^A(t) \quad \leftarrow \text{not possible with illiquid asset!}$$

$$\eta(t) = V^*(t) - A^*(t) - \xi^A(t)X(t) - \vartheta^A(t)Y(t)$$

Minimal risk process

$$R(t, \varphi) = E^Q \left[ (L^A(T) - L^A(t))^2 \mid \mathcal{F}(t) \right]$$

## Obtaining a decomposition with an illiquid asset

"Project" (decompose) illiquid asset on liquid assets:

$$dY(t) = \xi^Y(t)dX(t) + dL^Y(t)$$

**First term:** Financial risk, e.g. interest rate risk

**Second term:** Non-hedgeable insurance risk

Insert in market value decomposition

$$\begin{aligned} V^{*,Q}(t_j) &= V^{*,Q}(t_0) + \int_0^{t_j} \left( \xi^A(u) + \vartheta^A(u)\xi^Y(u) \right) dX(u) \\ &\quad + \sum_{j=1}^i \left( \int_{t_{j-1}}^{t_j} \vartheta^A(u) dL^Y(u) + \Delta L^A(t_j) \right) \end{aligned}$$

with  $\Delta L^A(t_j) = L^A(t_j) - L^A(t_{j-1})$ .

**Leads to decomposition with orthogonal terms**

# The mixed discrete and continuous strategy

Minimizing the risk process

$$R(t, \varphi) = \mathbb{E}^Q \left[ (C(T, \varphi) - C(t, \varphi))^2 \mid \mathcal{F}(t) \right]$$

leads to minimization of

$$\mathbb{E}^Q \left[ \left( \int_{t_i}^{t_{i+1}} \vartheta^A(u) dL^Y(u) - \vartheta(t_i) \Delta L^Y(t_{i+1}) \right)^2 \mid \mathcal{F}(t_i) \right]$$

Usual quadratic problem with solution

$$\hat{\vartheta}(t_i) = \frac{\mathbb{E}^Q \left[ \int_{t_i}^{t_{i+1}} \vartheta^A(u) dL^Y(u) \Delta L^Y(t_{i+1}) \mid \mathcal{F}(t_i) \right]}{\mathbb{E}^Q \left[ (\Delta L^Y(t_{i+1}))^2 \mid \mathcal{F}(t_i) \right]}$$

**Interpretation:** Optimal position  $\hat{\vartheta}(t_i)$  is a risk-adjusted average of the future dynamic strategy  $\vartheta^A(t)$  on interval  $(t_i, t_{i+1}]$

# Mixed dynamic and static hedging

## Theorem

The unique mixed discrete- and continuous-time risk-minimizing strategy associated with  $A^*$  is

$$\hat{v}^*(t) = \hat{v}^A(t_{j-1}) = \frac{\mathbb{E}^Q \left[ \int_{t_{j-1}}^{t_j} \vartheta^A(u) dL^Y(u) \Delta L^Y(t_j) \middle| \mathcal{F}(t_{j-1}) \right]}{\mathbb{E}^Q \left[ (\Delta L^Y(t_j))^2 \middle| \mathcal{F}(t_{j-1}) \right]}$$

$$\hat{\xi}^*(t) = \hat{\xi}^A(t) = \xi^A(t) + \xi^Y(t)(\vartheta^A(t) - \hat{v}^A(t_{j-1}))$$

$$\eta(t) = V^{*,Q}(t) - A^*(t) - \hat{\xi}^A(t)X(t) - \hat{v}^A(t_{j-1})Y(t)$$

$t \in (t_{j-1}, t_j]$ .

**Note:** Correction term for  $\xi(t)$  arises to adjust for modified risk from illiquid asset

# Modeling the mortality intensity:

## Known at time 0:

$\mu^\circ(x + t)$  is mortality intensity “today” at all ages  $x + t$

## Unknown at time 0:

$\zeta(x, t)$  is relative change in the mortality from 0 to  $t$ , age  $x$

## Stochastic mortality intensity:

$$\mu(x, t) = \mu^\circ(x + t)\zeta(x, t)$$

True survival probability from  $t$  to  $T$  given information  $\mathcal{I}(t)$ :

$$\mathcal{S}(x, t, T) = E^P \left[ e^{-\int_t^T \mu(x, \tau) d\tau} \middle| \mathcal{I}(t) \right]$$

## The model:

2-dim. time-inhomogeneous CIR model known from finance:

$$d\zeta(x, t) = (\gamma(x, t) - \delta(x, t)\zeta(x, t))dt + \sigma(x, t)\sqrt{\zeta(x, t)}dW^\mu(t)$$

### Survival probability in affine model

$$\mathcal{S}(x, t, T) = e^{A^\mu(x, t, T) - B^\mu(x, t, T)\mu(x, t)}$$

### Forward mortality intensity

$$f^\mu(x, t, T) = -\frac{\partial}{\partial T} \log \mathcal{S}(x, t, T)$$

### Financial market

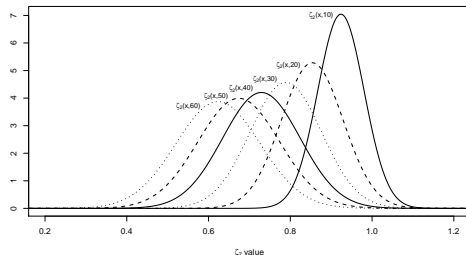
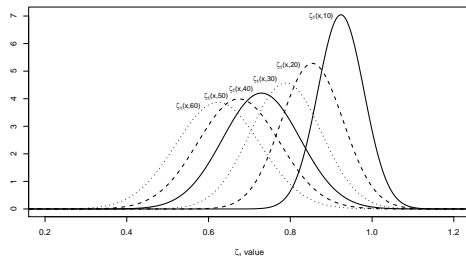
Vasiček model for short rate:

$$dr(t) = (\gamma^{r, \alpha} - \delta^{r, \alpha}r(t)) dt + \sqrt{\gamma^{r, \sigma}}dW^r(t)$$

### Zero coupon bond prices

$$P(t, T) = e^{A^r(t, T) - B^r(t, T)r(t)}$$

# Mortality improvement distribution



## Parameters:

Pf.	1	2
$\gamma_j$	0.0001800	0.0001805
$\delta_j$	0.0080	0.0081
$\sigma_{j,1}$	0.006	0.000
$\sigma_{j,2}$	0.018	0.019

## Initial values:

$$\zeta_j(x, 0) = 1$$

## Mean reversion levels:

$$\gamma_j / \delta_j$$

Portfolio 1: 0.0225

Portfolio 2: 0.0223

# Two portfolios of insured lives

## Counting processes and martingales

$$N_j(x, t) = \sum_{i=1}^{n_j} 1_{(T_{j,i} \leq t)}$$

$$M_j(x, t) = N_j(x, t) - \int_0^t (n_j - N_j(x, u-)) \mu_j(x, u) du$$

**Insurance payment process** (Benefits – premiums on pf 1)

$$\begin{aligned} dA(t) &= (n_1 - N_1(x, \bar{T})) \Delta A_0(\bar{T}) d1_{(t \geq \bar{T})} \\ &\quad + a_0(t)(n_1 - N_1(x, t)) dt + a_1(t) dN_1(x, t) \end{aligned}$$

( $a_i, A_0$  deterministic functions)

## Market value process

$$\begin{aligned}V^{*,Q}(t) &= E^Q \left[ \int_{[0,T]} e^{-\int_0^\tau r(u)du} dA(\tau) \middle| \mathcal{F}(t) \right] \\ &= \int_{[0,t]} e^{-\int_0^\tau r(u)du} dA(\tau) \\ &\quad + e^{-\int_0^t r(u)du} (n - N_1(x, t)) V^Q(t, r(t), \mu_1(x, t))\end{aligned}$$

where

$$\begin{aligned}V^Q(t, r(t), \mu_1(x, t)) &= \int_t^T P(t, \tau) S_1^Q(x, t, \tau) \left( a_0(\tau) + a_1(\tau) f^{\mu_1, Q}(x, t, \tau) \right) d\tau \\ &\quad + P(t, \bar{T}) S_1^Q(x, t, \bar{T}) \Delta A_0(\bar{T}) 1_{(t < \bar{T})}\end{aligned}$$

Survivor swap price process

$$\begin{aligned}\tilde{Z}_j^*(x, t) &= (n_j - N_j(x, t)) \int_t^T P^*(t, u) S_j^Q(x, t, u) du \\ &\quad - n_j {}_t\tilde{p}_x \int_t^T P^*(t, u) {}_{u-t}\tilde{p}_{x+t} du\end{aligned}$$

# Price and market value dynamics

Zero coupon bond price dynamics:

$$dP^*(t, T) = -\sigma^r B^r(t, T)P^*(t, T)dW^{r,Q}(t)$$

Survivor swap price dynamics:

$$dZ_j^{*,Q}(x, t) = \nu_j^{Z,Q}(t)dM_j^Q(x, t) + \eta_j^{Z,Q}(t)dW^{r,Q}(t) + \rho_j^{Z,Q}(t)dW^{\mu,Q}(t)$$

**Market risk:**  $\eta_j^{Z,Q}(t)dW^{r,Q}(t)$

**Remaining risk:** Unsystematic and systematic mortality risk

Market value dynamics:

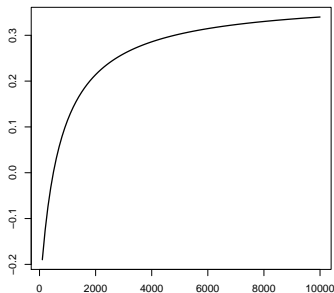
$$dV^{*,Q}(t) = \nu^{V,Q}(t)dM_1^Q(x, t) + \eta^{V,Q}(t)dW^{r,Q}(t) + \rho^{V,Q}(t)dW^{\mu,Q}(t)$$

Market value decomposition:

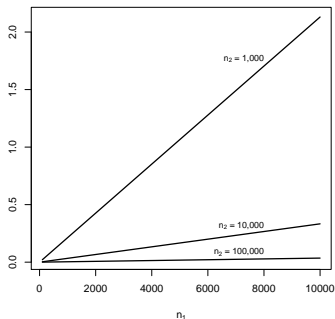
$$dV^{*,Q}(t) = \xi_j^A(t)dX(t) + \vartheta_j^A(t)dZ_j^*(x, t) + dL_j^A(t)$$

# Dependence on the number of policy-holders $n_1$

Survivor swaps on the insurance portfolio at time 0



Survivor swaps on the population at time 0



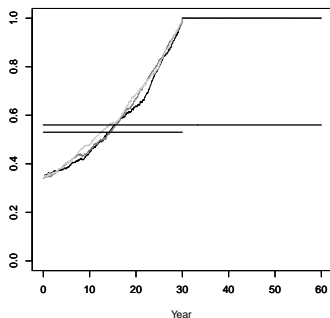
Survivor swaps on portfolio 1:

$$\vartheta_1^A(t) = \frac{\nu^{V,Q}(t) + \rho_1^{V,Q}(t)(\kappa_{1,1}^Q(t))^{-1} + \rho_2^{V,Q}(t)(\kappa_{1,2}^Q(t))^{-1}}{\nu_1^{Z,Q}(t) + \rho_{1,1}^{Z,Q}(t)(\kappa_{1,1}^Q(t))^{-1} + \rho_{1,2}^{Z,Q}(t)(\kappa_{1,2}^Q(t))^{-1}}$$

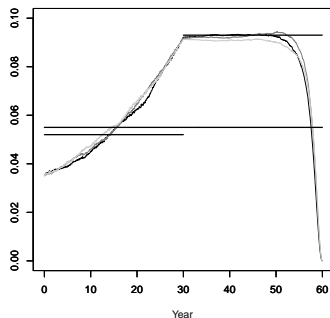
Here:  $\rho_j^{V,Q}(t)$  and  $\rho_{1,j}^{Z,Q}(t)$  are proportional to  $n_1$

# Comparison of discrete and continuous strategy

Number of survivor swaps on the insurance portfolio



Number of survivor swaps on the population



Discrete strategy:

$$\hat{\vartheta}_j^A(0) = \frac{E^Q \left[ \int_0^T \vartheta_j^A(u) dL^{Z_j^*, Q}(u) \Delta L^{Z_j^*, Q}(T) \middle| \mathcal{F}(0) \right]}{E^Q \left[ \left( \Delta L^{Z_j^*, Q}(T) \right)^2 \middle| \mathcal{F}(0) \right]}$$

# Intrinsic risk - continuous hedging

$n_1$	$n_2$	$\frac{\sqrt{R(0, \varphi_V^*)}}{n_1}$	$\frac{\sqrt{R(0, \varphi_B^*)}}{n_1}$	$\frac{\sqrt{R(0, \varphi_1^*)}}{n_1}$	$\frac{\sqrt{R(0, \varphi_2^*)}}{n_1}$
100	1,000	0.634	0.111	0.048	0.103
100	10,000	0.634	0.111	0.048	0.098
1,000	10,000	0.627	0.062	0.032	0.037
1,000	100,000	0.627	0.062	0.032	0.035
10,000	100,000	0.626	0.055	0.013	0.020

## Notation

$\varphi_V^*$ : No hedging (only savings account)

$\varphi_B^*$ : Bond market

$\varphi_1^*$ : Bond market + survivor swap on portfolio 1

$\varphi_2^*$ : Bond market + survivor swap on portfolio 2

# Intrinsic risk - mixed hedging

$n_1$	$n_2$	$\frac{\sqrt{R(0, \varphi_{D_1^*}^*)}}{n_1}$	$\frac{\sqrt{R(0, \varphi_{D_2^*}^*)}}{n_1}$	$\frac{\sqrt{R(0, \varphi_{D_1^*}^*)}}{n_1}$	$\frac{\sqrt{R(0, \varphi_{D_2^*}^*)}}{n_1}$
100	1,000	0.105	0.104	0.073	0.104
100	10,000	0.105	0.100	0.073	0.100
1,000	10,000	0.045	0.040	0.038	0.039
1,000	100,000	0.045	0.039	0.038	0.037
10,000	100,000	0.022	0.026	0.019	0.024

## Notation

$D_j^1$ : Constant swap for portfolio  $j$  on  $(0, 60]$

$D_j^2$ : Trading swap for portfolio  $j$  on  $(0, 30]$  and  $(30, 60]$