AFIR LIFE MUNICH 2009

Section Finance de l'AAI
IAA Finance Section
AFIR

Section Vie de l'AAI
IAA Life Section
LIFE

DAV
DEUTSCHE AKTUARVEREINIGUNG e.V.

Mean-variance inefficiency of CRRA and CARA utility functions for portfolio selection in defined contribution pension schemes[1]

Elena Vigna
Università di Torino and CeRP, Collegio Carlo Alberto

2009 AFIR Colloquium
Munich, 6-11 September 2009

[1] Carlo Alberto Notebook No. 108/2009

Outline

Financial risk in DC pension schemes

In DC pension schemes the financial risk is borne by the member.

The financial risk can be split in two parts: investment risk and annuity risk.
Investment risk: is the risk that low investment returns in the accumulation phase lead to lower than expected accumulated fund at retirement.
Annuity risk: is the risk that low conversion rates (driven by low bond yields at retirement) produce lower than expected annuity rate.

The investment risk can be approached searching appropriate investment strategies in the **accumulation phase**, while the annuity risk can be approached via the so-called ”income drawdown option”, in the **decumulation phase**. Therefore, papers of the actuarial literature on DC pension schemes focus either on the accumulation phase or on the decumulation phase.

We here focus on the accumulation phase, where the member has freedom on the investment strategy.

Literature on investment strategies in the accumulation phase in DC pension schemes

The main approach is maximization of expected utility of final wealth, see:

- Battocchio and Menoncin (IME, 2004)
- Boulier, Huang and Taillard (IME, 2001)
- Cairns, Blake and Dowd (JEDC, 2006)
- Deelstra, Grasselli and Koehl (IME, 2003)
- Devolder, Bosch Princep and Dominguez Fabian (IME, 2003)
- Di Giacinto, Federico and Gozzi (FS, 2009)
- Gao (IME, 2008)
- Haberman and Vigna (IME, 2002)
- Xiao, Zhai and Qin (IME, 2007)

Expected utility (EU) maximization vs mean-variance (M-V) approach

An alternative approach to expected utility maximization is mean-variance approach. However, the mean-variance approach in multi-period discrete time or in continuous time has not been adopted for decades.

Reason: the extension from single-period to multi-period or continuous time is difficult.

The extension has been developed via the martingale approach by Richardson (MS, 1989) and by Bajeux-Besnainou and Portait (MS, 1998). More recently, Li and Ng (MF, 2000) in multi-period discrete time and Zhou and Li (AMO, 2000) in continuous-time use an embedding technique to transform a M-V problem into a stochastic control LQ problem.

Comparison EU vs M-V: Empirical studies

There is little literature about comparison between the two leading approaches for portfolio selection. Before the seminal paper by Li & Ng and Zhou & Li, some empirical papers found inefficiency of optimal portfolios derived with expected utility maximization.

Regarding the logarithmic utility function

- Hakansson (JFQA, 1971) found that the characteristics of mean-variance portfolios and optimal portfolio are different
- Grauer (JFQA, 1981) found that with normal asset returns the composition of the two portfolios are different, both in the mix and in the choice of assets.

Regarding the power utility function,

- Grauer and Hakansson (MS, 1993) found that mean-variance portfolio does not approximate the power utility portfolio.

Comparison EU vs M-V: some theoretical results

- In the single-period framework the two approaches coincide if either the utility function is quadratic or the assets return are normal
- In the continuous-time when prices are log-normal there is consistency between EU and M-V at instantaneous level (Merton, JET 1971, Campbell and Viceira, 2002)
- This does **not** imply that the EU optimal policy should **remain** efficient on a time interval greater than the instant. In fact, in general it does not. This has been mentioned e.g. by Bajeux-Besnainou and Portait (MS, 1998).
- This is relevant, because investors should care more about behaving efficiently on the entire time horizon rather than in each single instant.

Main results of this paper

- In a Black-Scholes model, we prove that the widely used CRRA and CARA utility functions are not efficient in the mean-variance setting.
- We propose a measure of inefficiency, based on the distance between optimal portfolio variance and minimal variance (VIM: Variance Inefficiency Measure).
- We show how the inefficiency measure VIM depends on some relevant parameters of the model, such as risk aversion, Sharpe ratio of the risky asset, time horizon, initial wealth and contribution rate.
- Inefficiency is a decreasing function of risk aversion and an increasing function of time horizon and Sharpe ratio of the risky asset. This is relevant in the context of pension funds, given the long-term horizon involved.

The problem faced by the member

The member joins the scheme at time $t = 0$ and contributes $c \geq 0$ a year for $T > 0$ years until retirement (c and T are fixed). She can choose the investment strategy to be adopted at each time ($y(t)$ is the proportion of portfolio to be invested in the risky asset).

Assumptions:

- riskless asset:

$$dB(t) = rB(t)dt, \tag{1}$$

- risky asset:

$$dS(t) = \lambda S(t)dt + \sigma S(t)dW(t), \tag{2}$$

- fund evolution:

$$dX(t) = \{X(t)[y(t)(\lambda - r) + r] + c\}dt + X(t)y(t)\sigma dW(t) \tag{3}$$

The mean-variance approach

We assume that the member chooses the mean-variance approach. She wants to solve

$$(4)\qquad \begin{array}{ll} \textit{Minimize} & (J_1(y(\cdot)), J_2(y(\cdot))) \equiv (-E(X(T)), Var(X(T))) \\ \text{subject to} & \begin{cases} y(\cdot) & \text{admissible} \\ X(\cdot), y(\cdot) & \text{satisfy (3).} \end{cases} \end{array}$$

An admissible strategy $\bar{y}(\cdot)$ is called an *efficient strategy* if there exists no admissible strategy $y(\cdot)$ such that

$$(5)\qquad J_1(y(\cdot)) \leq J_1(\overline{y(\cdot)}) \quad J_2(y(\cdot)) \leq J_2(\bar{y}(\cdot)),$$

and at least one of the inequalities holds strictly. In this case, the point $(J_1(\bar{y}(\cdot)), J_2(\bar{y}(\cdot))) \in \mathbf{R}^2$ is called an *efficient point* and the set of all efficient points is called the *efficient frontier*.

The mean-variance approach

Applying the result by Zhou and Li, the problem is transformed in the equivalent one

$$(6)\quad \begin{aligned} &Minimize \quad (J(y(\cdot)),\alpha,\mu) \equiv E[\alpha X(T)^2 - \mu X(T)], \\ &\text{subject to} \quad \begin{cases} y(\cdot) & \text{admissible} \\ X(\cdot),y(\cdot) & \text{satisfy (3).} \end{cases} \end{aligned}$$

The derivation of the solution of this standard LQ control problem in a DC pension scheme is fully contained in Højgaard and Vigna (2007).
The optimal investment allocation at time t, given that the fund is x, is given by

$$(7)\quad \bar{y}(t,x) = -\frac{\beta}{\sigma x}\left[x - \delta e^{-r(T-t)} + \frac{c}{r}(1 - e^{-r(T-t)})\right],$$

where

$$(8)\quad \delta = \frac{\mu}{2\alpha} \qquad \text{and} \qquad \beta = \frac{\lambda - r}{\sigma}.$$

Expected value and variance of final fund, efficient frontier

(9) $$E(\overline{X}(T)) = \overline{x}_0 + \frac{e^{\beta^2 T} - 1}{2\alpha}$$

and

(10) $$Var(\overline{X}(T)) = \frac{e^{\beta^2 T} - 1}{4\alpha^2},$$

where

(11) $$\overline{x}_0 := x_0 e^{rT} + \frac{c}{r}(e^{rT} - 1).$$

The efficient frontier in the mean-standard deviation diagram is the straight line

(12) $$E(\overline{X}(T)) = \overline{x}_0 + \left(\sqrt{e^{\beta^2 T} - 1}\right) \sigma(\overline{X}(T)).$$

The expected utility approach

We now assume that the member chooses the expected utility approach. She want to solve

(13)
$$\begin{aligned} &Maximize \quad (J(y(\cdot))) \equiv E[U(X(T))], \\ &\text{subject to} \quad \begin{cases} y(\cdot) & \text{admissible} \\ X(\cdot), y(\cdot) & \text{satisfy (3).} \end{cases} \end{aligned}$$

The optimal value function is defined as

(14)
$$V(t,x) := \sup_{y(\cdot)} J(y(\cdot);t,x)$$

By writing the HJB equation associated to this problem and plugging the optimal control into it, one gets the non-linear PDE for V:

(15)
$$V_t + (rx + c)V_x - \frac{1}{2}\beta^2 \frac{V_x^2}{V_{xx}} = 0.$$

CARA utility function

Consider the exponential utility function

(16) $$U(x) = -\frac{1}{k}e^{-kx},$$

with (constant) Arrow-Pratt coefficient of absolute risk aversion equal to

$$ARA(x) = -\frac{U''(x)}{U'(x)} = k.$$

The optimal amount invested in the risky asset at time t if the wealth is x is:

(17) $$xy^*(t,x) = \frac{\beta}{\sigma k}e^{-r(T-t)}.$$

The expected value and the variance of the final fund under optimal control are

(18) $$E(X^*(T)) = \overline{x}_0 + \frac{\beta^2 T}{k} \qquad Var(X^*(T)) = \frac{\beta^2 T}{k^2}.$$

CRRA utility function, logarithmic utility

Consider the logarithmic utility function

(19) $$U(x) = \ln x,$$

with (constant) Arrow-Pratt coefficient of relative risk aversion equal to

$$RRA(x) = -\frac{U''(x)}{U'(x)}x = 1.$$

The optimal amount invested in the risky asset at time t if the wealth is x is:

(20) $$xy^*(t,x) = \frac{\beta}{\sigma}\left(x + \frac{c}{r}(1 - e^{-r(T-t)})\right).$$

The expected value and variance of the final fund under optimal control are

(21) $$E(X^*(T)) = \overline{x}_0 e^{\beta^2 T} \qquad Var(X^*(T)) = (E(X^*(T)))^2(e^{\beta^2 T} - 1).$$

CRRA utility function, power utility function

Consider the power utility function

$$U(x) = \frac{x^\gamma}{\gamma}, \tag{22}$$

with (constant) Arrow-Pratt coefficient of relative risk aversion equal to

$$RRA(x) = -\frac{U''(x)}{U'(x)}x = 1 - \gamma.$$

The optimal amount invested in the risky asset at time t if the wealth is x is:

$$xy^*(t,x) = \frac{\beta}{\sigma(1-\gamma)}\left(x + \frac{c(1-e^{-r(T-t)})}{r}\right). \tag{23}$$

The expected value and variance of the final fund under optimal control are

$$E(X^*(T)) = \overline{x}_0 e^{\frac{\beta^2 T}{1-\gamma}}, \qquad Var(X^*(T)) = (e^{\frac{\beta^2 T}{(1-\gamma)^2}} - 1)(E(X^*(T)))^2. \tag{24}$$

Notation

- $\overline{X}(T)$: final fund under mean-variance efficient control
- $X_U^*(T)$: final fund under optimal control associated to the utility function U

Sketch of proof

In order to prove that the utility function U produces optimal portfolios that are not efficient, one should prove either

1. $E(X_U^*(T)) = E(\overline{X}(T)) \quad \Rightarrow \quad Var(X_U^*(T)) > Var(\overline{X}(T))$;
   or
2. $Var(X_U^*(T)) = Var(\overline{X}(T)) \quad \Rightarrow \quad E(X_U^*(T)) < E(\overline{X}(T))$.

Theorem

Assume that the financial market and the wealth equation are as described above. Assume that the portfolio selection problem is solved via maximization of the expected utility of final wealth at time T, with preferences described by the utility function $U(x)$. Then, the couple $(Var(X_U^*(T)), E(X_U^*(T)))$ associated to the final wealth under optimal control $X_U^*(T)$ is not mean-variance efficient in the following cases:

i) $U(x) = -\frac{1}{k}e^{-kx}$;

ii) $U(x) = \ln x$;

iii) $U(x) = \frac{x^\gamma}{\gamma}$.

Proof

We follow the procedure as in (1) above.

i) Exponential. The two expected final funds are equal if and only if

(25) $$\frac{\beta^2 T}{k} = \frac{e^{\beta^2 T} - 1}{2\alpha}.$$

We need to prove that, under (25)

(26) $$\frac{\beta^2 T}{k^2} - \frac{e^{\beta^2 T} - 1}{4\alpha^2} > 0.$$

Using (25) it is easy to see that

(27) $$\frac{\beta^2 T}{k^2} - \frac{e^{\beta^2 T} - 1}{4\alpha^2} = \frac{e^{\beta^2 T} - 1}{2\alpha}\left(\frac{1}{k} - \frac{1}{2\alpha}\right) > 0 \Leftrightarrow \frac{k}{2\alpha} < 1.$$

The last inequality is true, since, due to (25),

(28) $$\frac{k}{2\alpha} = \frac{\beta^2 T}{e^{\beta^2 T} - 1} < 1 \qquad \mathit{for} \qquad \beta^2 T \neq 0.$$

ii) Logarithmic. The expected final funds are equal if and only if

$$e^{\beta^2 T} - 1 = \frac{e^{\beta^2 T} - 1}{2\alpha\overline{x}_0}, \tag{29}$$

which happens if and only if

$$\alpha = \frac{1}{2\overline{x}_0}. \tag{30}$$

Proving that, under (30), the variance in the logarithmic case is higher than the variance in the mean-variance efficient case is straightforward. In fact:

$$(\overline{x}_0 e^{\beta^2 T})^2 (e^{\beta^2 T} - 1) - \frac{e^{\beta^2 T} - 1}{4\alpha^2} = \overline{x}_0^2 (e^{\beta^2 T} - 1)^2 (e^{\beta^2 T} + 1) > 0. \tag{31}$$

.

iii) Power. The expected final funds are equal if and only if

$$e^{\frac{\beta^2 T}{1-\gamma}} - 1 = \frac{e^{\beta^2 T} - 1}{2\alpha\overline{x}_0}. \tag{32}$$

We intend to prove that, under (32), the variance in the power case is higher than the variance in the mean-variance efficient case, i.e. we have to prove that

$$(\overline{x}_0 e^{\frac{\beta^2 T}{1-\gamma}})^2 (e^{\frac{\beta^2 T}{(1-\gamma)^2}} - 1) - \frac{e^{\beta^2 T} - 1}{4\alpha^2} > 0. \tag{33}$$

With the change of variable

$$b := e^{\beta^2 T} \qquad a := \frac{1}{1-\gamma}, \tag{34}$$

the claim (33) is equivalent to

$$b^{2a}(b^{a^2}-1)(b-1)-(b^a-1)^2 = b^{a^2+2a+1}-b^{a^2+2a}-b^{2a+1}+2b^a-1 > 0 \tag{35}$$

for all $(a,b) \in (0,+\infty) \times (1,+\infty)$.

A special case: the usual portfolio selection problem

Corollary

Assume that an investor wants to invest a wealth of $x_0 > 0$ for the time horizon $T > 0$ in a financial market as before and wealth equation (3) with $c = 0$. Assume that she maximizes expected utility of final wealth at time T. Then, the couple $(Var(X_U^*(T)), E(X_U^*(T)))$ associated to the final wealth under optimal control $X_U^*(T)$ is not mean-variance efficient in the following cases:

i) $U(x) = -\frac{1}{k}e^{-kx}$;

ii) $U(x) = \ln x$;

iii) $U(x) = \frac{x^\gamma}{\gamma}$.

Proof

The proof is obvious, by setting $c = 0$ in the proof of Theorem (1) and observing that all the inequalities still hold. □

Inefficiency Measures

We define the Variance Inefficiency Measure as

$$VIM(X_U^*(T)) := Var(X_U^*(T)) - Var(\overline{X}(T)). \tag{36}$$

Assuming the same portfolio variance, inefficiency can be measured by the Mean Inefficiency Measure:

$$MIM(X_U^*(T)) := E(\overline{X}(T)) - E(X_U^*(T)). \tag{37}$$

Dependence of inefficiency on relevant parameters

We analyze dependence of inefficiency on relevant parameters:

- risk aversion of the member
- Sharpe ratio β
- time horizon T
- initial wealth x_0
- contribution rate c.

Exponential utility function

$$(38)\qquad VIM(X^*(T)) = \frac{(e^{\beta^2 T}-1)}{2\alpha k}\left(1-\frac{k}{2\alpha}\right) = \frac{\beta^2 T}{k^2}\left(1-\frac{\beta^2 T}{e^{\beta^2 T}-1}\right).$$

So that

1. The inefficiency is a decreasing function of the absolute risk aversion coefficient $ARA = k$.
2. The inefficiency is an increasing function both of the Sharpe ratio β and the time horizon T.
3. The inefficiency does not depend on the initial fund x_0 and on the contribution rate c.

Logarithmic utility function

$$VIM(X^*(T)) = \overline{x}_0^2(e^{\beta^2 T} - 1)^2(e^{\beta^2 T} + 1). \tag{39}$$

So that

1. The inefficiency is an increasing function both of the Sharpe ratio β and the time horizon T.
2. The inefficiency is an increasing function of both the initial fund $x_0 \geq 0$ and the contribution rate $c \geq 0$.

Power utility function

(40)

$$VIM(X^*(T)) = \frac{\overline{x}_0^2}{e^{\beta^2 T} - 1}\left(e^{\frac{2\beta^2 T}{(1-\gamma)}} (e^{\frac{\beta^2 T}{(1-\gamma)^2}} - 1)(e^{\beta^2 T} - 1) - (e^{\frac{\beta^2 T}{1-\gamma}} - 1)^2 \right).$$

With the change of variables:

$$b := e^{\beta^2 T} \qquad a := \frac{1}{1-\gamma},$$

we have:

(41)

$$VIM(X^*(T)) = VIM(a,b) = \frac{\overline{x}_0^2}{b-1}(b^{a^2+2a+1} - b^{a^2+2a} - b^{2a+1} + 2b^a - 1).$$

Power utility function

From the proof of Theorem (1) it is clear that $\frac{\partial VIM}{\partial a} > 0$.
What is more difficult to prove is that $\frac{\partial VIM}{\partial b} > 0$ for all values of $a > 0$, so that is still an open problem. We are able to prove it for $b > 2 \wedge a > a_*$ where $a_* \simeq 0.45$ is the positive root of $2a^3 + 4a^2 - 1 = 0$. These value imply $RRA < \frac{1}{a_*} \simeq 2.22$ and $\beta^2 T > \ln 2 \simeq 0.69$. Our conjecture is that this holds for all possible values of $a > 0$, $b > 1$.

Therefore:

1. The inefficiency is a decreasing function of the relative risk aversion coefficient $RRA = 1 - \gamma$.
2. The inefficiency is an increasing function both of the Sharpe ratio β and the time horizon T, if $\beta^2 T > \ln 2$ and $a < a_*$, where $a_* > 0$ solves $2a^3 + 4a^2 - 1 = 0$.
3. The inefficiency is an increasing function of both the initial fund $x_0 > 0$ and the contribution rate $c > 0$.

Quadratic loss: the target-based approach

We show the expected result that the quadratic utility-loss function is consistent with the mean-variance approach. We consider a modified version of the simple quadratic utility function, considering a target-based approach induced by a quadratic loss function. Højgaard and Vigna (2007) solve the problem of a member of a DC pension scheme who chooses a target value at retirement F and chooses the optimal investment strategy that minimizes

$$E\left[(X(T)-F)^2\right]. \tag{42}$$

In these circumstances, we shall say that the member solves the portfolio selection problem with the **target-based (T-B) approach**.

T-B approach, optimal policy

The optimal amount invested in the risky asset at time t if the wealth is x is:

(43)
$$y_{tb}(t,x) = -\frac{\lambda - r}{\sigma^2 x}(x - G(t)),$$

where

(44)
$$G(t) = Fe^{-r(T-t)} - \frac{c}{r}(1 - e^{-r(T-t)}).$$

T-B approach, mean of final fund

The expected final fund under optimal control turns out to be

(45)
$$E(X^*(T)) = e^{-\beta^2 T}\overline{x}_0 + (1 - e^{-\beta^2 T})F,$$

i.e. turns out to be a weighted average of the target and of the fund that one would have by investing fully in the riskless asset.

Theorem

Assume that the financial market and the wealth equation are as described before. Assume that the portfolio selection problem is solved via minimization of expected loss of final wealth at time T, with preferences described by the loss function $L(x)$. Then,

i) the couple $(Var(X_L^*(T)), E(X_L^*(T)))$ associated to the final wealth under optimal control $X_L^*(T)$ is mean-variance efficient if $L(x) = (F - x)^2$;

ii) each point $(Var(\overline{X}(T)), E(\overline{X}(T)))$ on the efficient frontier as outlined before is the solution of an expected loss minimization problem with loss function $L(x) = (F - x)^2$.

Proof i) After some algebra, the expected final funds are equal iff

$$E(\overline{X}(T)) = F - \frac{1}{2\alpha}. \tag{46}$$

We now have:

$$\begin{aligned} y_{tb}(t,x) &= -\frac{\lambda - r}{\sigma^2 x}\left\{x - \left[Fe^{-r(T-t)} - \frac{c}{r}(1 - e^{-r(T-t)})\right]\right\} \\ &= -\frac{\lambda - r}{\sigma^2 x}\left\{x - \left[\left(F - \frac{1}{2\alpha}\right)e^{-r(T-t)} - \frac{c}{r}(1 - e^{-r(T-t)}) + \frac{e^{-r(T-t)}}{2\alpha}\right]\right\} \\ &= \overline{y}(t,x). \end{aligned} \tag{47}$$

ii) Consider a point $(Var(\overline{X}(T)), E(\overline{X}(T)))$ on the efficient frontier. Using (9) it is possible to find the corresponding $\overline{\alpha}$ which in turn defines the target via (46):

$$\overline{F} = E(\overline{X}(T)) + \frac{1}{2\overline{\alpha}}. \tag{48}$$

It is then obvious that the point $(Var(\overline{X}(T)), E(\overline{X}(T)))$ chosen on the efficient frontier can be found by solving the target-based optimization problem with target equal to $\overline{F}$.□

Positivity of M-V efficient investment strategy

By defining the process

$$Z(t) = G(t) - X^*(t) \tag{49}$$

one can see that the process $Z(t)$ follows a geometric Brownian motion given by:

$$Z(t) = Z(0)e^{(r-\frac{3}{2}\beta^2)t-\beta W(t)} > 0 \qquad \Longleftrightarrow \qquad Z(0) > 0. \tag{50}$$

But $Z(0) > 0$ for the problem to be interesting. Thus, recalling (43), it is easy to prove the following corollary:

Corollary

Consider the financial market and the wealth equation as before. Then, with the mean-variance approach the optimal amount invested in the risky asset at any time $0 \leq t < T$ is strictly positive.

Numerical application

We intend to show extent of inefficiency via numerical examples. We have selected low, medium and high constant relative risk aversion coefficient equal to 1, 2 and 5 respectively. For each of them and for MV approach, power and exponential utility functions, Table 2 reports the standard deviation of the optimal portfolio (that is the *x*-coordinate of the optimal point in the standard deviation-mean diagram) and, in the last column, the (common value of the) mean of the optimal portfolio (i.e. the *y*-coordinate).

RRA $1-\gamma$	MV $\sigma(\overline{X}(T))$	Power $\sigma(X^*(T))$	Exponential $\sigma(X^*(T))$	MV efficient $E(\overline{X}(T))$
1	13.08	120.76	25.18	42.1
2	3.32	11.96	6.24	13.86
5	0.89	2.17	1.71	7.12

Table 2.

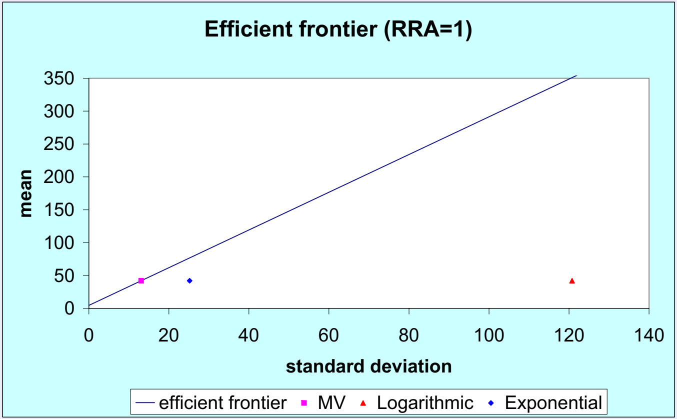
Efficient frontier (RRA=1)
350
300
250
200
150
100
50
0
mean
0
20
40
60
80
100
120
140
standard deviation
efficient frontier
MV
Logarithmic
Exponential

Figure 1

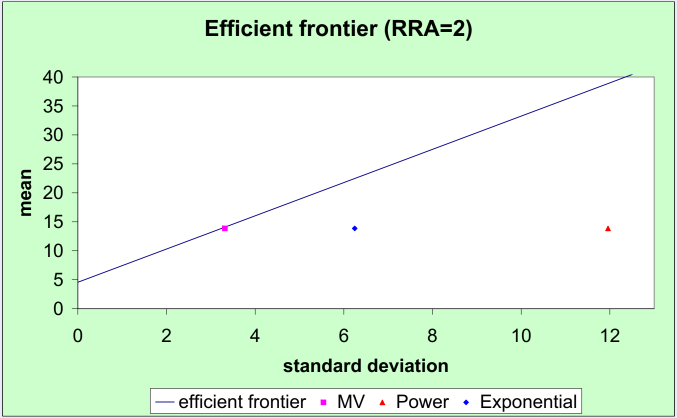
Efficient frontier (RRA=2)
mean
40
35
30
25
20
15
10
5
0
0
2
4
6
8
10
12
standard deviation
efficient frontier
MV
Power
Exponential

Figure 2

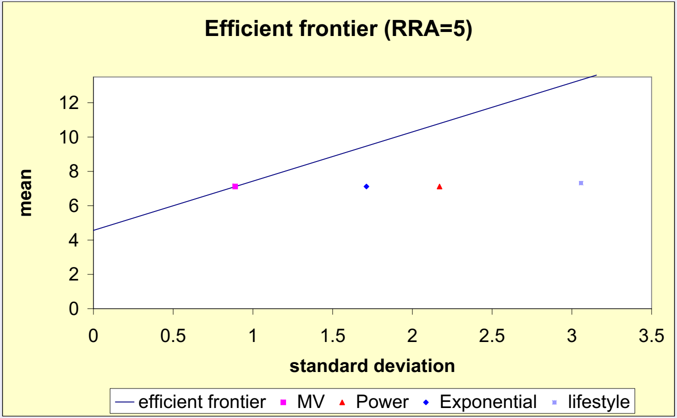
Efficient frontier (RRA=5)
mean
12
10
8
6
4
2
0
0
0.5
1
1.5
2
2.5
3
3.5
standard deviation
efficient frontier
MV
Power
Exponential
lifestyle

Figure 3

Numerical simulations with suboptimal cut-policies. Results.

Final wealth	MV cut	Power cut	Exponential cut	Lifestyle
5th perc.	3.65	4.05	4.05	3.8
25th perc.	6.36	5.28	5.6	5.13
50th perc.	7.1	6.45	6.71	6.61
75th perc.	7.32	7.93	7.88	8.72
95th perc.	7.4	10.63	9.57	13.57
Mean	6.54	6.78	6.77	7.32
Standard deviation	1.22	2.05	1.68	3.06
Prob reaching target	0	0.31	0.34	0.45
Mean shortfall	0.88	1.76	1.61	1.7

Table 3. Distribution of final wealth over 1000 Monte Carlo simulations. (Target $=F = 7.43$).

The case for M-V approach for DC pension schemes

One can argue that if the individual's preferences are represented by a CRRA utility function then she is not mean-variance optimizer. However, observe the following:

- It is evidently difficult for an agent to specify her own utility function and the corresponding risk aversion parameter, it is much easier to reason in terms of targets to reach. This was observed also by Kahneman and Tversky (Econometrica, 1979), by Bordley and Li Calzi (DEF, 2000) and by Zhou and Jin (AMaMeF, 2009).
- Considering that in DC schemes the financial risk is borne by the member, if two distributions of final wealth with same mean but different variances were to be compared, most workers would probably choose the distribution with lower variance, since it is wealth related to future pension.
- Last but not least, the mean-variance criterion is still the most used criterion to value and compare investment funds performances.

Thus, we believe that every further step in understanding the M-V efficient target-based approach should be encouraged in the context of DC pension schemes.

Further research

- Empirical investigations among active members of DC schemes to assess their preferences towards different distributions of final wealth.
- Inclusion of a stochastic interest rate in the financial market (crucial in a long time horizon context such as pension funds).
- Explore other utility functions of the HARA class.
- Time-dependent drift and volatility.
- Extension to multi-period discrete time framework.