Longevity Risk and Hedge Effects in a Portfolio of Life Insurance Products with Investment Risk

Ralph Stevens* Anja De Waegenaere† Bertrand Melenberg‡

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Abstract

Future payments of life insurance products depend on the uncertain evolution of survival probabilities. This uncertainty is referred to as longevity risk. Existing literature shows that the effect of longevity risk on single life annuities can be substantial, and that there exists a natural hedge potential from combining single life annuities with death benefits or survivor swaps. However, the effect of financial risk and portfolio composition on these hedge effects is typically ignored. The aim of this paper is to quantify the hedge potential of combining different life insurance products and mortality linked assets when an insurer faces both longevity and investment risk. We show that the hedge potential of combining different mortality-linked products depends significantly on both the product mix and the asset mix. First, ignoring the presence of other liabilities, such as survivor annuities, can lead to significant overestimation of the hedge effects of death benefits or survivor swaps on longevity risk in single life annuities. Second, investment risk significantly affects the hedge potential in a portfolio of life insurance products. When investment risk increases the hedge potential of survivor annuities increases and of death benefits decreases. Finally, we show that the hedge potential of survivor swaps is not only significantly affected by basis risk, but also by investment risk.

Keywords: Life insurance, life annuities, death benefits, survivor swaps, risk management, financial risk, longevity risk, insolvency risk, capital adequacy.

*Department of Econometrics and OR, Tilburg University, CentER for Economic Research and Netspar.

†Corresponding author: Department of Econometrics and OR Tilburg University, CentER for Economic Research and Netspar, PO Box 90153, 5000LE Tilburg, The Netherlands, Phone: +31 13 4662913, Fax: +31 13 4663280, Email: A.M.B.DeWaegenaere@uvt.nl.

‡Department of Econometrics and OR, Department of Finance, Tilburg University, CentER for Economic Research and Netspar.

1 Introduction

Our goal in this paper is to quantify longevity risk in portfolios of life insurance products, taking into account the potential effect of investment risk on the impact of longevity risk. Specifically, our focus is on the potential interactions between liability product mix effects, and investment mix effects.

Existing literature suggests that uncertainty regarding the future development of human life expectancy potentially imposes significant risk on pension funds and insurers (see, e.g., Olivieri and Pitacco, 2001; Brouhns, Denuit, and Vermunt, 2002; Cossette et al., 2007; Dowd, Cairns, and Blake, 2006). Existing literature also shows that the natural hedge potential that arises from combining life annuities and death benefits may be substantial (see, e.g., Cox and Lin, 2007; and Wang et al., 2008). These analyses quantify longevity risk in annuity portfolios by determining its effect on the probability distribution of the *present value* of all future payments, for a given, deterministic, and constant term structure of interest rates. A drawback of this approach, however, is that it does not allow to take into account the possible interaction between longevity risk and financial risk, i.e., it is a liability only approach. Hári et al. (2008) quantify longevity risk in portfolios of single life annuities in the presence of financial risk by determining its effect on the volatility of the *funding ratio* at a future date. They define the funding ratio as the ratio of the value of the assets over the best estimate value of the liabilities. They find that financial risk can significantly affect the impact of longevity risk on funding ratio volatility. A drawback of a funding ratio approach is that it requires specifying the probability distribution of the (fair) value of the liabilities at a future date. In recent years there has been considerable interest in developing pricing models for longevity linked assets and liabilities (see, e.g., Blake and Burrows, 2001; Dahl, 2004; Lin and Cox, 2005; and Denuit, Devolder, and Goderniaux, 2007). Unfortunately, however, the lack of liquidity for trade in longevity linked assets and/or liabilities makes it very difficult to calibrate these models.

Our goal in this paper is twofold. First, we quantify the impact of longevity risk, as well as interactions between financial risk and longevity risk, in a run-off approach. Specifically, we quantify risk by determining the minimal required buffer (i.e., asset value in excess of the best estimate value of the liabilities), such that the probability that the insurer or pension fund will be able to pay all future liabilities is sufficiently high (see, e.g., Olivieri and Pitacco, 2003). The size of the buffer will be affected by longevity risk, which arises due to uncertain deviations in the future liability payments from their current best estimates, and by financial risk, which arises due to uncertainty in future returns on assets. We show that, even when financial risk and longevity risk are assumed independent, financial risk significantly affects the effect of longevity risk on the minimal required buffer, i.e., interactions between financial and longevity risk should not be ignored.

Second, we quantify the effect of potential interactions between liability product mix effects and asset mix effects on the overall riskiness of a portfolio of life insurance products. Existing literature mainly focuses on the effect of longevity risk on single life annuities, for insureds with given characteristics (see, e.g., Olivieri, 2001; Olivieri and Pitacco, 2003; Cossette et al., 2007; and Hári et al., 2008). Life insurers and pension funds, however, may hold other longevity linked liabilities, such as, e.g., last survivor annuities and death benefit insurance.[1] Because the payments of these different life insurance products typically have different sensitivities to changes in mortality rates, insurers with a "diversified" portfolio of liabilities may be less sensitive to longevity risk. For example, Cox and Lin (2007) show empirically that a life insurer who has 95% of its business in annuities and 5% in death benefits prizes its annuities on average 3% higher than an insurer who has 50% of its business in annuities and 50% of its business in death benefits. This indicates that insurers with death benefits liabilities may have a competitive advantage. In addition, the impact of longevity risk on a portfolio of life insurance products may also depend significantly on the characteristics of the insured population. For example, because longevity trends for males and females are not perfectly correlated, insurers with a more "balanced" gender mix may be less affected by longevity risk.

The existing literature on liability mix effects focuses on the hedge potential of death benefits in portfolios of life annuities, and uses a liability only approach to quantify the risk reduction due to liability mix (see, e.g., Wang et al., 2008, 2009). We extend this analysis by quantifying the impact of investment risk on the natural hedge potential of combining life insurance products with different sensitivities to longevity risk, taking into account that other longevity linked liabilities may affect the hedge potential of death benefits in portfolios of life annuities. Analyzing the effect of product and asset mix on the overall risk is important for two reasons. First, taking into account interactions between financial and longevity risk may lead to more accurate solvency measures. For example, we find that while both partner pension liabilities and death benefits provide some hedge effect for longevity risk in old-age pension liabilities, the hedge potential of partner pension typically increases when investment risk increases, but the opposite

[1]Many defined benefit pension funds offer both old-age pension insurance and partner pension insurance. The latter consists of a survivor annuity that yields periodic payments if the partner of the insured is alive and the insured has passed away. The Retirement Equity Act of 1984 (REA) amended the Employee Retirement Income Security Act of 1974 (ERISA) to introduce mandatory spousal rights in pension plans.

holds for the hedge potential of death benefits. Second, insurers may be able to reduce their sensitivity to longevity risk by redistributing their risk. This may allow them to reduce longevity risk without a counterparty which takes over longevity risk. Our results indicate that the extent to which insurers may benefit from such mutual reinsurance depends not only on their liability portfolios, but also on their investment strategies.

Finally, the impact of longevity risk on the insurer's solvency may be reduced by investing in longevity-linked assets such as, e.g., a longevity bond or a survivor swap. Because the payments of such instruments are based on actual survival of a reference population, they may be used to partially hedge longevity risk. Existing literature shows that the hedge potential can be affected significantly by basis risk, i.e., residual risk due to differences in characteristics of the insured population and the reference population, see, e.g., Dowd, Cairns, and Blake (2006). In this paper we show that the hedge potential of such longevity linked assets not only depends on basis risk, but also on investment risk.

The paper is organized as follows. Section 2 presents the model and gives a formal definition of the risk measure. Section 3 shows how investment risk affects the impact of longevity risk in single life annuities, survivor annuities, and death benefits, respectively. In Section 4 we quantify the effect of the interaction between liability mix effects and asset mix effects. Section 5 deals with the effect of liability and asset mix on the hedge potential of survivor swaps. Section 6 concludes.

2 The model

This section is organized as follows. In Subsection 2.1 we define the life insurance liabilities that we consider. In Subsection 2.2 we define how we quantify risk in portfolios that are sensitive to both longevity risk and financial risk. Subsection 2.3 provides a brief discussion of the models that are used to forecast death rates, interest rates, and asset returns. A complete description of these models can be found in Appendices A and B.

2.1 Liabilities

In this subsection we define the life insurance liabilities that we consider. In addition to traditional old-age pensions, which take the form of a single life annuity, pension funds and insurers typically also offer different other types of life insurance products, such as partner pensions and death benefits. A partner pension provides the partner of a deceased participant with a life long annuity payment. The death benefit consists of a single payment at the moment the insured dies. Formally, we consider the following three types of liabilities:

i) A *single life annuity*, which yields a nominal yearly *old-age pension* payment of 1, with a last payment in the year the insured dies;

ii) A *survivor annuity*, consisting of a nominal yearly *partner pension* payment of 1 every year that the spouse outlives the insured;

iii) A *death benefit*, consisting of a nominal single *death benefit* payment of 1, in the year that the participant dies.

The liabilities consist of a stream of payments in future periods. Throughout the paper, we consider a fixed and given year $t = 0$, and quantify the risk in the payments in future years τ, for a portfolio with given characteristics. Because our focus is on the interaction between product mix and asset mix effects, we will consider portfolios consisting of several products, with varying weights. Specifically, let $\mathcal{P} = \{oa, pp, db\}$ denote the set of life insurance products, and let $\mathcal{I}$ denote the set of insureds.

Because in any future period, the level of the payment depends on whether the insured is alive, and, in case of partner pensions, whether the partner is alive, relevant characteristics are those that affect the probability distribution of their remaining lifetimes. In addition to age, remaining lifetime also significantly depends on gender and time. Therefore, we characterize a participant by a vector $(\overline{x}, \overline{g})$, where

$$\begin{array}{rclrcll} \overline{x} & = & x, & \overline{g} & = & g, & \text{if } p \in \{oa, db\}, \\ \overline{x} & = & (x, y), & \overline{g} & = & (g, g'), & \text{if } p = pp, \end{array}$$

where x denotes the age of the insured, $g \in \{m, f\}$ denotes the gender of the insured, and, in case of partner pension insurance, y denotes the age of the partner, and $g' \in \{m, f\}$ denotes her/his gender. Moreover, let

- $p_{x,t}^{(g)}$ denotes the probability that an x-year-old at date-t with gender g will survive at least another year;
- ${}_{\tau}p_x^{(g)} = p_{x,0}^{(g)} \cdot p_{x+1,1}^{(g)} \cdots p_{x+\tau-1,\tau-1}^{(g)}$ denotes the date-0 probability that an x-year-old at date-0 with gender g will survive at least another τ years.

The net cash outflow of life insurance products is affected by two types of longevity risk:

- *non-systematic longevity risk* because, conditional on given survival probabilities, whether an individual survives an additional year is a random variables;

- *systematic longevity risk* because the survival probabilities for future dates τ, $p_{x,\tau}^{(g)}$, are random variables.

While non-systematic longevity risk is diversifiable (i.e., the risk becomes negligible when portfolio size is large), this is not the case for systematic longevity risk. Therefore, throughout the paper we assume that portfolios are large enough for non-systematic longevity risk to be negligible, and focus on the impact of systematic longevity risk. Then, for any given future year τ, the liability payments for an old-age pension insurance, a partner pension insurance, and a death benefit insurance respectively, are given by (see, also, e.g., Gerber, 1997):

$$\begin{aligned} \widetilde{L}_{p,\tau}(\overline{x},\overline{g}) &= {}_{\tau}p_x^{(g)}, && \text{for } p = oa \text{ (old-age pension)},\\ &= \left(1 - {}_{\tau}p_x^{(g)}\right) \cdot {}_{\tau}p_y^{(g')}, && \text{for } p = pp \text{ (partner pension)},\\ &= {}_{\tau-1}p_x^{(g)} - {}_{\tau}p_x^{(g)}, && \text{for } p = db \text{ (death benefit)}. \end{aligned} \tag{1}$$

Then, the total payment of the life insurer in year τ is the sum of the payments for all products and for all insureds, i.e.,

$$\widetilde{L}_{\tau} = \sum_{i\in\mathcal{I}}\sum_{p\in\mathcal{P}} \overline{\delta}_{i,p} \cdot \widetilde{L}_{p,\tau}(\overline{x}_i, \overline{g}_i), \tag{2}$$

where $\overline{\delta}_{i,p}$ denotes the insured right of insured i for pension product p.

2.2 Quantifying longevity risk

We quantify longevity risk by determining the minimal size of the *buffer*, which is defined as the asset value in excess of the best estimate value of the liabilities, such that the probability that the insurer or pension fund will not be able to pay all future liabilities is sufficiently small. Specifically, let us denote T for the maximum number of years the insurer has to make payments,[2] and let $\widetilde{L}_s$ for the random payment in period s. Then, the current (i.e., date 0) best estimate of the liabilities equals expected present value of future payments, which is given by:

$$BEL = \sum_{s=1}^{T} \mathbb{E}\left[\widetilde{L}_s\right] \cdot P^{(s)}, \tag{3}$$

[2] For example, in a portfolio with annuities and death benefits, $T = x_{\max} + 1 - x_{min}$, where $x_{\max}$ is such that the probability that an individual reaches the age $x_{\max} + 1$ is zero, and x_{min} is the age of the youngest insured.

where $P^{(s)}$ denotes the current market value of a zero-coupon bond with maturity s, i.e., the present value of 1 paid out in period s. We express the initial asset value A_0 as the best estimate value of the liabilities plus a buffer that is a percentage of the best estimate value, i.e.,

$$A_0 = (1 + c) \cdot BEL. \tag{4}$$

The value of the insurer's assets at a future date depends on realized liability payments, and realized asset returns. Indeed, for any given s, the value of the assets at date $s + 1$ is given by:

$$A_{s+1} = \left(A_s - \widetilde{L}_s\right) \cdot (1 + R_{s+1}), \text{ for all } s = 0, \cdots, T - 1, \tag{5}$$

where R_{s+1} denotes the return on investments between time s and $s+1$. Consequently, the *terminal asset value*, i.e., the remaining asset value at time T, is given by:

$$A_T = A_0 \cdot \prod_{s=1}^{T} (1 + R_s) - \sum_{s=1}^{T} \widetilde{L}_s \cdot \prod_{\tau=s+1}^{T} (1 + R_\tau).$$

We quantify risk in portfolios of life insurance products with a given investment strategy by determining the minimal value of c such that the probability that the insurer or pension fund will not be able to pay all future liabilities is sufficiently small, i.e.,

$$\mathbb{P}\left(A_T < 0 \mid A_0 = (1 + c) \cdot BEL\right) \leqslant \varepsilon. \tag{6}$$

The minimal required buffer percentage c depends on the probability distribution of the terminal value of the assets, A_T, which, in turn, depends not only on on the initial asset value A_0, and the liability payments $\widetilde{L}_s$, but also on the investment strategy.

Recall that we express the initial asset value A_0 as the best estimate value of the liabilities, BEL, plus a buffer which is expressed as a percentage of the best estimate value. We allow for the case where the insurer uses a different investment strategy on both parts. The two portfolios will be referred to as the *best estimate portfolio* and the *buffer portfolio*, respectively. In addition, we want to take into account that pension funds or insurers may wish to do some form of duration matching. Therefore, we define the following strategies:

- best estimate portfolio: for every duration $s = 1, \cdots, T$,
  - the best estimate present value corresponding to duration s, i.e., the amount

$\mathbb{E}\left[\widetilde{L}_s\right] \cdot P^{(s)}$, is (re)invested in a portfolio that yields return $r_\tau^{be,(s)}$ in periods $\tau = 0, \cdots, s$;

– in period s, the value $\mathbb{E}\left[\widetilde{L}_s\right] \cdot P^{(s)} \cdot \prod_{\tau=0}^{s}\left(1 + r_\tau^{be,(s)}\right)$ is used to pay the liabilities in period s; any shortage or excess is taken from, or reinvested in, the buffer portfolio;

- buffer portfolio: is (re)invested in a portfolio that yields return r_τ^{bu} in periods $\tau = 0, \cdots, T$.

Note that whereas the value of the buffer portfolio is affected by both longevity risk and investment risk, the value of the best estimate portfolio is only affected by investment risk. For example, when the buffer portfolio is invested in equity and the best estimate portfolio in zero-coupon bonds, a lower return on the assets, or a higher than expected realization of the liabilities, leads to a smaller proportion of assets invested in equity.

With the above described investment strategy, we obtain the following result.

Proposition 1 *The required minimal buffer value is given by*

$$c = \frac{Q_{1-\epsilon}(L)}{BEL} - 1, \tag{7}$$

where

$$L = BEL + \sum_{s=1}^{T}\left(\frac{\widetilde{L}_s - \mathbb{E}\left[\widetilde{L}_s\right] \cdot P^{(s)} \cdot \prod_{\tau=1}^{s}\left(1 + r_\tau^{be,(s)}\right)}{\prod_{\tau=1}^{s}\left(1 + r_\tau^{bu}\right)}\right), \tag{8}$$

and $Q_{1-\varepsilon}(L)$ denotes the $(1-\varepsilon)-$quantile of L.

Proof. Given the above described investment strategy, and given (3), it follows from (5) that the terminal asset value satisfies:

$$\begin{aligned} A_T &= c \cdot BEL \cdot \prod_{s=1}^{T}\left(1 + r_s^{bu}\right) + \sum_{s=1}^{T}\left(\mathbb{E}\left[\widetilde{L}_s\right] \cdot P^{(s)} \cdot \prod_{\tau=1}^{s}\left(1 + r_\tau^{be,(s)}\right) - \widetilde{L}_s\right)\prod_{\tau=s}^{T}\left(1 + r_\tau^{bu}\right) \\ &= \left[(1+c) \cdot BEL - L\right] \cdot \prod_{s=1}^{T}\left(1 + r_s^{bu}\right), \end{aligned} \tag{9}$$

with L as defined in (8). Therefore, the terminal asset value A_T is nonnegative iff

$$A_0 = (1+c) \cdot BEL \geqslant L, \tag{10}$$

The result now follows immediately from (6). ■

The random variable L can be interpreted as follows. Conditional on any given future asset returns (r_τ^{bu} and $r_\tau^{be,(s)}$), and cash flows ($\widetilde{L}_s$), L represents the value of the assets needed at date 0 to pay all future liability payments. For the sake of intuition, consider for example the case where all assets would yield a deterministic and constant annual return r. Then, L is given by:

$$L = \sum_{s=1}^{T} \frac{\widetilde{L}_s}{(1+r)^s}, \tag{11}$$

i.e., the discounted present value of all future liabilities. Thus, the standard approach in which longevity risk is quantified by determining its effect on the probability distribution of the present value of liabilities can be seen as a special case of our model. Taking into account that asset return are uncertain, however, implies that L is not only affected by longevity risk, but also by financial risk. Therefore, we decompose L into four components, i.e.,

$$L = BEL + L^{long} + L^{inv} + L^{interact},$$

where

i) BEL is the deterministic component: BEL represents the value of the assets that would be needed at date 0 to pay all future *expected* liability payments, when these expected liabilities are cash-flow matched, i.e., for all durations s, the amount $\mathbb{E}\left[\widetilde{L}_s\right] \cdot P^{(s)}$ is invested in zero coupon bonds with maturity s. This implies that the aggregate return over s periods for this part of the best estimate portfolio equals $\prod_{\tau=1}^{s}\left(1 + r_\tau^{be,(s)}\right) = \frac{1}{P^{(s)}}$.

ii) L^{long} is the pure longevity risk component:

$$L^{long} = \sum_{s=1}^{T} \frac{\widetilde{L}_s - \mathbb{E}\left[\widetilde{L}_s\right]}{\prod_{\tau=1}^{s}\left(1 + \mathbb{E}\left[r_\tau^{bu}\right]\right)}.$$

This component represents the value of the assets that would be needed at date 0 to pay all future *unexpected* liability payments (i.e., payments in excess of the best estimate value), in absence of financial risk, i.e., when the asset return would be deterministic and equal to the expected return on the buffer portfolio, i.e., $\mathbb{E}\left[r_\tau^{bu}\right]$.

ii) L^{inv} is the pure investment risk component:

$$L^{inv} = \sum_{s=1}^{T} \frac{\mathbb{E}\left[\widetilde{L}_s\right] - \mathbb{E}\left[\widetilde{L}_s\right] \cdot P^{(s)} \cdot \prod\limits_{\tau=1}^{s} \left(1 + r_\tau^{be,(s)}\right)}{\prod\limits_{\tau=1}^{s} \left(1 + r_\tau^{bu}\right)}.$$

This component represents the value of the assets that is needed at date 0 in excess of BEL to pay all future *expected* liability payments due to deviations of the return on the best estimate portfolio from the cash-flow matching return, i.e., due to $\prod\limits_{\tau=1}^{s} \left(1 + r_\tau^{be,(s)}\right) \neq \frac{1}{P^{(s)}}$.

iii) $L^{interact}$ is the interaction investment and longevity risk component:

$$L^{interact} = \sum_{s=1}^{T} \left(\widetilde{L}_s - \mathbb{E}\left[\widetilde{L}_s\right]\right) \cdot \left(\frac{1}{\prod_{\tau=1}^{s} \left(1 + r_\tau^{bu}\right)} - \frac{1}{\prod_{\tau=1}^{s} \left(1 + \mathbb{E}\left[r_\tau^{bu}\right]\right)}\right).$$

This component represents the value of the assets that is needed at date 0 to pay all future *unexpected* liability payments due to deviations of the return on the buffer portfolio from its expected return, i.e., due to $r_\tau^{bu} \neq \mathbb{E}\left[r_\tau^{bu}\right]$. It reflects the interaction between longevity risk and financial risk; it is non zero only if there is both longevity risk and financial risk.

2.3 Modeling mortality rates and asset returns

In this subsection we briefly describe the stochastic forecast models we use to forecast the probability distribution of the future survival probabilities $p_{x,s}^{(g)}$ for $s \geq 0$, future returns on equity, and future term structure of interest rates. For the probability distribution of the future survival probabilities we include process risk, parameter risk, and model risk. To incorporate model risk, we estimate three classes of survival probability models, namely the Lee-Carter (1992) class of models, the Cairns-Blake-Dowd (2006) class of models, and the P-Splines model (Currie, Durbin, and Eilers, 2004), and generate 5000 scenarios for future survival rates from each class of models. To estimate the parameters in each model, we use age-, gender-, and time-specific number of death and exposures to death for the Netherlands, obtained from the Human Mortality Database. For a detailed description of the future survival probabilities models and estimation techniques, and for parameter estimates, we refer to Appendix B.

We generate 15000 scenarios for future survival probabilities; 5000 scenarios from Lee-Carter (1992)-type models with three different specifications, namely the Lee-Carter

(1992) model (1666 scenarios), the Brouhns, Denuit, and Vermunt (2002) model (1667 scenarios), and the Cossete et al. (2007) model (1667 scenarios); 5000 scenarios from Cairns-Blake-Dowd (2006) models with four different specifications, allowing for a quadratic term in the age effect, and/or constant/diminishing age effects in the cohort effects (each specification 1250 scenarios); and 5000 scenarios from the P-Splines model with one specification.

To forecast the future probability distribution of the asset returns, we use a Vasicek model to forecast the future probability distribution of the term structure of interest rates, combined with a Brownian motion with drift to model the stock prices. We include process risk and parameter risk. To incorporate parameter risk we estimate the parameters using Generalized Methods of Moments (GMM). We allow for dependence between the term structure of interest rates and the equity return in process risk by allowing the residual term in the equity returns and the term structure of interest rates to be dependent, and in parameter risk, by estimating the parameters for the future term structure of interest rates and stock prices simultaneously. To estimate the parameters of the probability distribution of future term structures of interest rates and equity returns, we use the daily instantaneous short rate, the daily interest rate on a 10 years Dutch government bond, and the daily return on the Dutch stock index "AEX", obtained from the Datastream. For a more detailed description of the term structure of interest rates and equity return model and estimation technique, and for parameter estimates, we refer to Appendix A.

3 Effect of interest rate risk

In this section we investigate how investment risk affects the impact of longevity risk in single life annuities, survivor annuities, and death benefits, respectively. In the traditional liability only approach, longevity risk is quantified by determining its effect on the discounted present value of the liability payments, given a constant and deterministic interest rate, i.e., by investigating the distributional characteristics of L as given in (11). As shown in the previous section, however, uncertain deviations from expected liabilities imply that financial risk cannot be fully hedged, and may affect the impact of longevity risk in a nontrivial way. In this section we illustrate the impact of financial risk on the required size of the buffer, by comparing the benchmark case in (11) to the case where interest rates are uncertain.

To quantify the effect of interest rate risk, we compare two investment strategies. The first investment strategy is a "naive one", where in every year, the remaining asset value after payment of the liabilities is reinvested in one-year zero-coupon bonds. Since pension liabilities often have a long duration, this naive investment strategy may have substantial investment risk. We therefore also consider an investment strategy that eliminates interest rate risk in the liabilities in the best estimate scenario. We refer to this investment strategy as the *expected liability hedge strategy.* Thus,

- In the *naive investment strategy*, both the best estimate value BEL, and the buffer $c \cdot BEL$ are (re)invested in one-year zero-coupon bonds, so that

  $$r_\tau^{be,(s)} = r_\tau^{bu} = r_\tau^{(1)}, \text{ for all } \tau = 0, \cdots, T, \text{ and } s = 1, \cdots, T,$$

  where $r_\tau^{(1)}$ denotes the one-year interest rate in year τ. Therefore, the required buffer percentage c is given by (7), with

  $$L = \sum_{s=1}^{T} \frac{\widetilde{L}_s}{\prod_{\tau=1}^{s} \left(1 + r_\tau^{(1)}\right)}. \tag{12}$$

- In the *expected liability hedge strategy*, the best estimate value, BEL, is invested in a portfolio of zero-coupon bonds that cash flow matches the best estimate value of the liabilities in each future period, i.e., for every duration s, the amount $\mathbb{E}\left[\widetilde{L}_s\right] \cdot P^{(s)}$ is invested in zero coupon bonds with maturity s. The buffer $c \cdot BEL$ is invested in one-year zero-coupon bonds. Specifically,

  $$\begin{aligned} &\textstyle\prod_{\tau=1}^{s} \left(1 + r_\tau^{be,(s)}\right) = \frac{1}{P^{(s)}}, \\ &1 + r_\tau^{bu} = 1 + r_\tau^{(1)}, \end{aligned} \tag{13}$$

  Then, the required buffer percentage c is given by (7), with

  $$L = BEL + \sum_{s=1}^{T} \frac{\widetilde{L}_s - \mathbb{E}\left[\widetilde{L}_s\right]}{\prod_{\tau=1}^{s} \left(1 + r_\tau^{(1)}\right)}. \tag{14}$$

  Under this strategy, the best estimate portfolio eliminates investment risk in the best estimate scenario for the liabilities. Indeed, for each time to maturity s, the face value of the zero-coupon bonds equals the current best estimate of the cash flow in year s. Thus, all hedgeable risk is eliminated, and investment risk only arises due to uncertain deviations in realized liability payments, which affect the value of the buffer portfolio.

We use the models described in the Appendix to simulate the probability distributions of future investment returns and survival probabilities. We then use these simulated distributions to determine the minimum value of the buffer that is needed in order to reduce the probability of ruin to 2.5% (i.e., $\varepsilon = 0.025$), for the two investment strategies, and single life annuities, survivor annuities, and death benefits, respectively. We consider two types of insureds, namely males and females aged 65. In case of survivor annuities, the partner of a male insured is a female aged 62; the partner of a female insured is a male aged 68.[3] The results are displayed in Table 1. It is intuitively clear that the effect of longevity risk as well as of financial risk on the required buffer may depend substantially on the duration of the liabilities. Therefore, the first column in Table 1 displays the duration of the best estimate of the liabilities, which is given by:

$$\text{Duration} = \frac{\sum_{s=1}^{T} s \cdot P^{(s)} \cdot \mathbb{E}\left[\widetilde{L}_s\right]}{\sum_{s=1}^{T} P^{(s)} \cdot \mathbb{E}\left[\widetilde{L}_s\right]}.$$

Table 1: Capital requirements for life insurance products

Product	Duration	c^{naive}	$c^{\text{elh,1y}}$	c^{LO}
Male single life annuity	8.2	27.6%	6.1%	4.4%
Female single life annuity	8.9	33.0%	6.9%	4.9%
Male survivor annuity	16.4	88.2%	20.1%	12.9%
Female survivor annuity	13.5	74.8%	37.8%	24.6%
Male death benefit	14.5	67.3%	7.6%	8.0%
Female death benefit	16.8	89.5%	15.7%	15.6%

The table displays the duration (first column), and the minimal required buffer for the naive investment strategy (only one-year zero-coupon bonds, second column), the *expected liability hedge strategy* as defined in (13) (third column), and for the benchmark liability-only case (last column). We observe that the minimal required buffer percentage depends heavily on the investment strategy. First, compared to the naive investment strategy with only one-year zero-coupon bonds, the expected liability hedge strategy with a buffer portfolio invested in one-year zero-coupon bonds significantly reduces the reserve requirements for each life insurance product. Second, the liability only approach may lead to significant underestimation of the minimum required buffer. Indeed, even when all hedgeable financial risk is avoided (i.e., under the expected liability hedge strategy), the required buffer is still significantly larger, except for death benefits, than using the liability only approach. Hence, even for an insurer who has this conservative investment strategy, there is still significant investment risk. This occurs because

[3] The age difference is based on the average age difference in married couples (see, e.g., Brown and Poterba, 2000).

longevity risk implies that the future cash flows cannot be fully cash-flow matched using zero-coupon bonds. Note that the high reserve requirements using the investment strategy with only one-year zero-coupon bonds is partly due to a lower expected yearly return on bonds with a short duration instead of a long duration.

4 Asset and liability mix effects

In this section we quantify the effect of product mix (ratios of insured rights for the different life insurance products) and gender mix (ratio of male insured rights over total insured rights) on the required solvency buffer, for different investment strategies. To highlight the effect of the interaction between longevity risk and investment risk, we consider the case where the insurer eliminates all hedgeable investment risk by investing the best estimate value in a portfolio of zero-coupon bonds that matches the best estimate value of the liabilities in each future period. Financial risk then only arises due to longevity risk, because investment risk in the assets needed to cover unexpected deviations from the best estimate value cannot be hedged. Therefore, we consider the following investment strategies:

- BEL is invested in the *expected liability hedge strategy,* as defined in (13), so that

$$L = BEL + \sum_{s=1}^{T} \frac{\widetilde{L}_s - \mathbb{E}\left[\widetilde{L}_s\right]}{\prod_{\tau=1}^{s}\left(1 + r_{\tau}^{bu}\right)}.$$

- the buffer is invested in one of the following portfolios:
  - 100% one-year zero-coupon bonds;
  - 67% one-year zero-coupon bonds, 33% equity;
  - 33% one-year zero-coupon bonds, 67% equity;
  - 100% equity.

We consider two types of insureds, male and female insureds aged 65. The partner of a male insured (if present) is aged 62; the partner of a female insured (if present) is aged 68. Now let us denote:

$$\begin{array}{lll} \delta_{p,g} & = \sum_{i\in\mathcal{I},g_i=g} \overline{\delta}_{i,p}, & \text{for } g \in \{m,f\}, \text{ and } p \in \mathcal{P}, \\ \gamma & = \frac{\delta_{oa,m}}{\delta_{oa,m}+\delta_{oa,f}} & \\ w_g & = \frac{\delta_{pp,g}}{\delta_{oa,g}} & \text{for } g \in \{m,f\}, \\ d_g & = \frac{\delta_{db,g}}{\delta_{oa,g}} & \text{for } g \in \{m,f\}, \end{array} \qquad (15)$$

i.e., $\delta_{p,g}$ denotes the total insured right for product p for insureds with gender g; γ is the fraction of male single life annuities right relative to the total single life annuities rights, w_g is the fraction of survivor annuities rights relative to single life annuities rights for gender g, and d_g is the fraction of death benefits relative to single life annuities rights for gender g. Then, it is verified easily that the aggregate liabilities in years s, as defined in (2) satisfy:

$$\frac{\widetilde{L}_s}{\delta_{oa,m}+\delta_{oa,f}} = (1-\gamma)\cdot\left[\widetilde{L}_{oa,s}(65,f) + w_f\cdot\widetilde{L}_{pp,s}(65,68,f,m) + d_f\cdot\widetilde{L}_{db,s}(65,f)\right] + \gamma\cdot\left[\widetilde{L}_{oa,s}(65,m) + w_m\cdot\widetilde{L}_{pp,s}(65,62,m,f) + d_m\cdot\widetilde{L}_{db,s}(65,m)\right],$$

where $\widetilde{L}_{oa,s}(\cdot)$, $\widetilde{L}_{pp,s}(\cdot)$, and $\widetilde{L}_{db,s}(\cdot)$ are as defined in (1). Thus the effect of product and gender mix is fully characterized by γ, w_g, and d_g.

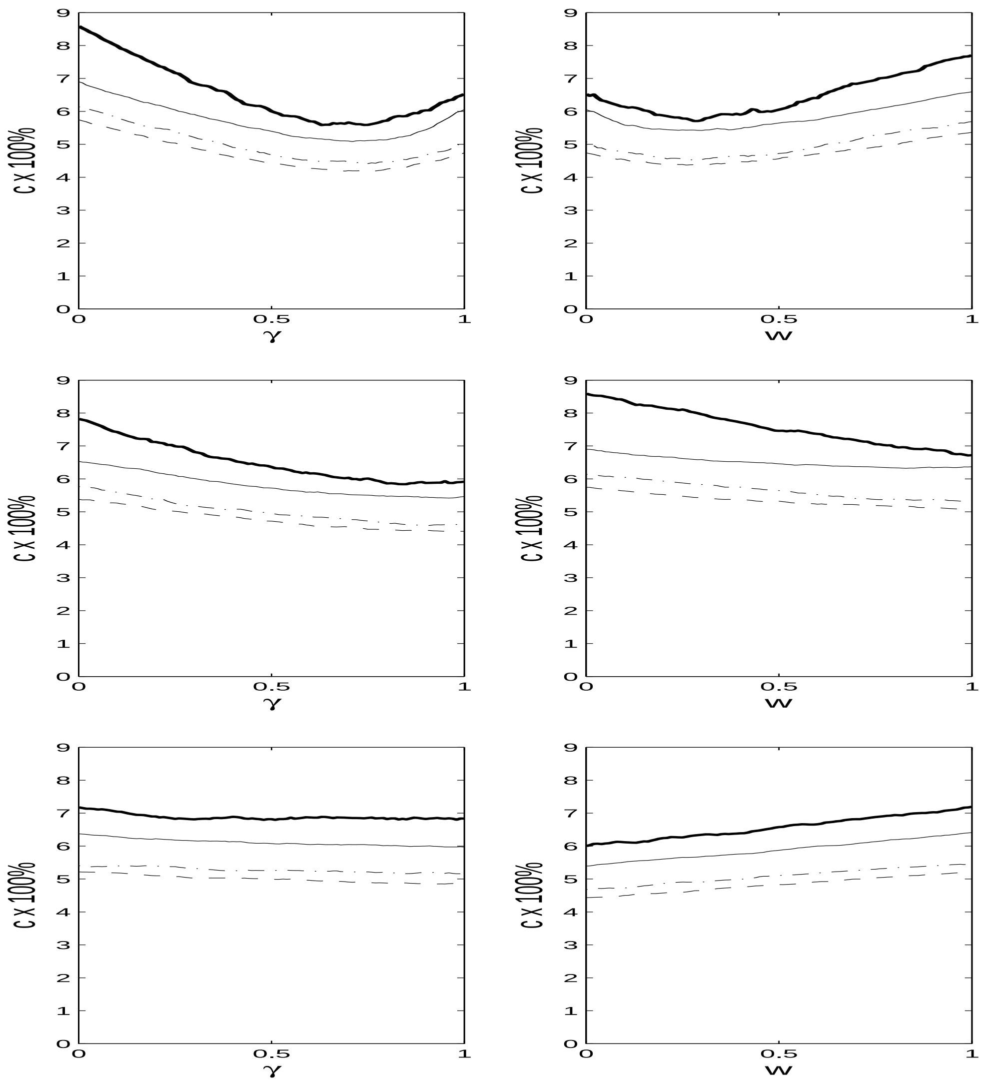
In Subsection 4.1 we investigate the hedge effects of product and gender mix in portfolios of single life and survivor annuities, without death benefits. In Subsection 4.2 we include death benefits.

4.1 Gender and product mix in life insurance products

In this subsection we investigate the effect of product and gender mix on longevity risk in portfolios of single life and survivor annuities without death benefits liabilities, i.e., $d_m = d_f = 0$. We also investigate how these product and gender mix effects are affected by the investment strategy. In order to reduce the number of parameters, we consider the case where product mix (i.e., the ratio of survivor annuity rights over single annuity rights) is the same for both genders, i.e., $w = w_m = w_f$.

The left panels in Figure 1 illustrate the effect of gender mix (i.e., the ratio γ of male insured rights over total insured rights) on the required buffer percentage c, in portfolios of single life and survivor annuities, for three different product mixes: $w = 0$ (top panel), $w = 0.35$ (middle panel), and $w = 0.7$ (bottom panel). The right panels in Figure 1 illustrate the effect of product mix (i.e., the ratio w of insured rights for survivor annuities over insured rights for single life annuities) on the required buffer percentage c, for three different gender mixes: 100% male insured rights ($\gamma = 1$, top panel), 100% female insured rights ($\gamma = 0$, middle panel), and 50% male insured rights and 50% female insured rights ($\gamma = 0.5$, bottom panel). In each case we consider four different investment strategies for the buffer portfolio, 100% equity (bold), 67% equity and 33% one-year zero-coupon bonds (dashed lines), 33% equity and 67% one-year zero-coupon bonds (dotted lines), and 100% one-year zero-coupon bonds (solid lines).
We observe the following:

Figure 1: Reserve requirements in portfolios of single life and survivor annuities.



Left column of panels: the effect of gender mix. The panels display the required buffer percentage c as a function of γ, in portfolios of single life and survivor annuities where the ratio of survivor annuity rights over single life annuity rights equals $w = 0$ (top panel), $w = 0.35$ (middle panel), and $w = 0.7$ (bottom panel). Right column of panels: the effect of product mix. The panels display required buffer percentage c as a function of w, in portfolios of single life and survivor annuities where the ratio of male insured rights over total rights equals $\gamma = 1$ (top row), $\gamma = 0$ (middle row), and $\gamma = 0.5$ (bottom row). In each case we consider four different investment strategies for the buffer portfolio, 100% equity (bold), 67% equity and 33% one-year zero-coupon bonds (dashed lines), 33% equity and 67% one-year zero-coupon bonds (dotted lines), and 100% one-year zero-coupon bonds (solid lines).

- For every liability mix, reserve requirements are significantly affected by unhedgeable investment risk. An increase in equity leads to a higher expected return, but it also yields a higher probability that the realized return is lower than expected. The first effect dominates when the fraction of equity is lower than 33%. The latter effect dominates for more risky investment strategies. Depending on the liability mix, the required buffer percentage when the buffer portfolio is fully invested in equity is almost 50% higher than when only 33% of the buffer portfolio is invested in equity.

- With regard to liability mix effects (i.e., effects of gender and product mix), we observe two effects:

  i) For every asset mix, combining male and female liabilities provides hedge effects, but these effects are only significant when the fraction of survivor annuity rights is sufficiently low.

  ii) The effect of including survivor annuities in a portfolio of single life annuities depends on gender mix. This occurs because there are two opposite effects. On the one hand survivor annuities can reduce reserve requirements because they are negatively correlated with single life annuities. On the other hand, they are more affected by the uncertainty in future survival probabilities because they have a longer duration. For a portfolio with half male and half female rights, the latter effect dominates.

- With regard to the interaction between longevity risk and investment risk, we observe two effects:

  i) as expected, the effect of investment risk increases when the duration is higher, i.e,. the effect is larger for single life annuities combined with survivor annuities, than for single life annuities only, and larger for annuities for females than for males;

  ii) liability mix effects (i.e., effects of gender and product mix) are stronger when investment risk is higher.

4.2 Natural hedge potential of death benefits

In this subsection we quantify the effect of death benefits on reserve requirements in portfolios with varying product mixes (i.e., varying ratios of insured rights for survivor annuities over insured rights for single life annuities), and investigate how these hedge effects are affected by investment risk. We focus on the case where product mix is identical for both genders, i.e., $w = w_m = w_f$ and $d = d_m = d_f$.

As discussed earlier, the existing literature typically quantifies longevity risk by investigating its effects on the probability distribution of the discounted present value of the liabilities, for a constant and deterministic interest rate. The following proposition shows that when financial risk is ignored, longevity risk in single life annuities can be completely hedged by death benefits.

Proposition 2 *Consider an immediate single life annuity for an x-year old with gender g, with an annual payment of 1, and a death benefit with a single payment of δ in the year of decease for an x-year old with gender g. If $R_s = r$ for all s, and $\delta = \frac{1+r}{r}$, then the terminal asset value A_T is unaffected by longevity risk.*

Proof. Let $R_\tau = r$ for all τ, and let $\widetilde{L}_\tau = {}_\tau p_x^{(g)} + \delta \cdot \left({}_{\tau-1}p_x^{(g)} - {}_\tau p_x^{(g)} \right)$. Then, it follows from (9) that:

$$
\begin{aligned}
\frac{A_T}{(1+r)^T} &= (1+c) \cdot BEL - \sum_{s=1}^{T} \frac{\widetilde{L}_s}{(1+r)^s} \\
&= (1+c) \cdot BEL - \sum_{\tau=1}^{T} \left(\frac{{}_\tau p_x^{(g)} + \delta \cdot \left({}_{\tau-1}p_x^{(g)} - {}_\tau p_x^{(g)} \right)}{(1+r)^\tau} \right) \\
&= (1+c) \cdot BEL - \sum_{\tau=1}^{T-1} \left(\frac{\left(1 - \delta + \frac{\delta}{1+r}\right)}{(1+r)^\tau} \right) {}_\tau p_x^{(g)} - (1-\delta) \cdot \frac{{}_T p_x^{(g)}}{(1+r)^T} - \delta \cdot {}_0 p_x^{(g)} \\
&= (1+c) \cdot BEL - \delta = c \cdot BEL.
\end{aligned}
$$

The last equality follows from $BEL = \mathbb{E}\left[\sum_{s=1}^{T} \frac{\widetilde{L}_s}{(1+r)^s}\right] = \delta$, $\delta = \frac{1+r}{r}$, ${}_0p_x^{(g)} = 1$, and ${}_Tp_x^{(g)} = 0$, because by assumption, the probability that the insured reaches age $x + T$ is negligibly small. Therefore, the terminal asset value is given by $A_T = c \cdot BEL \cdot (1+r)^T$, which is independent of survival rates. ■

Proposition 1 suggests that the hedge potential of including death benefits in portfolios of single life annuities may be significantly overestimated if financial risk is ignored.
Figure 2 shows that in order to properly quantify the hedge effects, it is important to take into account both product mix and asset mix effects. The figure displays the effect of death benefits on the minimal required buffer percentage c for portfolios of life insurance products, for given investment strategies. The left panels in Figure 2 display the minimum required buffer as a function of d, the ratio of the insured rights

for death benefits over single life annuities, in portfolios with only single life annuities. The right panels display the minimum required buffer as a function of d, for portfolios of single life annuities and survivor annuities with $w = 0.5$. In both cases, we consider three different gender mixes, 100% male insured rights ($\gamma = 1$, top panel), 100% female insured rights ($\gamma = 0$, middle panel), and 50% male insured rights and 50% female insured rights ($\gamma = 0.5$, bottom panel), and four different investment strategies for the buffer portfolio, 100% equity (bold), 67% equity and 33% one-year zero-coupon bonds (dashed lines), 33% equity and 67% one-year zero-coupon bonds (dotted lines), and 100% one-year zero-coupon bonds (solid lines).

As expected, death benefits can significantly reduce the reserve requirements. However, the effect depends strongly on product mix and asset mix. The hedge effect is lower in portfolios with both single life and survivor annuities than in portfolios with only single life annuities, since survivor annuities partially hedge longevity risk in single life annuities, and is lower when the investment strategy is more risky.

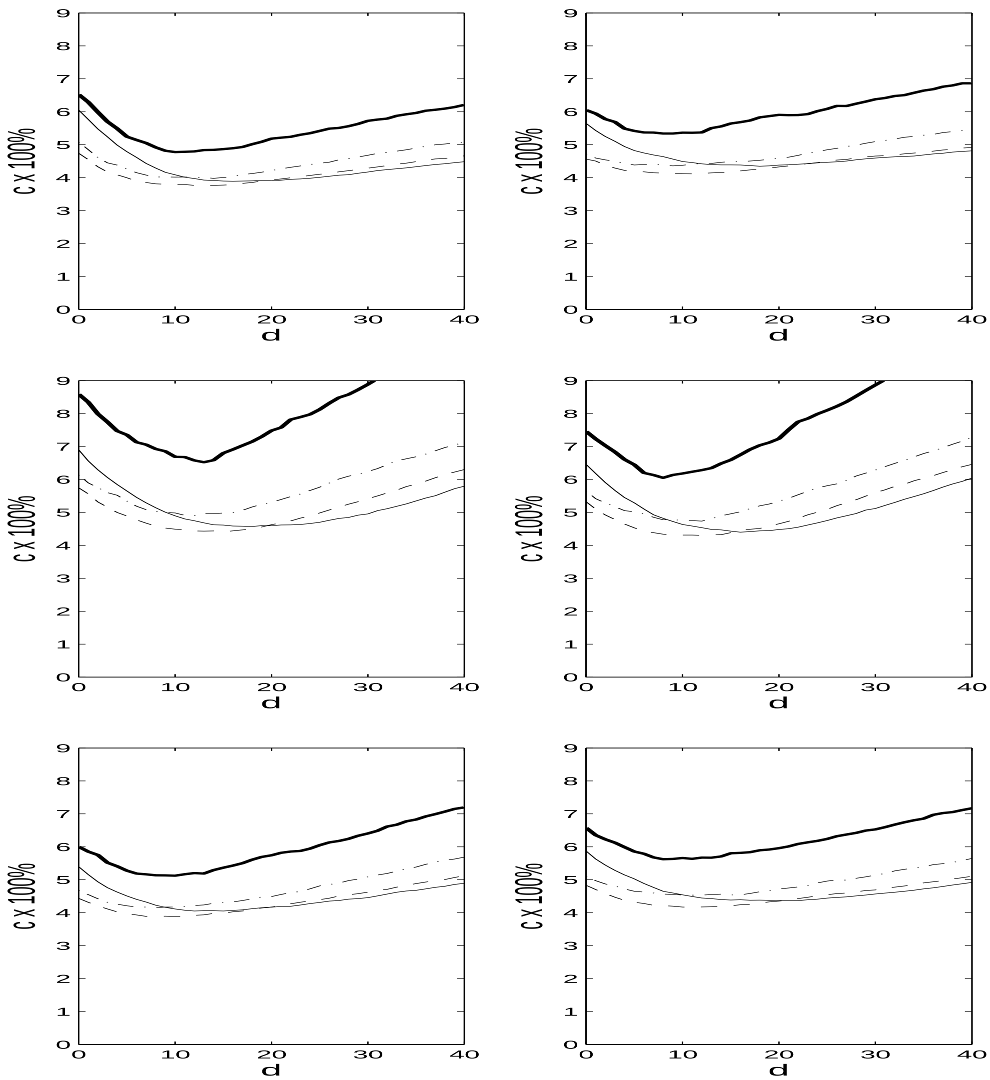
5 Hedge effects of mortality linked assets

In this section we investigate the effect of the mortality linked assets in the asset portfolio on the probability of ruin for a life insurer. More precisely, we investigate the effect of longevity swaps on the reserve requirements in a portfolio of life insurance products. Dowd et al. (2006) discuss the mechanism and use of survivor swaps as an instruments for managing, hedging, and trading mortality-dependent risks instead of longevity bonds. A survivor swap can be defined as a swap involving at least one future (stochastic) mortality-dependent payment. Given this definition, the most basic case of a survivor swap is an exchange of a single fixed payment for a single mortality-dependent payment. More precisely, let ref denote a reference population. Then, at time $t = 0$, party A agrees with party B that A pays to B at time $\tau > 0$ the amount $K(\tau, ref)$ known at time 0 and B pays to A at the amount $S(\tau, ref)$, which depends on mortality rates and are thus currently stochastic. The payments made in this agreement are that party B pays A if $K(\tau, ref) < S(\tau, ref)$ the amount $S(\tau, ref) - K(\tau, ref)$ and party B pays A if $K(\tau, ref) > S(\tau, ref)$ the amount $K(\tau, ref) - S(\tau, ref)$. Hence, the payment of the survival swap equals:

$$SS(\tau, ref) = S(\tau, ref) - K(\tau, ref), \tag{16}$$

where $S(\tau, ref)$ is the random mortality-dependent payment and $K(\tau, ref)$ is the fixed payment. Typically, $K(\tau, ref)$ and $S(\tau, ref)$ are determined such that there is no cash

Figure 2: Reserve requirements in portfolios of single life and survivor annuities and death benefits



The panels display the required buffer percentage c as a function of d in portfolios of life insurance products, where the ratio of survivor annuity rights over single life annuity rights equals $w = 0$ (left column), and $w = 0.50$ (right column). In both cases, we consider three gender mixes, $\gamma = 1$ (top row), $\gamma = 0$ (middle row), and $\gamma = 0.5$ (bottom row), and four asset mixes for the *buffer portfolio*, 100% equity (bold), 67% equity and 33% one-year zero-coupon bonds (dashed lines), 33% equity and 67% one-year zero-coupon bonds (dotted lines), and 100% one-year zero-coupon bonds (solid lines).

transfer at the time of the issue. However, there is currently no publicly traded market in longevity-linked products and hence we do not observe the market price of longevity risk.[4] To avoid making assumptions regarding the price of the swap, we set $K(\tau, ref)$ equal to the current expected payment in period τ, and assume that there is a cash transfer at the time of issue which equals the current (over the counter) price of the survivor swap. The survival swap we consider in this paper is one where the payment is linear in the number of survivors in the underlying population. Assuming that non–systematic longevity risk is negligible, this yields:

$$S(\tau, ref) = \frac{1}{N} \sum_{i \in ref} {}_{\tau}p_{x(i)}^{(g(i))}$$

$$K(\tau, ref) = \frac{1}{N} \mathbb{E}\left[\sum_{i \in ref} {}_{\tau}p_{x(i)}^{(g(i))}\right],$$

where $x(i)$ and $g(i)$ denotes the age and the gender of individual i in the reference population.

A problem with survivor swaps is to obtain a reference population. A natural reference group from the point of view of the insurer is the population of the insurer. However, the insurer may then have more information about the population than the seller of the survivor swap. Since the insurer may have this private information, buying a survivor swap can be interpreted as a signal that the reference group has low mortality probabilities, and hence the price of the survivor swaps would be high, see Biffis and Blake (2009). Another problem with the natural reference group from the point of view of the insurer is the tradeability of the survivor swaps, i.e., when every life insurer has a different reference group, many different survivor swaps are needed. This would lead to much higher transaction costs for the seller of the survivor swap, since he has to put extra efforts in estimating the size of longevity risk in the survivor swaps. In order to eliminate the private information problem and to increase the tradeability, the whole population of a country is often chosen as reference group, since the information on this reference group is the same for the issuer and buyer of the swap. An example is the first longevity bond[5] issued by European Investment Bank/Bank National de Paris

[4] In recent years there has been considerable interest in developing pricing models for longevity linked assets and liabilities, see, e.g., Blake and Burrows (2001), Dahl (2004), Lin and Cox (2005), and Denuit, Devolder, and Goderniaux (2007). Unfortunately, however, the lack of liquidity for trade in longevity linked assets and/or liabilities makes it very difficult to calibrate these models.

[5] The longevity bond was issued by the EIB and managed by BNP Paribas. The face value was £540 million, and was primarily intended for purchase by U.K. pension funds. The survivor swap involved yearly coupon payments that were tied to an initial annuity payment of £50 million indexed to the survivor rates of English and Welsh males aged 65 years in 2003. The longevity bond was withdrawn prior issue.

announced in November 2004, which had as reference population the English and Welsh males at age 65 in 2003.

In Subsection 5.1 we investigate how the hedge effect of survivor swaps depends on liability mix and asset mix. To isolate these effects, we assume no basis risk. In Subsection 5.2 we investigate how the hedge effect of survivor swaps is affected by basis risk. In order to focus on the effect of unhedgeable financial risk on the reduction in longevity risk from the mortality linked assets, we will consider the expected liability hedge strategy with buffer portfolios as defined in Section 4.

5.1 Vanilla survivor swaps and product mix

In this subsection we investigate the effect of vanilla survivor swaps on buffer requirement for an insurer with a portfolio of life insurance products. The vanilla survivor swap $VSS(ref)$ consists of a portfolio of survivor swaps with all times to maturity. We use two different vanilla survivor swaps, namely one with reference group the whole male population aged 65 (i.e., $ref = m$) and another with reference group the whole female population aged 65 (i.e., where $ref = f$).

Let s_m be the number of vanilla survivor swaps with reference population males, and s_f with reference group females. Then, the minimal required initial value of the assets in order to limit the probability of ruin to ε is given by:

$$A_0 = BEL + \overline{c}(s_m, s_f) \cdot BEL + V_{VSS}(s_m, s_f),$$

where $\overline{c}(s_m, s_f) \cdot BEL$ denotes the required buffer in excess of the best estimate value and the price of the survivor swaps, and $V_{VSS}(s_m, s_f)$ is the date-0 price of the vanilla survivor swap. Because the liability payments, net of payoff from longevity swaps is given by:

$$\widetilde{L}_s = \widetilde{L}_s - s_m \cdot SS(s, m) - s_f \cdot SS(s, f),$$

it follows from Proposition 1 and (7) that

$$\overline{c}(s_m, s_f) = \frac{Q_{1-\varepsilon}(L(s_m, s_f))}{BEL} - 1,$$

where

$$L(s_m, s_f) = BEL + \sum_{s=1}^{T} \frac{\widetilde{L}_s - \mathbb{E}\left[\widetilde{L}_s\right] - s_m \cdot SS(s, m) - s_f \cdot SS(s, f)}{\prod_{\tau=1}^{s}\left(1 + r_\tau^{(bu)}\right)}.$$

Because we do not make assumptions regarding the price of the swap, we cannot determine the "optimal" fraction of survivor swaps, i.e., the one that minimizes the required asset value A_0. However, for any given portfolio of survivor swaps (s_m, s_f), we can determine the relative attractiveness of the vanilla survivor swaps for different liability mixes and asset mixes. Moreover, for any given asset mix, we can determine the maximum price of the portfolio of survivor swaps under which a lower asset value is sufficient to cover all future liabilities with probability at least $1-\varepsilon$ with survivor swaps than without survivor swaps. This maximum price is given by:

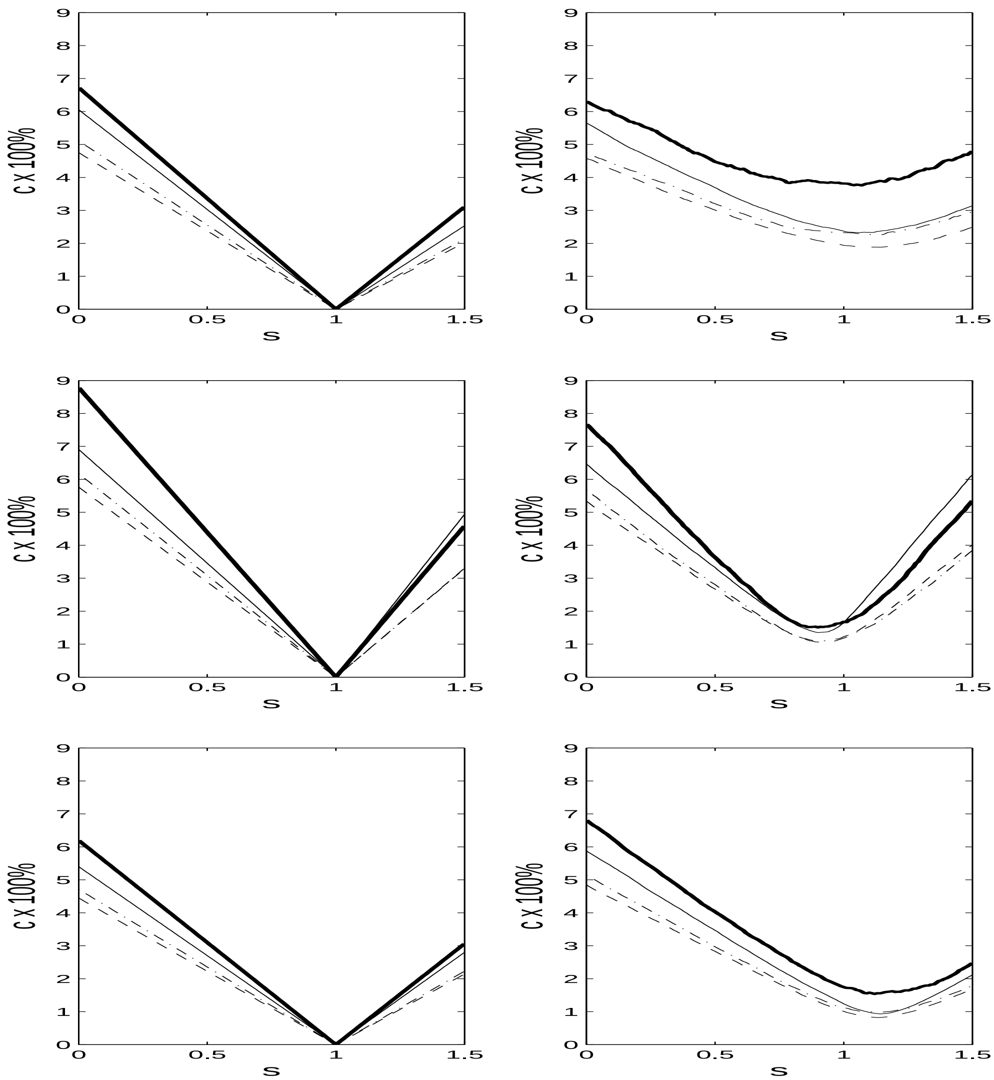
$$V_{VSS}^{\max}(s_m, s_f) = [\overline{c}(0,0) - \overline{c}(s_m, s_f)] \cdot BEL. \tag{17}$$

We now investigate the hedge potential survivor swaps in portfolios with varying product and asset mixes. We also determine the maximum price under which investing in survivor swaps leads to lower capital requirements in each case. In order to reduce the number of parameters, let $s = \frac{s_m}{\delta_{oa,m}} = \frac{s_f}{\delta_{oa,f}}$ with $\delta_{oa,\cdot}$ as defined in (15). This implies that (when there is no basis risk), s equals the fraction of single life annuity rights for which longevity risk is fully hedged by vanilla survivor swaps, for both males and females.

Figures 3 and 4 display the minimum required buffer, and the maximum price as defined in (17) (both as percentage of the best estimate value of the liabilities), respectively, as a function of s. We consider two product mixes, only single life annuities (left panels), and single life and survivor annuities with $w = 0.5$ (right panels), three gender mixes, $\gamma = 1$ (top row), $\gamma = 0$ (middle row), and $\gamma = 0.5$ (bottom row), and four asset mixes, 100% equity (bold), 67% equity and 33% one-year zero-coupon bonds (dashed lines), 33% equity and 67% one-year zero-coupon bonds (dotted lines), and 100% one-year zero-coupon bonds (solid lines).

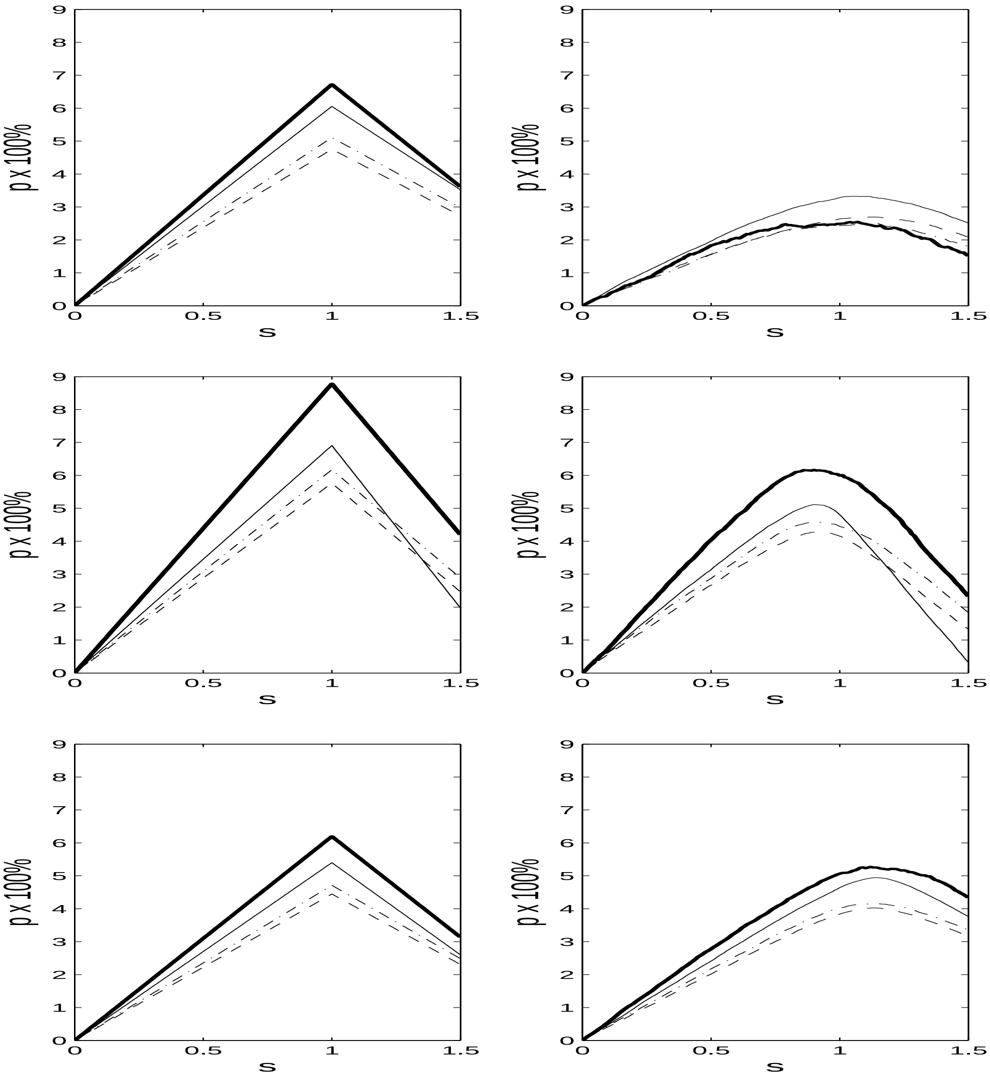
From Figure 3 we observe that survivor swaps can significantly reduce reserve requirements in portfolios of life insurance products. However, the effect depends strongly on product mix and asset mix. Not surprisingly, the fact that there is no basis risk implies that for a portfolio of only single life annuities, longevity risk can be fully eliminated by survivor swaps (with $s = 1$). For a portfolio with also survivor annuities, however, the maximal risk reduction is attained at either $s < 1$ or $s > 1$. This occurs because survivor annuities to some extent can provide a natural hedge for single life annuities, but on the other hand are also affected more strongly by longevity risk because they have longer duration. The first effect dominates for a portfolio with only female insureds, whereas the second effect dominates for a portfolio consisting of half male and half female insured rights.

Figure 3: Reserve requirements in portfolios of annuities and vanilla survivor swaps without basis risk



The figure displays the minimal required buffer percentage, $\overline{c}(s, s)$ as a function of s for portfolios with only single life annuities ($w = 0$, left panels), and for portfolios with single life and survivor annuities ($w = 0.5$; right panels). In both cases, we consider three gender mixes, $\gamma = 1$ (top row), $\gamma = 0$ (middle row), and $\gamma = 0.5$ (bottom row), and four asset mixes for the *buffer portfolio*, 100% equity (bold), 67% equity and 33% one-year zero-coupon bonds (dashed lines), 33% equity and 67% one-year zero-coupon bonds (dotted lines), and 100% one-year zero-coupon bonds (solid lines).

Figure 4: Maximum price of vanilla survivor swaps without basis risk



The figure displays $\overline{c}(0,0)-\overline{c}(s,s)$, i.e., the maximum price as a percentage of BEL, as a function of s for portfolios with only single life annuities ($w=0$, left panels), and for portfolios with single life and survivor annuities ($w=0.5$; right panels). In both cases, we consider three gender mixes, $\gamma=1$ (top row), $\gamma=0$ (middle row), and $\gamma=0.5$ (bottom row), and four asset mixes for the *buffer portfolio*, 100% equity (bold), 67% equity and 33% one-year zero-coupon bonds (dashed lines), 33% equity and 67% one-year zero-coupon bonds (dotted lines), and 100% one-year zero-coupon bonds (solid lines).

5.2 Vanilla survivor swaps with basis risk

In the previous subsection we have observed that vanilla survivor swaps can reduce the reserve requirements in a portfolio of life insurance products substantially. For a portfolio consisting of only single life annuities it can even eliminate longevity risk. In these calculations we have assumed that there is no basis risk, i.e., the mortality rates of the individuals in the reference group for the vanilla survivor swap are equal to the mortality rates of the insureds in the portfolio of life insurance products. Dowd, Cairns, and Blake (2006) investigate the hedge effect of a longevity bond where there exists basis risk, because the mortality rates of the insureds differ from the reference group of the longevity bond. They use a longevity bond which is based on 60-year-old males to reduce longevity risk for an annuity for a 65-year-old male. Another type of basis risk in the hedge effects of these mortality linked assets arises from the construction of the product. A mortality linked asset with reference group the whole population in a country, for example the announced EIB longevity bond which would depend on survivor probabilities of English and Welsh males aged 65 years in 2003, may also lead to basis risk, since it is known that survival probabilities for insureds are generally higher than for the whole population (see, e.g., Denuit, 2008). In this paper we quantify the basis risk of a survivor swap which is based on the whole population instead of the population of the insurer. As pointed out in Brouhns et al. (2002) the mortality rates of individuals with a life insurance product are generally lower than for individuals without a life insurance product. We use the Cox-type relational model that has been successfully applied in Brouhns et al. (2002) and Denuit (2008) to account for adverse selection. Specially, we let

$$\log(\mu_{x,t}^{(h)}) = \alpha^{(h)} + \beta^{(h)} \cdot \log(\mu_{x,t}^{(g)}), \tag{18}$$

where $\alpha^{(h)}$ and $\beta^{(h)}$ is the speed of the future mortality improvements of the group h relative to the general population with gender g. We use the estimated parameter values from Denuit (2008) which are given in Table 2 for group insureds and individual insureds, for both males and females.[6]

[6]Notice that $\beta^{(h)} < 1$, which implies that the speed of the future mortality improvements in the insured population is small er than the corresponding speed for the general population. This occurs because the adverse selection observed in the Belgian individual life market is so strong that the future improvements for the insured population are weaker than for the general population.

	$m, group$	$f, group$	$m, individual$	$f, individual$
$\alpha^{(h)}$	-0.71755	-0.577829	-1.54351	-1.024695
$\beta^{(h)}$	0.79180	0.843850	0.81849	0.906784

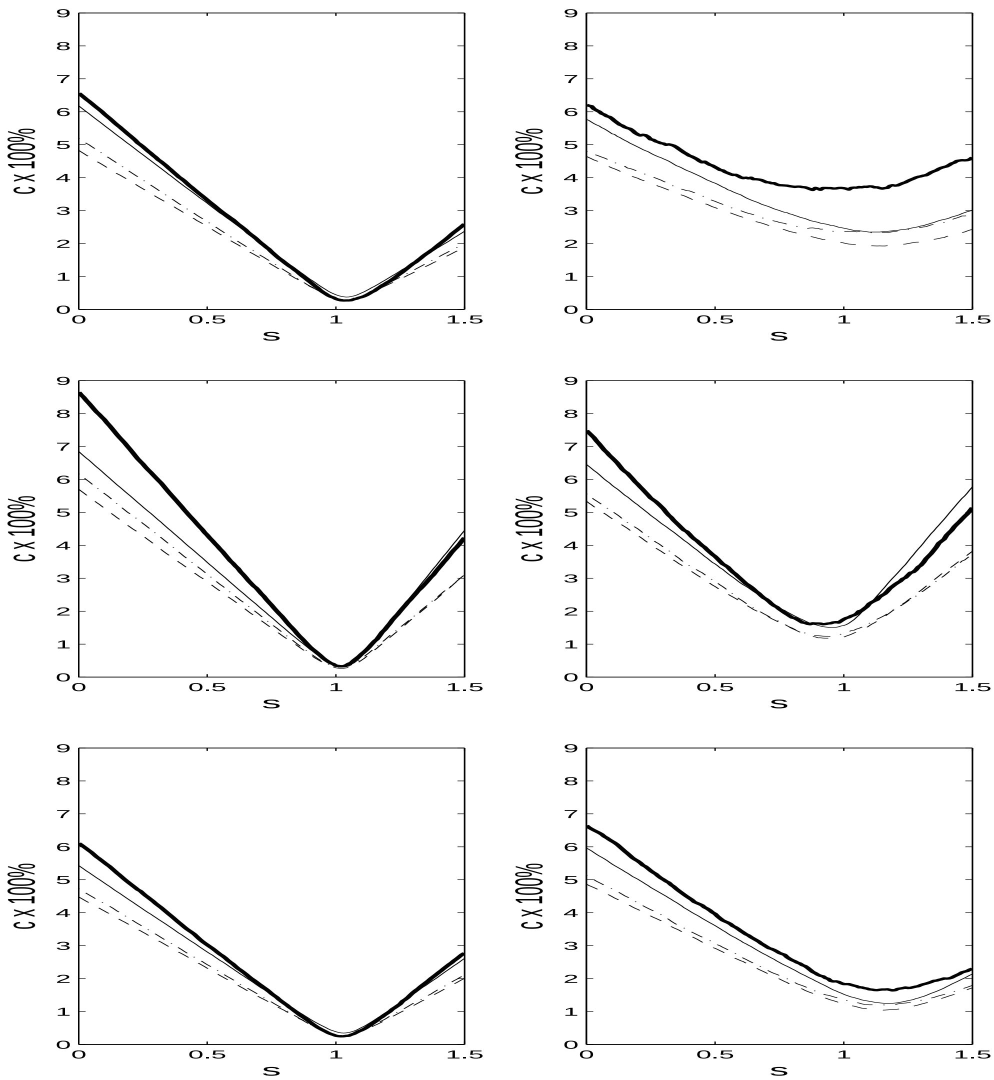
Table 2: Parameters estimates of the Cox-type relational model. Source: Denuit (2008).

We assume that the reference population of the vanilla survivor swap is the general population of males and the general population of females. We adjust the mortality rates of the insureds in the portfolio using equation (18). In Figures 5 and 6 we display the effect of the vanilla survivor swap based on a reference group of the general population for group insureds and for individual insureds, respectively. If the insured belongs to group (individual) insureds, we assume that the partner also belongs to group (individual) insureds.[7]

As expected, the hedge potential of survivor swaps reduces significantly when there is basis risk. However, it is also significantly affected by investment risk. For example, in portfolios with survivor annuities for male insureds, the hedge potential of survivor swaps reduces significantly when investment risk becomes higher.

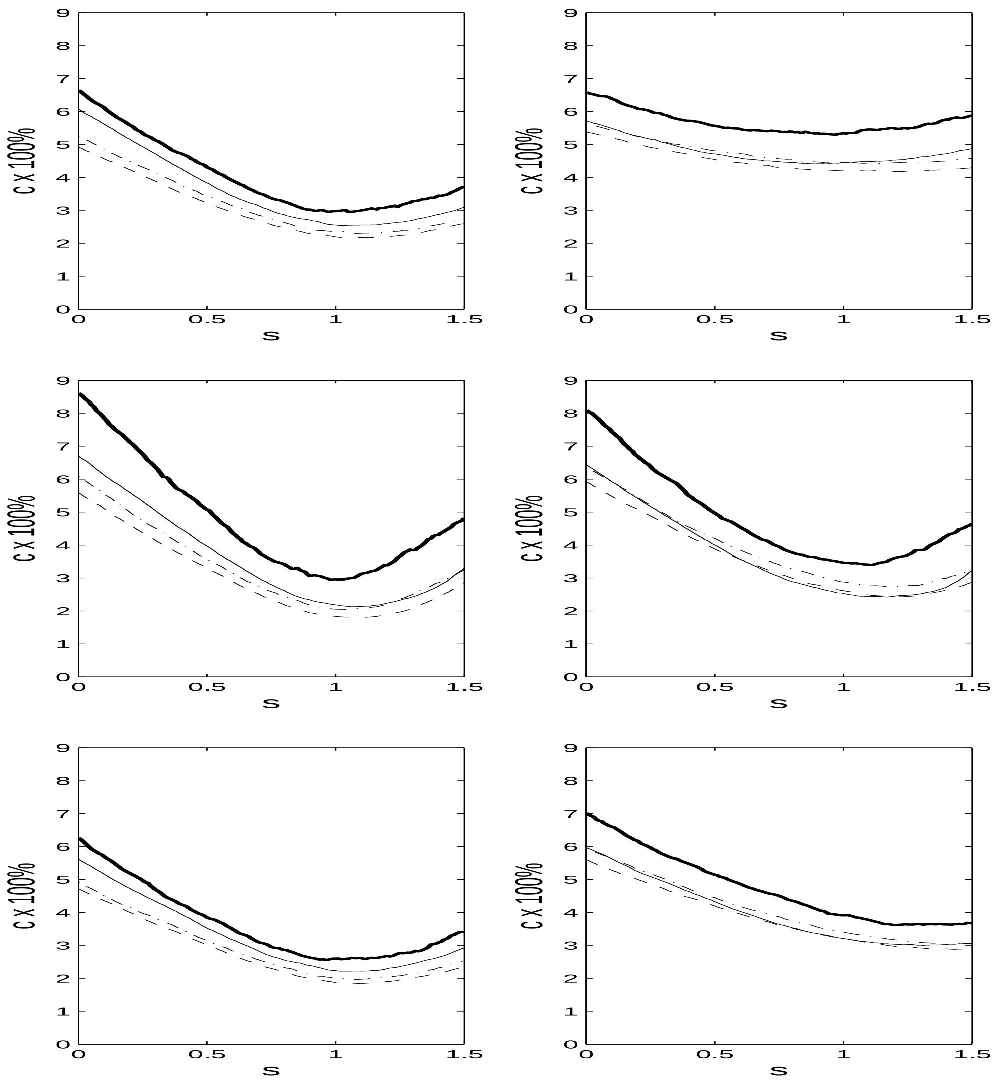
[7]The difference in survivor probabilities may be due to for instance social economic status, living conditions, income, which are typically the same for both spouses.

Figure 5: Reserve requirements in portfolios of annuities and vanilla survivor swaps with basis risk: group insureds



The figure displays the reserve requirements, $\overline{c}(s, s)$ as a function of s for portfolios with only single life annuities ($w = 0$, left panels), and for portfolios with single life and survivor annuities ($w = 0.5$; right panels). In both cases, we consider three gender mixes, $\gamma = 1$ (top row), $\gamma = 0$ (middle row), and $\gamma = 0.5$ (bottom row), and four asset mixes for the *buffer portfolio*, 100% equity (bold), 67% equity and 33% one-year zero-coupon bonds (dashed lines), 33% equity and 67% one-year zero-coupon bonds (dotted lines), and 100% one-year zero-coupon bonds (solid lines).

Figure 6: Reserve requirements in portfolios of annuities and vanilla survivor swaps with basis risk: individual insureds



The figure displays the reserve requirements, $\overline{c}(s, s)$ as a function of s for portfolios with only single life annuities ($w = 0$, left panels), and for portfolios with single life and survivor annuities ($w = 0.5$; right panels). In both cases, we consider three gender mixes, $\gamma = 1$ (top row), $\gamma = 0$ (middle row), and $\gamma = 0.5$ (bottom row), and four asset mixes for the *buffer portfolio*, 100% equity (bold), 67% equity and 33% one-year zero-coupon bonds (dashed lines), 33% equity and 67% one-year zero-coupon bonds (dotted lines), and 100% one-year zero-coupon bonds (solid lines).

6 Conclusions

[TO BE WRITTEN]

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A The distribution of the financial returns

In this section we briefly describe the estimation procedure for the returns in the financial market. The financial market consists of zero-coupon bonds with different times to maturity and one equity stock. The parameters are estimated using GMM, to make forecasts of the distribution of the returns of the assets in this financial market we include parameter risk.

We assume that the instantaneous spot rate, r_t evolves as an Ornstein-Uhlenbeck process with constant coefficients:

$$dr_t = (a - br_t)\, dt + \sigma dZ_t^1, \tag{19}$$

where a, b, and σ are model parameters, and Z_t^1 is a standard Brownian motion. The stock price, S_t follows a Brownian motion with drift:

$$\begin{aligned} dS_t =& \mu_t S_t dt + \sigma_S S_t dZ_t^2, \\ \mu_t =& r_t + \lambda_S \sigma_S, \end{aligned} \tag{20}$$

where μ_t are σ_S are model parameters, and Z_t^2 is a standard Brownian motion. The correlation between the standard Brownian motions Z_t^1 and Z_t^2 is equal to ρ.

Let $P_t^{(n)}$ be the price at time t of a zero-coupon bond with face value of one which matures at time $t + n$, and let the yield to maturity $R_t^{(n)}$ be the internal rate of return at time t on a bond maturing at time $t + n$. Then we have:

$$R_t^{(n)} \equiv \frac{-\log\left(P_t^{(n)}\right)}{n}, \tag{21}$$

with

$$\begin{aligned} P_t^{(n)} &= \exp\left(D_t^{(n)}\right) \cdot \exp\left(-B_t^{(n)} \cdot r_t\right), \\ B_t^{(n)} &= \frac{1 - \exp\left(-b \cdot n\right)}{b}, \\ D_t^{(n)} &= \left(B_t^{(n)} - n\right) \cdot \left(\frac{(a - \sigma\lambda) \cdot 2b - \sigma^2}{2b^2}\right) - \frac{\sigma^2}{4b} \cdot \left(B_t^{(n)}\right)^2. \end{aligned}$$

To estimate the parameters of the Ornstein-Uhlenbeck process we discretize the stochastic differential equation (SDE) of equation (19). Let Δt be the time step, then $\alpha = a\Delta t$,

$\beta = b\Delta t$, and $\sigma_{\Delta t} = \sigma\sqrt{\Delta t}$:

$$
\begin{aligned}
r_{t+\Delta t} - r_t &= \alpha - \beta r_t + \epsilon_{t+\Delta_t}, \\
\frac{S_{t+\Delta t} - S_t}{S_t} &= (r_t + \lambda_S \sigma_S)\,\Delta_t + \epsilon^S_{t+\Delta_t}, \\
\begin{pmatrix} \epsilon_{t+\Delta_t} \\ \epsilon^S_{t+\Delta_t} \end{pmatrix} &\equiv \begin{pmatrix} \sigma_{\Delta t} & 0 \\ 0 & \sigma^S_{\Delta t} \end{pmatrix} N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \right),
\end{aligned}
$$

where N is a bivariate random shock with mean zero and unit variance, we use five implied moment conditions:

$$
\begin{array}{llllll}
\mathbb{E}\left[\epsilon_{t+\Delta_t}\right] & = & 0, & \mathbb{E}\left[\epsilon^2_{t+\Delta_t}\right] & = & \sigma^2_{\Delta t}, \\
\mathbb{E}\left[\epsilon^S_{t+\Delta_t}\right] & = & 0, & \mathbb{E}\left[\left(\epsilon^S_{t+\Delta_t}\right)^2\right] & = & \left(\sigma^S_{\Delta t}\right)^2, \\
\mathbb{E}\left[\epsilon_{t+\Delta_t}\epsilon^S_{t+\Delta_t}\right] & = & \rho\sigma_{\Delta t}\sigma^S_{\Delta t}. & & &
\end{array}
\tag{22}
$$

Using the Vasicek model where the parameters are known, the price zero-coupon bond any time to maturity follows from equation (21). In practice there may be a difference between the observed price of the zero-coupon bond and the theoretical price obtained from the Vasicek model. This may be due to for instance, that the Vasicek model is a simple model which does not capture the full dynamics of the term structure of interest rates over time. IN order to estimate the parameter λ we assume that the yield on a zero-coupon bond maturing in n years from time t is given by:

$$
R^{(n)}_t = -\frac{D^{(n)}_t}{n} + \frac{B^{(n)}_t r_t}{n} + \epsilon^R_t, \tag{23}
$$

which implies an additional moment condition:

$$
\mathbb{E}\left[\epsilon^R_t\right] = 0. \tag{24}
$$

The set of moment conditions is derived by the conditions given in equations (22) and (24) together with the assumption that that the error is uncorrelated with the explanatory variable r_t:

$$f_t(\theta) = \begin{pmatrix} \epsilon_{t+1} \\ \epsilon_{t+1} r_t \\ \epsilon_{t+1}^2 - \sigma_{\Delta t}^2 \\ \left(\epsilon_{t+1}^2 - \sigma_{\Delta t}^2\right) r_t \\ \epsilon_t^R \\ \epsilon_t^R r_t \\ \epsilon_{t+1}^S \\ \left(\epsilon_{t+1}^S\right)^2 - \left(\sigma_{\Delta t}^S\right)^2 \\ \epsilon_{t+1}\epsilon_{t+1}^S - \rho\sigma_{\Delta t}\sigma_{\Delta t}^S \end{pmatrix}.$$

The population moment vector is constructed such that $\mathbb{E}\left[f_t(\theta)\right] = 0$.
We define the sample moments by:

$$g_T(\theta) = \frac{1}{T}\sum_{t=1}^{T} f_t(\theta),$$

where T is the sample size. The GMM objective function is defined by minimizing

$$J_T = g_T'(\theta) W_T g_T(\theta), \tag{25}$$

where W_T is a positive definite weighting matrix. We use the Newey-West estimator of W_T which is given by:

$$\begin{aligned} W_T^{-1} =& \widehat{S}_0 + \sum_{j=1}^{k}\left(1 - \frac{j}{k+1}\right)\left(\widehat{S}_j + \widehat{S}_j^{,}\right), \\ \widehat{S}_j =& \frac{1}{T}\sum_{t=j+1}^{T} f_t(\theta) f_{t-j}(\theta), \end{aligned}$$

where k is the lag length. We set the lag length in the Newey-West estimator first equal to 5 to estimate the parameters and their standard errors and then re-estimate everything using increasing lag lengths until the lag length has negligible effect on the standard errors of the parameters to be estimated.
Under the efficient GMM we have asymptotic normality for the estimated parameters:

$$\sqrt{T}\left(\widehat{\theta} - \theta\right) \xrightarrow{d} N\left(0, \left(d^{,} W_T d\right)^{-1}\right), \tag{26}$$

where d is the Jacobian of the population moment vector $\mathbb{E}\left[f_t(\theta)\right]$.

Table 3: Parameter estimates of distribution of the financial returns

Parameter	a	b	σ	λ	λ_S	σ_S	ρ
Estimate	0.0045908	0.10399	0.0042971	-0.81134	0.40832	0.23663	-0.028284
Standard dev	0.0011086	0.026058	0.0005669	0.3307	0.17706	0.017793	0.008286

The table displays the estimates and the standard deviation of the estimates of the model parameters for the distribution of the returns of the assets in the financial market.

We use daily Dutch financial data obtained from *Datastream* from January 31, 1997 till January 1, 2007 to estimate the parameters. We use three time series, namely the instantaneous short rate, the interest rate on a 10 years Dutch government bond, and the return on the Dutch stock index "AEX". Table 3 displays the estimates and the standard deviation of the estimates of the model parameters. To obtain simulation of the financial returns we assume that the bivariate random shock N has a bivariate normal distribution.

B The distribution of the mortality probabilities

In this appendix we briefly describe the models used to quantify the macro-longevity risk affecting $p_{x,t}^{(g)}$. Let $\mu_{x,t}^{(g)}$ denote the force of mortality of a person with age x and gender g at time t, i.e., $\mu_{x,t}^{(g)} = \lim_{\Delta t \to 0} P\left(0 \leqslant T_{x,t}^{(g)} \leqslant \Delta t\right)/\Delta t$. We assume that for any integer age x, any gender g, and any time t, it holds that $\mu_{x+u,t}^{(g)} = \mu_{x,t}^{(g)}$, for all $u \in [0,1)$. Then, one can verify

$$p_{x,t}^{(g)} = \exp\left(-\mu_{x,t}^{(g)}\right).$$

Next, let $D_{x,t}^{(g)}$ denote the observed number of deaths in year t in a cohort with gender g and aged x at the beginning of year t, and let $E_{x,t}^{(g)}$ denote the number of person years during year t in a cohort with gender g and aged x at the beginning of year t, the so-called exposure. Then, under appropriate regularity conditions, as discussed by Gerber (1997), the Maximum Likelihood estimator for the force of mortality is given by $\widehat{\mu}_{x,t}^{(g)} = D_{x,t}^{(g)}/E_{x,t}^{(g)}$. We assume that there is no (sampling) risk involved in this relationship, so that the macro-longevity risk is fully captured by the risk in $D_{x,t}^{(g)}/E_{x,t}^{(g)}$. We use the three (variants of) the Lee-Carter model, a P-Spline model, and four (variants of) the CBD-model to quantify the macro-longevity risk in $D_{x,t}^{(g)}/E_{x,t}^{(g)}$. The three variants of the Lee-Carter model are the models proposed by Lee and Carter (1992), Brouhns, Denuit, and Vermunt

(2002), and Cossette et al. (2007), which are described in Appendix B.1. In Appendix B.2 we describe the P-splines model. In Appendix B.3 we describe the four models for the CBD-model. For each of the three different classes of models (i.e., the Lee-Carter models, the P-spline model, and the CBD-model) we use 5,000 simulation. Within the Lee-Carter class of models we use 1666 simulations for the Lee and Carter (1992), 1667 simulations for the Brouhns, Denuit, and Vermunt (2002), 1667 simulations for the and Cossette et al. (2007). For each of the four different specifications of the CBD-model we use 1250 simulations.

Age, gender, and time specific numbers of death and exposed to death are obtained from the Human Mortality Database.[8] In our case $x \in \{0, 1, 2, ..., 99, 100^+\}$, with 100^+ the age group of people aged 100 years or more. We use the time period 1977–2006, so that $T = 2006$. This time period minimizes the statistic proposed by Booth et al. (2002) to test the hypothesis that the age components in the Lee-Carter model are invariant over time. We use this selection, since mortality experience in the industrialized world seems to suggest a substantial age-time interaction in the twentieth century.

B.1 Lee-Carter (1992) model

In this section we describe the tree variants of the Lee-Carter model, namely the models proposed by Lee and Carter (1992), Brouhns, Denuit, and Vermunt (2002), and Cossette et al. (2007).

The model by Lee and Carter (1992) is given by

$$\log\left(D_{x,t}^{(g)}/E_{x,t}^{(g)}\right) = a_x^{(g)} + b_x^{(g)} k_t^{(g)} + \epsilon_{x,t}^{(g)}, \tag{27}$$

where $k_t^{(g)}$ is an index of the level of mortality, $a_x^{(g)}$ is an age-specific constant describing the general pattern of mortality by age, $b_x^{(g)}$ is an age-specific constant describing the relative speed of the change in mortality by age, and where $\epsilon_{x,t}^{(g)}$ represents the measurement error, assumed to satisfy $\epsilon_{x,t}^{(g)} \mid \mathcal{K}_t \sim N\left(0, \sigma_{x,g}^2\right)$, conditional on $\mathcal{K}_t = \left\{k_\tau^{(g)} \mid g \in \{m, f\}, \tau = t, t-1, ...\right\}$. Moreover, we assume that the $\epsilon_{x,t}^{(g)}$ are independent for different x and g, conditional on $\mathcal{K}_t$.

To model the process for $\left(k_t^{(m)}, k_t^{(f)}\right)$ over time, we use an ARIMA(0,1,1) model

$$k_t^m = k_{t-1}^m + c^m + \theta^m e_{t-1}^m + e_t^m, \tag{28}$$
$$k_t^f = k_{t-1}^f + c^f + \theta^f e_{t-1}^f + \rho e_t^m + e_t^f, \tag{29}$$

[8]See www.mortality.org.

where c^g is the gender g specific drift term which indicates the average annual change of k_t^g, θ^g is the gender specific moving average coefficient, and e_t^g is the gender specific innovation such that

$$\begin{pmatrix} e_t^m \\ e_t^f \end{pmatrix} \mid \mathcal{K}_{t-1} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_m^2 & 0 \\ 0 & \sigma_f^2 \end{pmatrix}\right).$$

The parameter ρ captures the correlation between k_t^m and k_t^f over time.
In case of the model by Brouhns, Denuit, and Vermunt (2002), the age and gender specific numbers of deaths are modeled by a Poisson process,

$$D_{x,t}^{(g)} \mid \widetilde{\mathcal{K}}_t \sim \text{Poisson}\left(E_{x,t}^{(g)} e^{a_x^{(g)} + b_x^{(g)} k_t^{(g)}}\right), \tag{30}$$

with $\widetilde{\mathcal{K}}_t = \mathcal{K}_t \bigcup \left\{E_{x,\tau}^{(g)} \mid g \in \{m, f\}, \text{all } x, \tau = t, t-1, ...\right\}$ We assume that the $D_{x,t}^{(g)}$ are independent for different x and g, conditional on $\widetilde{\mathcal{K}}_t$. The process for $\left(k_t^{(m)}, k_t^{(f)}\right)$ is modeled as in case of the Lee and Carter (1992)-model, i.e., via equations (28)–(29).
As third model, we consider Cossette et al. (2007). These authors model the age specific numbers of deaths $D_{x,t}^{(g)}$ via the Binomial Gumbel process,

$$D_{x,t}^{(g)} \mid \widetilde{\mathcal{K}}_t \sim \text{Bin}\left(E_{x,t}^{(g)}, 1 - exp\left(-e^{a_x^{(g)} + b_x^{(g)} k_t^{(g)}}\right)\right), \tag{31}$$

where we again assume that the $D_{x,t}^{(g)}$ are independent for different x and g, conditional on $\widetilde{\mathcal{K}}_t$, and where we model the process for $\left(k_t^{(m)}, k_t^{(f)}\right)$ via equations (28)–(29).
To forecast the future mortality rates, we use the current time (defined in the appendix as time T) mortality table and a reduction factor. In this way the mortality rates are set such that the estimation error in the last year of the mortality data is zero, so that we avoid a jump-off bias in the forecasts. Let $q_{x,T}^{(g)} = 1 - p_{x,T}^{(g)}$ be based on the last year of mortality data. We forecast $\mu_{x,T+s}^{(g)}$ as follows

$$\widehat{\mu}_{x,T+s}^{(g)} = \mu_{x,T}^{(g)} \times \widehat{RF}_{x,T,s}^{(g)}. \tag{32}$$

The reduction factor is given by

$$\widehat{RF}_{x,T,s}^{(g)} = e^{\widehat{b}_x^{(g)} \times (\widehat{k}_{T+s}^{(g)} - \widehat{k}_T^{(g)})}, \tag{33}$$

where $\widehat{b}_x^{(g)}$ and $\widehat{k}_T^{(g)}$ denote the (model specific) estimated $b_x^{(g)}$ and $k_T^{(g)}$, respectively, and where $\widehat{k}^{(g)}_{T+s}$ denotes the $s \geqslant 1$ periods ahead forecast. For the latter we use

$$\begin{pmatrix} \widehat{k}^{m}_{T+s} \\ \widehat{k}^{f}_{T+s} \end{pmatrix} \mid \widehat{\mathcal{K}}_{T+s-1} \sim N\left(\begin{pmatrix} \widehat{k}^{m}_{T+s-1} + \widehat{c}^{m} \\ \widehat{k}^{f}_{T+s-1} + \widehat{c}^{m} \end{pmatrix}, \begin{pmatrix} \widehat{\sigma}^2_m & \widehat{\rho}^2\widehat{\sigma}^2_m \\ \widehat{\rho}^2\widehat{\sigma}^2_m & \widehat{\rho}^2\widehat{\sigma}^2_m + \widehat{\sigma}^2_f \end{pmatrix} \right), \tag{34}$$

with $\widehat{\mathcal{K}}_T = \mathcal{K}_T$, and $\widehat{\mathcal{K}}_{T+s} = \widehat{\mathcal{K}}_{T+s-1} \bigcup \left\{ \widehat{k}^{g}_{T+s}, \widehat{k}^{f}_{T+s} \right\}$, for $s = 1, 2, 3, ...$, employing (model specific) estimates.

In order to avoid localized age induced anomalies in $\widehat{b}^{(g)}_x$ in the three models, we follow Renshaw and Haberman (2003). These authors proposed to smooth the age specific estimated parameters $\widehat{b}^{(g)}_x$ using cubic B-splines, with internal knots,

$$\zeta^{(g)}_0 + \zeta^{(g)}_1 x + \zeta^{(g)}_2 x^2 + \zeta^{(g)}_3 x^3 + \sum_{j=1}^{r} \zeta^{(g)}_{3+j} (x - x_j)^3_+, \tag{35}$$

where $(x - x_j)^3_+ = (x - x_j)^3$, in case $x - x_j > 0$, and zero otherwise. As internal knots we use $x_1 = 9.5$, $x_2 = 20.5$, $x_3 = 50.5$, $x_4 = 60.5$, and $x_r = x_5 = 80.5$. The cubic B-splines are fitted to the (model specific) estimated $\widehat{b}^{(g)}_x$ using the method of least squares.

The model-specific parameters are estimated imposing the required normalizations and using the estimation techniques as described in the corresponding papers. For the Lee and Carter (1992)-model we first estimate the parameters $a^{(g)}_x$, $b^{(g)}_x$, and $k^{(g)}_t$ using a singular value decomposition (SVD). Secondly, for all $t \leqslant T$ and $g \in \{m, f\}$, we re-estimate $k^{(g)}_t$ such that the estimated number of deaths using the estimates of $a^{(g)}_x$ and $b^{(g)}_x$ in equation (1) (with $\epsilon^{(g)}_{x,t} = 0$) equals the observed number of deaths. These re-estimated $k^{(g)}_t$, $t \leqslant T$, are used to estimate the process for $\left(k^{(m)}_t, k^{(f)}_t\right)$ using equations (28)–(29). For the Brouhns, Denuit, and Vermunt (2002)-model and the Cossette et al. (2007)-model we use the iterative procedure proposed by Goodman (1979) to obtain the Maximum Likelihood estimates, where the criterium to stop the procedure is a very small (i.e., 10^{-10}) increase of the log-likelihood.

The parameter estimates relevant for the quantification of the macro-longevity risk are plotted in Figure 7 (the $\widehat{b}^{(g)}_x$) and Table I (the parameter estimates of equations (28)–(29)). The estimation results for the different models are quite comparable, with the estimation results of the Lee and Carter (1992) model slightly deviating from the other two models. At younger ages males and females are more sensitive to changes in the time trend, while also males around 60 and females around 77 show an increased sensitivity, see Figure 7. In terms of the $k^{(g)}_t$-processes, we find that the male drift term is more negative than the female drift term, but in case of females the first order moving average term is more negative. The risk in the female process (given by $\widehat{\rho}^2\widehat{\sigma}^2_m + \widehat{\sigma}^2_f$) is also substantial higher than in case of the males (given by $\widehat{\sigma}^2_m$). Finally, there is a substantial

Table 4: Estimation results for the Lee-Carter models

Model	g	$c^{(g)}$	$\theta^{(g)}$	σ_g	ρ
Lee-Carter	m	-1.854	-0.131	1.612	0.583
	f	-1.576	-0.373	1.511	
Brouhns, Denuit, and Vermunt	m	-1.849	-0.096	1.376	0.799
	f	-1.519	-0.148	1.123	
Cossette et al.	m	-1.854	-0.097	1.386	0.799
	f	-1.529	-0.160	1.147	

Parameter estimates of equations (28)–(29)). Lee-Carter: Lee and Carter (1992)-model; Brouhns, Denuit, and Vermunt: Brouhns, Denuit, and Vermunt (2002)-model; Cossette et al.: Cossette et al. (2007)-model.

correlation between the male and female process.

We include three sources of macro-longevity risk: process risk, parameter risk, and model risk. First, using (32) and (33), given a specific model and given the corresponding model specific estimates, there is *process risk* due to fact that future values of $\widehat{k}_{T+s}^{(g)}$ are risky, see equation (34). Next, given a specific model, the forecasts (32)–(34) are based on model specific estimates, and these estimates are sensitive to estimation inaccuracy. The corresponding risk is referred to as *parameter risk*. Finally, different models might be used to calculate the forecasts (32)–(34). Assuming that some prior distribution is used to do the forecast calculations, there is also *model risk*.

To quantify the macro-longevity risk, we proceed as follows. Given the initial data

$$\left\{\left(D_{x,t}^{(g)}, E_{x,t}^{(g)}\right) \mid x \in \{0, 1, 2, ..., 99, 100^+\}, g \in \{m, f\}, t \in \{1977, \ldots, T = 2006\}\right\},$$

the following steps are taken.

1) For each of the three models, the parameters $\widehat{a}_x^{(g)}$, $\widehat{b}_x^{(g)}$, and $\widehat{k}_t^{(g)}$ are estimated and the corresponding residuals $r_{x,t}^{(g)}$ are computed.[9] Let R_t be the matrix with components $r_{x,t}^{(g)}$, for $g \in \{m, f\}, x \in \{0, ..., 100^+\}$.

2) Next, for each model, we generate B replications $\overline{R}_t(b)$, $b = 1, ..., B$, of the residual matrix R_t, by sampling with replacement.[10] Using these residual matrices, the corresponding (model specific) bootstrapped numbers of death $\overline{D}_{x,t}^{(g)}(b)$, $b = 1, ..., B$, are determined.[11]

[9]In case of the Brouhns, Denuit, and Vermunt (2002)- and the Cossette et al. (2007)-model, we calculated the deviance residuals.

[10]This corresponds to the residual bootstrap percentile interval-method of Efron and Tibshirani (1998). See also Koissi, Shapiro, and Högnäs (2006).

[11]In case of deviance residuals, this requires the use of the inverse relationship between numbers of

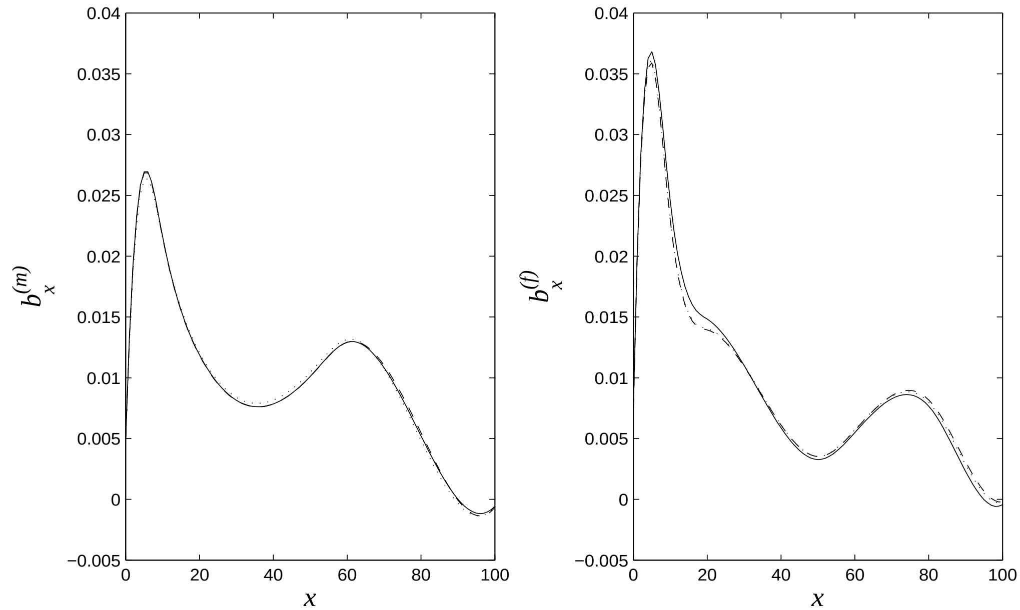

Figure 7: Estimated $b_x^{(g)}$ after smoothing using cubic B-splines. Left panel: $g = m$; right panel: $g = f$. The solid curve corresponds to the Lee and Carter (1992)-model; the dashed curve corresponds to the Brouhns, Denuit, and Vermunt (2002)-model, and the dotted curve corresponds to the Cossette et al. (2007)-model.

3) Given the bootstrapped numbers of death $\overline{D}_{x,t}^{(g)}(b)$, we compute the (model specific) bootstrap estimates $\widehat{a}_x^{(g)}(b)$, $\widehat{b}_x^{(g)}(b)$ $\widehat{k}_t^{(g)}(b)$, $b = 1, ..., B$, using the described estimation techniques.

4) Given the bootstrap estimates $\widehat{a}_x^{(g)}(b)$, $\widehat{b}_x^{(g)}(b)$, $\widehat{k}_t^{(g)}(b)$, we generate $\widehat{k}_{T+s}^{m}(b)$ and $\widehat{k}_{T+s}^{f}(b)$, using the model specific version of (34), for $s = 1, ..., 85$ and $b = 1, ..., B$. This allows us to calculate the corresponding $p_{x,T+s}^{(g)}(b)$ via (32) and (33), resulting in $\mathcal{F}_t(b)$ for appropriate t.

5) Finally, for some quantity of interest $F = F(\mathcal{F}_t)$, we calculate the (model specific) bootstrap values $F(b) \equiv F(\mathcal{F}_t(b))$, for $b = 1, ..., B$, for each of the three models. On the basis of the distribution of all bootstrap values of $F(b)$, merged over the three models, we are able to quantify the macro-longevity risk.

B.2 P-Splines

In this section we briefly describe the P-spline model to forecast future forces of mortality. We use the model proposed by Currie, Durbin, and Eilers (2004) to quantify

death and deviance residuals.

longevity risk in the forces of mortality.
Let $B_y = B_y(x_y)$, be a $n_y \times c_y$ regression matrix of B-splines based on explanatory variable x_y and let $B_a = B_a(x_a)$, be a $n_a \times c_a$ regression matrix of B-splines based on explanatory variable x_a. The regression matrix for our model is the Kronecker product:

$$B = B_y \otimes B_a.$$

For the general population we regress:

$$\log\left(\underline{D}^{(m)} + \underline{D}^{(f)}\right) = \log\left(\underline{E}^{(m)} + \underline{E}^{(f)}\right) + B\underline{\alpha}^{(p)} + \underline{\epsilon}^{(p)}, \tag{36}$$

where, for the purpose of regression the data are arranged in column order, that is $\underline{D}^{(g)} = \text{vec}\left(D^{(g)}\right)$ and $\underline{E}^{(g)} = \text{vec}\left(E^{(g)}\right)$, and the log of a vector is the log applied componentwise to each element in the vector. The general trend in the force of mortality of the whole population is given by $B\underline{\alpha}^{(p)}$. For the difference in the forces of mortality between the general population and the gender specific forces of mortality we regress for both $g = m$ and $g = f$:

$$\log\left(\underline{D}^{(g)}\right) = \log\left(\underline{E}^{(g)}\right) + B\widehat{\underline{\alpha}}^{(p)} + B\underline{\alpha}^{(g)} + \underline{\epsilon}^{(g)}, \tag{37}$$

where $B\underline{\alpha}^{(p)}$ is estimated in the previous step and thus know in this regression. The difference in the general trend in the force of mortality of the whole population and the gender specific trend in the forces of mortality are given by $B\underline{\alpha}^{(g)}$. The gender specific trend in the force of mortality are given by the sum of the general trend in the force of mortality of the whole population and the gender specific trend in the forces of mortality, i.e., the gencer specific trend in the force of mortality is given by $B\underline{\alpha}^{(p)} + B\underline{\alpha}^{(g)}$.
We introduce a penalty on α, the penalty matrix is:

$$P = \lambda_a I_{c_y} \otimes D_a^{'} D_a + \lambda_y D_y^{'} D_y \otimes I_{c_a},$$

where λ_a and λ_y are smoothing parameters, I_{c_y} is an identity matrix of size c_y, D_a is a difference matrix with dimension $(c_a - p_a) \times c_a$ where p_a is the order of the penalty on age, similar definitions apply for I_{c_a} and D_y. The parameters λ_a and λ_y are set such that they optimize the Bayesian Information Criterion (BIC).
The fitted penalty likelihood can be obtained by the penalized version of the scoring algorithm (corresponding to IWLS):

$$(B^{'}\widetilde{W}B + P)\hat{\alpha} = B^{'}\widetilde{W}\widetilde{\alpha} + B^{'}(y - \widetilde{\mu}),$$

where $\widetilde{\alpha}, \widetilde{\mu}$, and $\widetilde{W} = \text{diag}(\widetilde{\mu})$ denotes current estimates, $\hat{\alpha}$ denotes updated estimates of α, and y denotes $\underline{D}^{(m)} + \underline{D}^{(f)}$ in case of the general trend given in equation (36), and $\underline{D}^{(g)}$ in case of the difference between the pattern between the general trend and the gender specific forces of mortality given in equation (37).
To forecast we use $V = \text{blockdiag}\{I, 0\}$, where I is of size $n_a n_y$ and 0 is a squared matrix of size $n_a n_f$, where n_f is the number of time period to forecast.

$$\left(B^{'}V\widetilde{W}B + P\right)\hat{\alpha} = B^{'}V\widetilde{W}\widetilde{\alpha} + B^{'}V\left(y - \widetilde{\mu}\right), \tag{38}$$

The uncertainty in α is approximately given by:

$$\text{Var}(\alpha) \approx \left(B^{'}VWB + P\right)^{-1}.$$

Using this covariance matrix of α we can simulate different paths of the future gender specific forces of mortality.
For $g = p, m$, and f the deviance residuals, Dev, are given by

$$Dev^{(g)} = 2 \cdot \left(\sum_{x,t} D_{x,t}^{(g)} \cdot \log\left(\frac{D_{x,t}^{(g)}}{\widehat{D}_{x,t}^{(g)}}\right) - D_{x,t}^{(g)} + \widehat{D}_{x,t}^{(g)}\right),$$

where $\widehat{D}_{x,t}^{(g)}$ is the estimate of the number of deaths. The number of effective parameters, Tr, are given by

$$Tr = \text{trace}\left(B\left(B^{'}\widetilde{W}B + P\right)^{-1}B^{\prime}\widetilde{W}\right).$$

Then we have that the Baysian Information Criterium, BIC is given by

$$BIC = 2 \cdot Dev + \log(n_a \cdot n_y) \cdot Tr.$$

Table 5 displays the help parameter values using Dutch mortality data from 1977 till 2006 for the ages 20 till 110. Following Currie, Durban, and Eilers (2006) we use for both the age splines and the time splines cubic B-splines, i.e., $bdeg = 3$ with a second order penalty, i.e., $pord = 2$. Also following Currie, Durban, and Eilers (2006), for the number of knots c, we use the rule of thumb that one internal knot for every four or five observations is sufficient for equally spaced data.

Table 5: Help parameter values of the P-spline model

	General	Males-general	Females-general
$bdeg_a$	3	3	3
$pord_a$	2	2	2
n_a	91	91	91
c_a	21	21	21
λ_a	15	1400	820
$bdeg_a$	3	3	3
$pord_y$	2	2	2
n_y	30	30	30
c_y	8	8	8
λ_y	72	3000	1800
Dev	3992	3308	3207
Tr	85.6	27.9	31.4
BIC	8662	6836	6662

This table displays the help parameter values for the P-spline model.

B.3 CBD model

In this section we briefly describe the the third class of models, the CBD-models to forecast future forces of mortality, which are members of the family of generalized CBD-Perks models. The CBD class of models was first introduced in Cairns, Blake, and Dowd (2006), later there several extension have been proposed to the original model, see e.g. Cairns et al. (2009). The CBD model fits the one-year mortality probability to a simple parametric form. The one-year mortality probabilities are obtained form the number of deaths and the number exposed to death by the following equation:

$$q_{x,t}^{(g)} = 1 - \exp\left(-\mu_{x,t}^{(g)}\right) = 1 - \exp\left(-\frac{D_{x,t}^{(g)}}{E_{x,t}^{(g)}}\right). \tag{39}$$

The general specification of the CBD-model is given by

$$\log\left(\frac{q_{x,t}^{(g)}}{1-q_{x,t}^{(g)}}\right) = \beta_x^{(g)}(1)\kappa_t^{(g)}(1)+\beta_x^{(g)}(2)\kappa_t^{(g)}(2)+\beta_x^{(g)}(3)\kappa_t^{(g)}(3)+\beta_x^{(g)}(4)\gamma_{t-x}^{(g)}(4)+\epsilon_{x,t}^{(g)}, \tag{40}$$

where $\beta_x^{(g)}(1), \beta_x^{(g)}(2), \beta_x^{(g)}(3)$, and $\beta_x^{(g)}(4)$ are given constants depending on age and time, $\kappa_t^{(g)}(1), \kappa_t^{(g)}(2)$, and $\kappa_t^{(g)}(3)$ are time-specific regression coefficients, $\gamma_{t-x}^{(g)}(4)$ is the cohort-specific regression coefficient, and $\epsilon_{x,t}^{(g)}$ is the residual with zero mean.
In the original CBD-model, proposed in Cairns, Blake and Dowd (2006), the parameters

are set

$$\begin{array}{rclrcl}\beta_x^{(g)}(1) & = & 1, & \beta_x^{(g)}(3) & = & 0, \\ \beta_x^{(g)}(2) & = & (x-\overline{x}), & \beta_x^{(g)}(4) & = & 0,\end{array}$$

where $\overline{x} = \frac{1}{n_a}\sum_{i=x_{\min}}^{x_{\max}} i$ is the mean age in the sample range, with n_a the number of age groups, $x_{\min}$ the youngest age group in the mortality probabilities, and $x_{\max}$ the oldest age group in the mortality probabilities. We refer to this version of the CBD-model as "CBD 1", which leads to the following characterization of the mortality probabilities:

$$\log\left(\frac{q_{x,t}^{(g)}}{1-q_{x,t}^{(g)}}\right) = \kappa_t^{(g)}(1) + (x-\overline{x})\,\kappa_t^{(g)}(2). \tag{41}$$

As a first generalization of the CBD model a cohort effect as included, the other specifications of the CBD model was the same as the original CBD-model:

$$\begin{array}{rclrcl}\beta_x^{(g)}(1) & = & 1, & \beta_x^{(g)}(3) & = & 0, \\ \beta_x^{(g)}(2) & = & (x-\overline{x}), & \beta_x^{(g)}(4) & = & 1.\end{array}$$

We call the CBD-model with these parameters "CBD 2", which leads to the following characterization of the mortality probabilities:

$$\log\left(\frac{q_{x,t}^{(g)}}{1-q_{x,t}^{(g)}}\right) = \kappa_t^{(g)}(1) + (x-\overline{x})\,\kappa_t^{(g)}(2) + \gamma_{t-x}^{(g)}(4). \tag{42}$$

To avoid identifiability problems, we impose the following two identification constraints:

$$\begin{array}{rclrcl}\sum_{c\in C}\gamma_{t-x}^{(g)}(4) & = & 0, & \sum_{c\in C} c\cdot\gamma_{t-x}^{(g)}(4) & = & 0,\end{array}$$

where C is the set of all cohort years of birth that have been included in the analysis, and $c = t - x$.
The next model is a generalization of the previous one, which adds a quadratic term into the age effect.

$$\begin{array}{rclrcl}\beta_x^{(g)}(1) & = & 1, & \beta_x^{(g)}(3) & = & (x-\overline{x})^2 - \widehat{\sigma}_x^2, \\ \beta_x^{(g)}(2) & = & (x-\overline{x}), & \beta_x^{(g)}(4) & = & 1,\end{array}$$

where $\widehat{\sigma}_x^2 = \frac{1}{n_a}\sum_{i=x_{\min}}^{x_{\max}}(i-\overline{x})^2$. We refer to this version of the CBD-model as "CBD

3", which leads to the following characterization of the mortality probabilities:

$$\log\left(\frac{q_{x,t}^{(g)}}{1-q_{x,t}^{(g)}}\right) = \kappa_t^{(g)}(1) + (x-\overline{x})\,\kappa_t^{(g)}(2) + \left((x-\overline{x})^2 - \widehat{\sigma}_x^2\right)\kappa_t^{(g)}(3) + \gamma_{t-x}^{(g)}(4). \quad (43)$$

To avoid identifiability problems, we impose the following three identification constraints:

$$\textstyle\sum_{c\in C}\gamma_{t-x}^{(g)}(4) = 0, \quad \sum_{c\in C} c\cdot\gamma_{t-x}^{(g)}(4) = 0, \quad \sum_{c\in C} c^2\cdot\gamma_{t-x}^{(g)}(4) = 0.$$

The last generalization of the original CBD-model assumes that the age effects of the cohort effects are diminishing over time for any specific cohort. This results in the following characterization:

$$\begin{aligned} \beta_x^{(g)}(1) &= 1, & \beta_x^{(g)}(3) &= 0, \\ \beta_x^{(g)}(2) &= (x-\overline{x}), & \beta_x^{(g)}(4) &= \left(x_c^{(g)} - x\right), \end{aligned}$$

for some constant parameter $x_c^{(g)}$ to be estimated, using optimization of the BIC. We call the CBD-model with these parameters "CBD 4", which leads to the following characterization of the mortality probabilities:

$$\log\left(\frac{q_{x,t}^{(g)}}{1-q_{x,t}^{(g)}}\right) = \kappa_t^{(g)}(1) + (x-\overline{x})\,\kappa_t^{(g)}(2) + \left(x_c^{(g)} - x\right)\gamma_{t-x}^{(g)}(4). \quad (44)$$

To avoid identifiability problems, we impose the following identification constraint:

$$\sum_{c\in C}\gamma_{t-x}^{(g)}(4) = 0.$$

In order to make forecasts of the mortality probabilities we have to forecast the process $\kappa(t) = \left[\kappa_t^{(m)}(1)\ \kappa_t^{(m)}(2)\ \kappa_t^{(m)}(3)\ \kappa_t^{(f)}(1)\ \kappa_t^{(f)}(2)\ \kappa_t^{(f)}(3)\right]$. It is assumed that the process is subject to i.i.d. multivariate normal shocks with mean μ and covariance matrix V. Let $D(t) = \kappa(t) - \kappa(t-1)$, and $D = \{D(1), \ldots, D(n)\}$. In the absence of any clear prior beliefs about the values of μ and V Cairns, Blake, and Dowd (2006) use a non-informative prior distribution. A common prior for the multivariate normal distribution in which both μ and V are unknown is the Jeffreys prior:

$$p(\mu, V) \propto |V|^{-3/2},$$

where $|V|$ is the determinant of the covariance matrix V. The posterior distribution for

Table 6: Parameter estimates of the CBD-models

	$\mu_1^{(m)} \cdot 10^2$	$\mu_2^{(m)} \cdot 10^4$	$\mu_3^{(m)} \cdot 10^5$	$\mu_1^{(f)} \cdot 10^2$	$\mu_2^{(f)} \cdot 10^4$	$\mu_3^{(f)} \cdot 10^5$	$LogL$	# par	BIC
CBD 1	-1.3723	8.3578		-1.1211	1.3925		-12042	120	24905
CBD 2	-1.3203	3.9099		-1.0141	17.7359		-9344	236	20302
CBD 3	-1.3708	8.0533	2.1365	-0.87047	1.7736	-5.8667	-9220	294	20449
CBD 4	-3.9336	-1.7694		6.7977	36.4297		-9431	240	20503

The table displays the estimation of the parameter μ and the log likelihood, number of parameter, and the Bayesian Information Criterion (BIC) for the different CBD-models. For model CBD 4 we have used $x_c^{(m)} = 74$ and $x_c^{(f)} = 75$.

$(\mu, V|D)$ satisfies

$$\begin{aligned} V^{-1}|D \sim & \text{Wishart}\left(n-1, n^{-1}\widehat{V}^{-1}\right), \\ \mu|V, D \sim & MVN\left(\widehat{\mu}, n^{-1}V\right), \\ \text{where } \widehat{\mu} = & n^{-1}\sum_{t=1}^{n} D(t), \\ \text{and } \widehat{V} = & n^{-1}\sum_{t=1}^{n} \left(D(t)-\widehat{\mu}\right)\left(D(t)-\widehat{\mu}\right)'. \end{aligned}$$

We estimate the parameters using Dutch mortality data from 1977 till 2006 for the ages $x_{\min} = 60$ till $x_{\max} = 90$. The log-likelihood is given by:

$$LogL = \sum_{x,t,g} D_{x,t}^{(g)} \cdot \log\left(E_{x,t}^{(g)} \cdot \widehat{\mu}_{x,t}^{(g)}\right) - E_{x,t}^{(g)} \cdot \widehat{\mu}_{x,t}^{(g)} - \log\left(D_{x,t}^{(g)}!\right),$$

where $\widehat{\mu}_{x,t}^{(g)} = -\log\left(1-\widehat{q}_{x,t}^{(g)}\right)$ is the force of mortality using the parameter estimates. Table 6 displays the estimates of μ for the different models.