Mortality when Converting from Conventional Life Insurance to Universal Life Insurance Policies

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ABSTRACT

This paper considers different aspects of conversion from conventional life insurance policies into universal life policies.

Finding formulas for conventional policies on an annual basis is typically quite straightforward, but this paper analyses and discusses also monthly mortalities.

This paper introduces a concept "discount factor preserving method" and "risk premium preserving method" which ensures the compatibility of old and new formulas.

The main focus of this paper is conversion. The results, especially when viewed from or analyzed on a monthly basis, are different than those referred to in actuarial literature.

Keywords:
Universal life techniques, Conversion, Mortality, Discount factor preserving method, Risk premium preserving method

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1 INTRODUCTION

1.1 Purpose of this document

This paper describes conventional model conversion into universal life models.

> I have chosen to use the terms
> - *conventional model* for the prospective model where the liability is the present value of the future net outgoing cash flows
> - *universal life* model for the retrospective model where the liability is calculated as the accumulation of account entries over the years up to the balance sheet date
>
> The universal life model can sometimes also been called the "recursive method" (see e.g. Gerber [3], p. 68). However, recursive methods are used also in other contexts. Using the name "recursive method" is justified especially when the conventional formulas are expressed in universal life formulas.

This paper concentrates on modeling the reserve calculation using universal life models.

> By "reserve" in this document I mean the policy savings and not "liabilities" as typically used in general actuarial literature.

This paper is based on my previous study "Conversion from conventional life insurance policies into universal life policies" (see Niittuinperä [9]). The previous study covers more comprehensively the conversion, including also some proofs that I have not shown in this paper.

1.2 Used notation

The notations used in this document are mainly based on the International Actuarial Notation, published in the Encyclopedia of Actuarial Science (see Wolthuis [12]).

However, in this document I do not always differentiate between discrete and continuous models because sometimes the same formulas may be used for both models. The difference is only specifically cited in situations where there is some relevant difference between the models.

About detailed notations see chapter 4.

The terminology and notations used may differ from one reference to another, but when cited in this document, uniform terminology and symbols are used.

2 MOTIVATION FOR THE PAPER

2.1 Understanding old products

Using universal life models may help in the process of fulfilling IFRS and solvency II requirements. Also in Principles for the Conduct of Insurance Business IAIS has stated about the disclosure principles that the insurance undertaking shall have to inform the policyholder of costs and associated charges.

As a result of these requirements insurance undertakings should have more detailed information about their portfolios. This paper does not discuss the simulation of future cash flows but concentrates on modeling the reserve calculation using universal life models.

New requirements laid down for the insurance undertakings do not require universal life models. Some approximation methods can also be used, but they should be based on analysis of cash flows.

2.2 Efficiency of the insurance undertaking processes

It is well documented that old insurance undertakings have applied many techniques during their product generations and life cycles. It is also well known that the new ones are universal life -type policies. This means that the products in conventional techniques are often run-off portfolios that will still be in force for many years to come.

It is inevitable that sooner or later, the leadership of insurance undertakings will have to ask themselves whether it is commercially viable or indeed is it even wise to maintain these old techniques because of the financial implications involved of running separate computer systems, improving and increasing their efficiency and then needing and also requiring and employing several interfaces to consolidate the data.

The costs are not exclusively software or exclusively personnel costs but rather a mixture of them. Solving the problems of old portfolios requires often significant management involvement, even though the revenue is relatively small compared to other products.

One option is to convert the policies with conventional techniques to policies with universal life techniques. Then, if the software is parameter driven, sometimes the undertaking may be able to manage several products with the same software.

3 PROCESS

3.1 Goal of the process

The goal in itself is very simple. We have two models:

In **conventional model** prospective calculation is used:

$$V_{x+t} = A_{x+t:w}(*) \cdot S_{x+t} - B_{x+t:k-t]} \cdot \ddot{a}_{x+t:k-t]}$$

In **universal life model** retrospective calculation is used:

$$V_{x+t} = \sum_{k=0}^{t} \left\{ (1-\beta) \cdot B_{x+k} + \frac{i}{1-q_{x+t}} \cdot (V_{x+k-1} + B_{x+k}) - \frac{q_{x+t}}{1-q_{x+t}} \cdot \left[(1-\gamma) \cdot S_{x+t} - (V_{x+k-1} + B_{x+k}) \right] \right\}$$

The goal is to find a one-to-one relationship between the models.

In practical solutions recursive formulas can always be found. (see Koller [6], pp. 49 – 51). As a complex example, I have derived formulas for some Finnish sickness insurance policies where the risk functions are continuous and force of mortality is defined by Makeham model. When solving the fifth degree function Trapezoidal rule was used. The three-pages-long derivation of the formula may be found from my previous study (see Niittuinperä [9], pp. 32 – 35).

Typically the formulas are written for a sum insured equal to 1, which entails that the reserve has to be multiplied by the sum insured. However, in practice the formulas are calculated based on the actual sum insured, and this convention is therefore used in this paper.

3.2 Deriving formulas

I have used the following step by step approach to the conversion rules:

1) Defining a general annual model without any loadings. In this context annual model means that the calculations are performed annually only at the end of an insurance year.
2) Defining a general annual expense-loaded model.
3) Defining a general monthly model. In this model the calculations are performed monthly at the end of an insurance month.
4) Defining a monthly expense-loaded model.

> In this paper, by "insurance year" I mean the one-year-long time period starting the same month and day as the policy becomes effective. By "insurance month" I mean any one-month-long time period starting the same day as the policy becomes effective. If the month does not have that day, the last day of the month is chosen.

In this paper I have not covered expense models, but concentrated on mortality. About the expense models I refer to my study (see Niittuinperä [9], chapters 7, 9.1, 9.2 and 13.5).

If we know the value either at the end of the month, the other one can also be calculated. The formulas are not as simple as at the end of the month. You may find some examples from my previous study (see Niittuinperä [9], pp. 32 – 35).

3.3 Testing

Testing is closely related to deriving the formulas.

The formulas in this paper have been tested in practice by calculating the same numerical input data using both the conventional model and the universal life model.

4 CONVENTIONAL FORMULAS

4.1 General

In this paper I do not derive the conventional formulas because the derivations are readily available from the literature (see e.g. Gerber [3] pp. 119 – 123, Neill [8] pp. 38 – 71 and Schmidt [11] pp. 123 – 129).

4.2 Age notifications

Unless otherwise mentioned, for ages I have used the following notifications:

- x	the age at the beginning of the insurance year
- x+t	the age t whole years after x
- x+t+m/12	the age m whole months ($0 \le m < 12$) from x+t (before x+t+1)
- x+t+u	the age time u ($0 \le u < 1$) from x+t

4.3 Discrete model

In a discrete model mortalities are calculated separately for each age. For each age x a number l_x is estimated. This number represents people alive at age x (it can be assumed e.g. that $l_0 = 10^6$).

Let us define

$$d_x = l_x - l_{x+1}$$

$$q_x = \frac{d_x}{l_x} = 1 - \frac{l_{x+1}}{l_x}$$

$$v = \frac{1}{1+i}$$

where

x	is age
d_x	is number of death (at age x)
q_x	is mortality (at age x)
i	is technical interest rate
v	is discount coefficient

The commutation numbers are as follows:

$$D_x = l_x \cdot v^x$$

$$N_x = \sum_{i=x}^{w'} D_i$$

$$C_x = d_x \cdot v^{x+1} = (l_x - l_{x+1}) \cdot v^{x+1} = q_x \cdot v \cdot D_x = \frac{q_x}{1+i} \cdot D_x$$

$$M_x = C_x + ... + C_{w'}$$

l_x may be calculated from q_x -numbers as follows: $l_{x+1} = (1 - q_x) \cdot l_x$.

The annuity will be

$$\ddot{a}_{x:n]} = \frac{N_x - N_{x+n}}{D_x}$$

where

n   is duration of annuity and
w '   is last age of tables

Define also monthly N_x -numbers as follows:

$$N_x^{(12)} = \frac{1}{12} \cdot \sum_{i=x}^{w} D_i^{(12)}$$

where $D_i^{(12)}$ is monthly D_x -number

4.4 Continuous model

In a continuous model l_x is defined using continuous force of mortality:

$$l_x = l_0 \cdot e^{-\int_0^x \mu_s ds}$$

From this we may calculate the q_x -numbers as in discrete case.

$$q_x = 1 - \frac{l_{x+1}}{l_x} = 1 - \frac{l_0 \cdot e^{-\int_0^{x+1} \mu_s ds}}{l_0 \cdot e^{-\int_0^x \mu_s ds}} = 1 - e^{-\int_0^{x+1} \mu_s ds + \int_0^x \mu_s ds} = 1 - e^{-\int_x^{x+1} \mu_s ds}$$

Several mortality models can be used (see e.g. Gerber [3], p. 17 – 18 and Bowers et al. [1] pp. 77 – 79). Later on I will assume that the functions have the required derivatives and integrals.

By using the continuous model it is possible to calculate a risk for any period. We consider first the annual case, but later also the monthly case is discussed.

By using an Euler summation we get annual representations between the continuous model and discrete model:

$$\overline{N}_x = N_x - D_x \cdot \left[\frac{1}{2} + \frac{1}{12} \cdot (\ln(1+i) + \mu_x) \right]$$

$$\overline{M}_x = D_x - \ln(1+i) \cdot \overline{N}_x$$

where D_x and N_x are calculated as in the discrete case (see Neill (3.2.1, 3.2.8, 3.3.3) p. 78, 81, 102). Neill writes that it is typical to use a shorter approximation:

$$\overline{N}_x = N_x - \frac{1}{2} \cdot D_x$$

The continuous examples that I will give later are based on the first formulas.

I concentrate on the discrete model and describe separately the behavior of the continuous model. Note that a bar above the basic symbol denotes continuous actuarial functions.

4.5 Accumulation and discount factors

In this paper I use terms "accumulation factor" and "discount factor" also for cases where not only the interest but also mortality is taken into account.

The **accumulation factor** including effect of interest and mortality can be written as follows:

$$\frac{D_{x+t}}{D_{x+t+1}} = \frac{1+i}{1-q_{x+t}} = 1 + \frac{i}{1-q_{x+t}} + \frac{q_{x+t}}{1-q_{x+t}}$$

The results are found as follows:

$$\frac{D_{x+t}}{D_{x+t+1}} = \frac{l_{x+t} \cdot v^{x+t}}{l_{x+t+1} \cdot v^{x+t+1}} = \frac{1+i}{\frac{l_{x+t+1}}{l_{x+t}}} = \frac{1+i}{1-q_{x+t}} = 1 + \left(\frac{1+i}{1-q_{x+t}} - 1 \right) = 1 + \frac{i+q_{x+t}}{1-q_{x+t}} = 1 + \frac{i}{1-q_{x+t}} + \frac{q_{x+t}}{1-q_{x+t}}$$

Sometimes also the respective **discount factor** is needed. It is equal to

$$\frac{D_{x+t+1}}{D_{x+t}} = \frac{1-q_{x+t}}{1+i} = 1 - \frac{i}{1+i} - \frac{q_{x+t}}{1+i}.$$

In the continuous case the discount factor may also be expressed as follows:

$$\frac{D_x}{D_{x+1}} = \frac{l_0 \cdot (1+i)^{-x} \cdot e^{-\int_0^x \mu_s ds}}{l_0 \cdot (1+i)^{-(x+1)} \cdot e^{-\int_0^{x+1} \mu_s ds}} = \frac{1}{1+i} \cdot e^{-\int_x^{x+1} \mu_s ds}$$

5 GENERAL CONVENTIONAL MODEL

5.1 Premium and reserve

Let us consider the following general premium model:

$$B_{x:k]} = \frac{A_{x:w}(*)}{\bar{a}} \cdot S_x$$

where

$A_{x:w}(*)$	= net single premium coefficient depending on the product (*) in question
S_x	= sum insured at time x for a period of n years
$\bar{a}$	= 1, for single premium
	= $\ddot{a}_{x:k]}$ for annual premiums for a period of k years

Reserve for sum insured S_{x+t} at time x+t in this case would be:

$$V_{x+t} = A_{x+t:w}(*) \cdot S_{x+t} - B_{x+t:k-t]} \cdot \ddot{a}_{x+t:k-t]}.$$

5.2 Universal life representation for discrete model

It is quite easy to proof that the reserve may be calculated by the following formula:

$$V_{x+t+1} = V_{x+t}$$
$$+ B_{x+t:k-t]} - E_{x+t} + \frac{i}{1-q_{x+t}} \cdot (V_{x+t} + B_{x+t:k-t]} - E_{x+t}) - \frac{q_{x+t}}{1-q_{x+t}} \cdot (S_{x+t} - (V_{x+t} + B_{x+t:k-t]} - E_{x+t}))$$

The last term can be positive or negative depending on whether the risk sum is positive or negative.

The different components are as follows:

- annual premium	$B_{x+t:k-t]}$
- annual annuity	E_{x+t}
- interest	$\frac{i}{1-q_{x+t}} \cdot (V_{x+t} + B_{x+t:k-t]} - E_{x+t})$
- mortality	$-\frac{q_{x+t}}{1-q_{x+t}} \cdot \left[S_{x+t} - (V_{x+t} + B_{x+t:k-t]} - E_{x+t})\right]^+$
- compensation	$-\frac{q_{x+t}}{1-q_{x+t}} \cdot \left[S_{x+t} - (V_{x+t} + B_{x+t:k-t]} - E_{x+t})\right]^-$

The whole proof for the formulas can be found from my previous study (See Niittuinperä, pp. 12 – 16). The general result without division to the components and slightly differently expressed has been proofed also in the literature. The proofs do not directly show the relationship with the commutation numbers (see e.g. Neill [8] (4.4.1), p. 124 and Bowers et al. [1] (8.3.10) p. 235, Schmidt [11] (5.5.9) p. 124 and Gerber (6.3.4) [3] p. 61).

As an example, pure endowment single premium can be derived using commutation numbers as follows:

> Assume that the reserve for sum insured S_{x+t} at moment x+t is equal to (see e.g. Schmidt [11], example 5.4.5 (5), p. 126):
>
> $$V_{x+t} = \frac{D_w}{D_{x+t}} \cdot S_{x+t} \,.$$
>
> After one year the reserve is equal to
>
> $$V_{x+t+1} = \frac{D_w}{D_{x+t+1}} \cdot S_{x+t} = \frac{D_{x+t}}{D_{x+t+1}} \cdot \frac{D_w}{D_{x+t}} \cdot S_{x+t} = \frac{D_{x+t}}{D_{x+t+1}} \cdot V_{x+t}$$
>
> $$= V_{x+t} + \frac{i + q_{x+t}}{1 - q_{x+t}} \cdot V_{x+t} = V_{x+t} + \frac{i}{1 - q_{x+t}} \cdot V_{x+t} + \frac{q_{x+t}}{1 - q_{x+t}} \cdot V_{x+t}$$

The result is natural because it shows that the reserve is the previous reserve corrected by interest increase and mortality compensation.

Assuming first that mortality $q_{x+t} = 0$ and then that guaranteed interest i = 0, we obtain the following results:

1) $\frac{i}{1-q_{x+t}} \cdot V_{x+t}$ is the effect of guaranteed interest increase

2) $\frac{q_{x+t}}{1-q_{x+t}} \cdot V_{x+t}$ is the compensation due to mortality

5.3 Universal life representation for continuous models

If the continuous model does not have continuity correction as defined in chapter 4.4., then the formulas defined in the discrete model apply.

In the continuous model it is possible to use continuity corrections for the discrete values. They act like loadings. For example payment corrections are taken into account only when payments are paid.

The reserve may be calculated by

$$V_{x+t+1} = V_{x+t}$$

$$+ B_{x+t:k-t]} - E_{x+t} + \frac{i}{1-q_{x+t}} \cdot (V_{x+t}^A + B_{x+t:k-t]} - E_{x+t})$$

$$- \frac{q_{x+t}}{1-q_{x+t}} \cdot (S_{x+t} - (V_{x+t}^A + B_{x+t:k-t]} - E_{x+t})) + B_{x+t}^C + E_{x+t}^C + Q_{x+t}^C$$

where

B_{x+t}^{C} is the correction related to payments

$$-\left\{\frac{1}{12}\cdot(\mu_{x+t}-\mu_{x+t+1})+\frac{i+q_{x+t}}{1+i}\cdot\left[\frac{1}{2}+\frac{1}{12}\cdot\left(\ln(1+i)+\mu_{x+t+1}\right)\right]\right\}\cdot B_{x+t}$$

E_{x+t}^{C} is the correction related to annuities

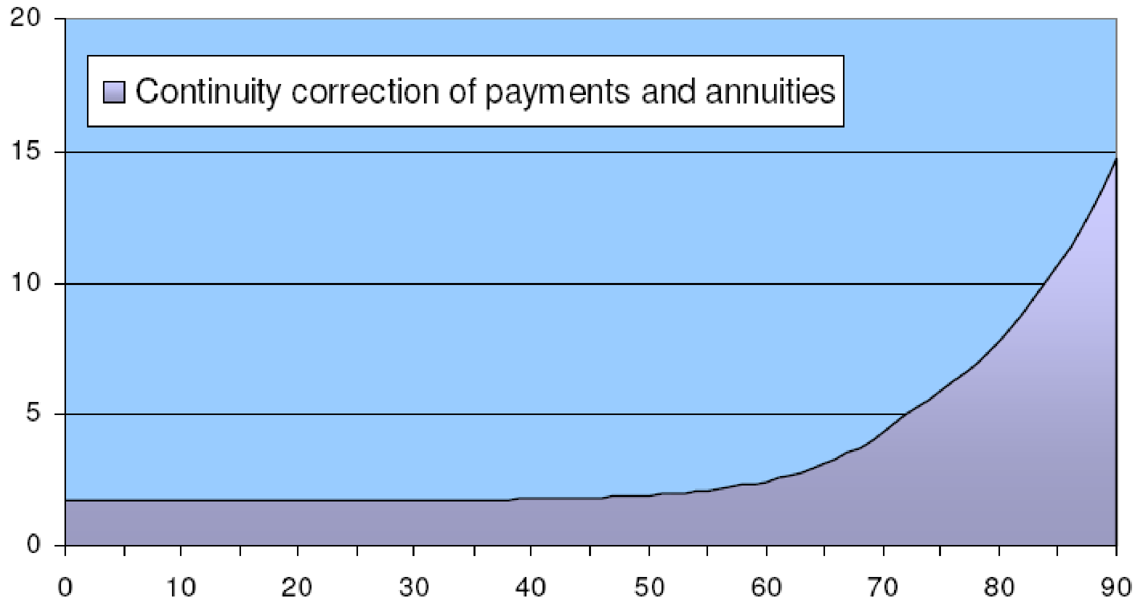
$$\left\{\frac{1}{12}\cdot(\mu_{x+t}-\mu_{x+t+1})+\frac{i+q_{x+t}}{1+i}\cdot\left[\frac{1}{2}+\frac{1}{12}\cdot\left(\ln(1+i)+\mu_{x+t+1}\right)\right]\right\}\cdot E_{x+t}$$

Q_{x+t}^{C} is the correction related to mortality charges

$$\left(-\frac{i}{1+i}+\ln(1+i)\cdot\left\{1-\frac{1}{12}\cdot(\mu_{x+t}-\mu_{x+t+1})-\frac{i+q_{x+t}}{1+i}\cdot\left[\frac{1}{2}+\frac{1}{12}\cdot\left(\ln(1+i)+\mu_{x+t+1}\right)\right]\right\}\right)\cdot S_{x+t}$$

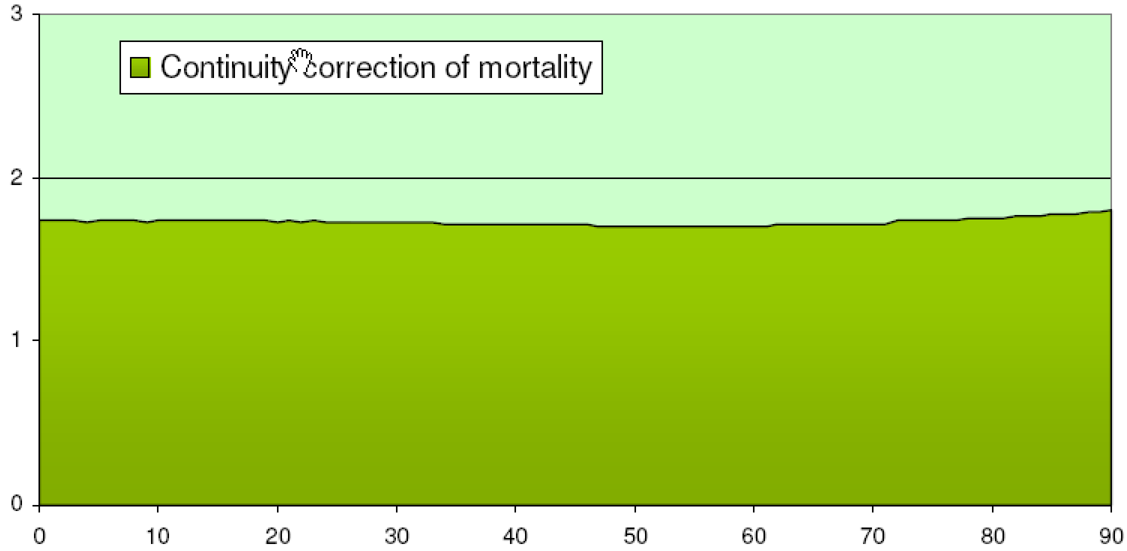
The proofs may be found from my previous study (see Niittuinperä [9], chapter 6.4, pp. 17 – 20).

The correction related to annuities and payments for interest rate 3,5 % is shown in picture 5.1. The correction is almost constant until age 40 and then rises to 13,8 % until age 90.



Picture 5.1 Continuity correction effect of payments and annuities (%) for interest rate 3,5 % at the end of the year

Mortality charges are charged each year. The continuity correction of the mortality can be added to the value of the discrete model because it has originally defined to be a component of the mortality. The correction does not vary great deal, especially if charged at the end of the year as shown in picture 5.2. During the same period as above the correction ranges from 1,71 % to 1,80 % with lowest value at age 56.



Picture 5.2 Continuity correction effect of mortality (%) for interest rate 3,5 % if charged at the end of the year

If the charges were charged in the beginning of the year, then the discount factor should have been taken into account. The factor decreases the value the more the older a person is.

6 CALCULATION AT THE END OF AN INSURANCE MONTH

6.1 General

The methods defined above give exact values for the end of an insurance year.

It is common that the insurance undertaking has defined the reserve formulas of the conventional products at least for each insurance anniversary. It is almost as common that some approximation formula is used between the insurance anniversaries.

In this chapter 6 I shall concentrate on defining the exact values of the reserves. Mortality will be adjusted so that the reserve of the pure endowment will be preserved. With this mortality assumption and common mortality charges, the reserve is no longer preserved. Therefore I shall define different mortality functions for such cases.

6.2 Mortality assumption at non-integer ages

6.2.1 General

It is common that the mortality tables are defined for integer ages. In continuous models the mortalities for non-integer ages can be easily calculated.

> The term "non-integer ages" has been used e.g. by Forfar (see Forfar [2], p. 1007). Sometimes this is also called "fractional ages" (see e.g. Bowers et al. [1] p. 74 and Jones et al. [5] and [6]).

The mortality in non-integer ages has been defined in literature in different ways. The most common models are the following:

- uniform distribution of deaths (called also UDD or linearity of mortality)
- constant force of mortality
- Balducci model (called also hyperbolic model)

(See Jones et al. [5], p. 261 – 276, Jones et al. [4], p. 363 – 370, where the authors unify and extend the mentioned models and see Bowers et al. [1] pp. 74 – 76, Gerber [3] p. 21 – 22 and Forfar [2] p. 1007.)

I shall refer the above-mentioned models, but I also propose some modifications to them.

Jones and Mereu have criticized the above models: "While this has the advantage of simplicity, all three assumptions result in force of mortality and probability density functions with implausible discontinuities at integer ages." (See Jones et al. [4], p. 363.)

My point of view is the conversion and applied models. In conversion the insurance undertaking is bound to the promises it has given. I am not concerned about eventual discontinuities. I introduce here a new concept called "discount factor preserving method" and derive some mortality functions based on that concept. The proposed modifications that I mentioned above are based on this method.

6.2.2 Discount factor preserving method

One possible goal for the universal life model is that each year the reserve is exactly the same as if it were calculated by the conventional formulas. This means that accumulation and discount factors should be the same on an annual basis:

$$\frac{D_x}{D_{x+1}} = \prod_{m=0}^{11} \frac{D_{x+m/12}}{D_{x+(m+1)/12}} = \frac{1+i}{1-q_x}$$

I shall later call this as "**discount factor preserving method**".

If we assume that the interest rate is constant, then in accordance with the annual accumulation factor, the monthly accumulation factor is as follows:

$$\frac{D_{x+m/12}}{D_{x+(m+1)/12}} = \frac{\sqrt[12]{1+i}}{1-q_{x+m/12}}$$

So, the monthly interest rate is equal to $\sqrt[12]{1+i}-1$.

In principle it is possible to find a discount factor preserving method by adjusting the mortality, the interest rate or both. In practice I propose to adjust mortality because the interest rate has normally been fixed.

Note that if interest is constant, then, because $D_x = l_x \cdot \left(\frac{1}{1+i}\right)^x$, discount factor preserving method preserves also l_x-numbers at the end of the year.

> Jones and Mereu write about the model that I have called linear D_x-model: "Strictly speaking, this is not an FAA... (fractional age assumption) ... because different age at death distributions arise for differenct choices of the interest rate." (See Jones et al. [5], p. 262.) In discount factor preserving method the mortality may depend on the chosen interest rate, but normally not vice versa. There are some arguments against linear D_x-model that I shall consider in summary section.

6.2.3 Constant force of mortality

Let us denote constant force of mortality by μ^c and the respective mortality by $q^c_{x+m/12}$.

In this case l_{x+1} is equal to

$$l_{x+1} = l_x \cdot e^{-\mu_x}$$

and

$$\mu_x = -\ln\left(\frac{l_{x+1}}{l_x}\right)$$

Accordingly $l_{x+m/12}$ is equal to

$$l_{x+(m+1)/12} = l_{x+m/12} \cdot e^{-\mu^c/12}$$

and

$$\prod_{m=0}^{11} \frac{D_{x+m/12}}{D_{x+(m+1)/12}} = \prod_{m=0}^{11} e^{\mu^c/12} = e^{\sum_{m=0}^{12} \mu^c/12} = e^{\mu^c} = e^{\mu_x} = \frac{D_x}{D_{x+1}}.$$

Hence, the mortality is

$$q^c_{x+m/12} = 1 - e^{-\mu^c/12}$$

This means that the same force of mortality for non-integer years may be used as for the integer years. In fact constant force of mortality implies also that the mortality is constant in non-integer years. Let us denote the constant mortality by q^c_x. Its value depends on q_x and may be found as follows:

$$\prod_{m=0}^{11} \frac{D_{x+m/12}}{D_{x+(m+1)/12}} = \frac{\left(\sqrt[12]{1+i}\right)^{12}}{\left(1-q^c_x\right)^{12}} = \frac{1+i}{1-q_x} = \frac{D_x}{D_{x+1}}$$

So, we obtain

$$\left(1-q^c_x\right)^{12} = 1-q_x .$$

This yields the following result:

$$q^c_x = 1 - \sqrt[12]{1-q_x}$$

So, it is possible to choose whether to use constant mortality or force of mortality.

6.2.4 Uniform distribution of deaths

The unified mortality means that the deaths are uniformly distributed. In the UDD model the l_x-numbers are interpolated as follows:

$$l_{x+m/12} = \left(1-\frac{m}{12}\right)\cdot l_x + \frac{m}{12}\cdot l_{x+1}$$

When dividing by l_x, the following result is obtained:

$$\frac{l_{x+m/12}}{l_x} = 1-\frac{m}{12}+\frac{m}{12}\cdot\frac{l_{x+1}}{l_x} = 1-\frac{m}{12}\cdot\left(1-\frac{l_{x+1}}{l_x}\right) = 1-\frac{m}{12}\cdot q_x .$$

From this we obtain

$$\frac{l_{x+(m+1)/12}}{l_{x+m/12}} = \frac{1-\frac{m+1}{12}\cdot q_x}{1-\frac{m}{12}\cdot q_x} = 1-\frac{\frac{1}{12}\cdot q_x}{1-\frac{m}{12}\cdot q_x} = 1-\frac{q_x}{12-m\cdot q_x}$$

which yields the result

$$q_{x+m/12} = \frac{q_x}{12 - m \cdot q_x}$$

However, this method does not preserve the discount factor. So, I shall define a modified UDD as such a mortality q_x^u that the mortality in month x+m is equal to $\frac{q_x^u}{12 - m \cdot q_x^u}$ and preserves the discount factor. Then we obtain the following equation:

$$\prod_{m=0}^{11} \left(1 - \frac{q_x^u}{12 - m \cdot q_x^u} \right) = 1 - q_x$$

Here q_x^u-number can be found by iteration (see about iteration e.g. Kreyszig, pp. 838 – 848).

6.2.5 Balducci assumption

The Balducci assumption (see Jones et al. [5], p. 261 – 276, Jones et al. [4], p. 363 – 370, Gerber [3] p. 21 – 22 and Forfar [2] p. 1007) assumes that the monthly mortality is determined by

$$\frac{1}{l_{x+m/12}} = \frac{1 - \frac{m}{12}}{l_x} + \frac{\frac{m}{12}}{l_{x+1}}$$

Because of this it is sometimes called hyperbolic model.

In this case we obtain

$$\frac{l_{x+1}}{l_{x+m/12}} = \frac{l_{x+1} \cdot \left(1 - \frac{m}{12}\right) + l_x \cdot \frac{m}{12}}{l_x} = \frac{l_{x+1} + \frac{m}{12} \cdot \left(l_x - l_{x+1}\right)}{l_x} = \frac{l_x - \left(l_x - l_{x+1}\right) + \frac{m}{12} \cdot \left(l_x - l_{x+1}\right)}{l_x}$$
$$= \frac{l_x - \left(1 - \frac{m}{12}\right) \cdot \left(l_x - l_{x+1}\right)}{l_x} = 1 - \left(1 - \frac{m}{12}\right) \cdot q_x$$

which is the Balducci assumption for one month.

From this we obtain

$$\frac{l_{x+(m+1)/12}}{l_{x+m/12}} = \frac{1 - \left(1 - \frac{m+1}{12}\right) \cdot q_x}{1 - \left(1 - \frac{m}{12}\right) \cdot q_x} = 1 - \frac{\frac{1}{12} \cdot q_x}{1 - \left(1 - \frac{m}{12}\right) \cdot q_x} = 1 - \frac{q_x}{12 - \left(12 - m\right) \cdot q_x}$$

which yields to the result

$$q_{x+m/12} = \frac{q_x}{12-(12-m)\cdot q_x}$$

This mortality does not, however, preserve the discount factor.

Let us now define modified Balducci assumption as such a mortality q_x^b that the mortality in month x+m is equal to $\frac{q_x^b}{12-(12-m)\cdot q_x^b}$ and preserves the discount factor. Then we obtain the following equation:

$$\prod_{m=0}^{11}\left[1-\left(1-\frac{m}{12}\right)\cdot q_x^b\right] = 1-q_x$$

In this case q_x^b -number can be found by iteration (see about iteration e.g. Kreyszig, pp. 838 – 848).

6.2.6 Continuous model

In the continuous model case the monthly mortalities may be calculated from the mortality function. The same formulas as on annual level may be applied for the calculations at the end of an insurance month.

In this case l_x is defined by using continuous force of mortality:

$$l_x = l_0 \cdot e^{-\int_0^x \mu_S ds}$$

From this we obtain also a value for each month:

$$l_{x+m/12} = l_0 \cdot e^{-\int_0^{x+m/12} \mu_S ds}$$

In this case

$$D_{x+m/12} = l_0 \cdot (1+i)^{-x+m/12} \cdot e^{-\int_0^{x+m/12} \mu_S ds}$$

and the accumulation factor is

$$\frac{D_{x+m/12}}{D_{x+(m+1)/12}} = \frac{l_0 \cdot (1+i)^{-x-m/12} \cdot e^{-\int_0^{x+m/12} \mu_S ds}}{l_0 \cdot (1+i)^{-x-(m+1)/12} \cdot e^{-\int_0^{x+(m+1)/12} \mu_S ds}} = (1+i)^{-\frac{1}{12}} \cdot e^{-\int_{x+m/12}^{x+(m+1)/12} \mu_S ds}$$

Thus we obtain the desired result:

$$\prod_{m=0}^{11}\frac{D_{x+m/12}}{D_{x+(m+1)/12}}=\prod_{m=0}^{11} l_0\cdot(1+i)^{-1/12}\cdot e^{-\int_{x+m/12}^{x+(m+1)/12}\mu_s ds}$$

$$=l_0\cdot(1+i)^{-12/12}\cdot e^{-\sum_{m=0}^{11}\int_{x+m/12}^{x+(m+1)/12}\mu_s ds}=l_0\cdot(1+i)^{-1}\cdot e^{-\int_{x}^{x+1}\mu_s ds}=\frac{D_x}{D_{x+1}}$$

From this we may calculate the q_x-numbers as in annual case.

$$q_{x+m}=1-\frac{l_{x+(m+1)/12}}{l_{x+m/12}}=1-e^{-\int_{x+m/12}^{x+(m+1)/12}\mu_s ds}$$

6.2.7 Linear D_x-model

Let us assume that D_x -numbers change linearly across non-integer years (see also the comment of Jones and Mereu that I mentioned in chapter 6.2.2). This means that for all m = 0,…,11

$$D_{x+(m+1)/12}=D_{x+m/12}-\frac{1}{12}\cdot(D_x-D_{x+1})=D_x-\frac{m+1}{12}\cdot(D_x-D_{x+1})=\left(1-\frac{m+1}{12}\right)\cdot D_x+\frac{m+1}{12}\cdot D_{x+1}$$

$$=\left(\frac{11-m}{12}+\frac{1-q_x}{1+i}\cdot\frac{m+1}{12}\right)\cdot D_x$$

On the one hand,

$$\frac{D_{x+m/12}}{D_{x+(m+1)/12}}=\frac{\sqrt[12]{1+i}}{1-q_{x+m/12}}$$

On the other hand,

$$\frac{D_{x+m/12}}{D_{x+(m+1)/12}}=\frac{\left(\frac{12-m}{12}+\frac{1-q_x}{1+i}\cdot\frac{m}{12}\right)\cdot D_x}{\left(\frac{11-m}{12}+\frac{1-q_x}{1+i}\cdot\frac{m+1}{12}\right)\cdot D_x}=\frac{(12-m)\cdot(1+i)+(1-q_x)\cdot m}{(11-m)\cdot(1+i)+(1-q_x)\cdot(m+1)}$$

From this we obtain

$$q_{x+m/12}=1-\frac{(12-m)\cdot(1+i)+(1-q_x)\cdot m}{(11-m)\cdot(1+i)+(1-q_x)\cdot(m+1)}\cdot\sqrt[12]{1+i}$$

6.2.8 Linear discount factor model

Let us assume that the reserve of pure endowment changes linearly across non-integer years. This means that the monthly change is equal to

$$\frac{1}{12}\cdot\left(\frac{D_x}{D_{x+1}}-1\right)=\frac{1}{12}\cdot\left(\frac{1+i}{1-q_x}-1\right)=\frac{1}{12}\cdot\frac{i+q_x}{1-q_x}$$

and

$$D_{x+m/12}=\frac{D_x}{1+\frac{m}{12}\cdot\frac{i+q_x}{1-q_x}}=12\cdot\frac{1-q_x}{12-12\cdot q_x+m\cdot(i+q_x)}\cdot D_x=12\cdot\frac{1-q_x}{12+m\cdot i-(12-m)\cdot q_x}\cdot D_x$$

Now for all m = 0,…,11

$$\frac{D_{x+m/12}}{D_{x+(m+1)/12}}=\frac{\sqrt[12]{1+i}}{1-q_{x+m/12}}=\frac{12\cdot\frac{1-q_x}{12+m\cdot i-(12-m)\cdot q_x}\cdot D_x}{12\cdot\frac{1-q_x}{12+(m+1)\cdot i-(12-(m+1))\cdot q_x}\cdot D_x}=\frac{12+(m+1)\cdot i-(11-m)\cdot q_x}{12+m\cdot i-(12-m)\cdot q_x}$$

From this equation we obtain the following result:

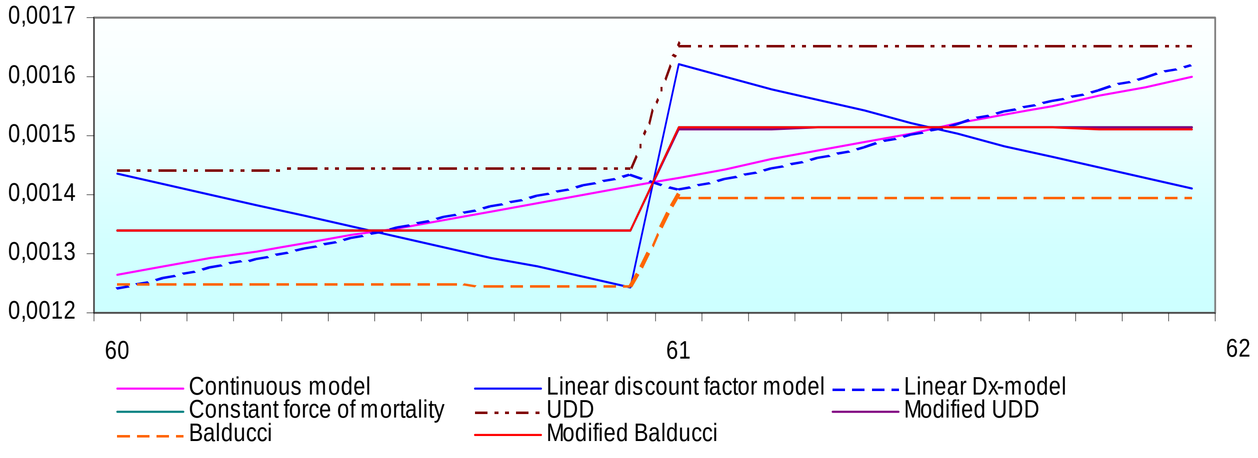
$$q_{x+m/12}=1-\frac{12+m\cdot i-(12-m)\cdot q_x}{12+(m+1)\cdot i-(11-m)\cdot q_x}\cdot\sqrt[12]{1+i}$$

6.2.9 Summary

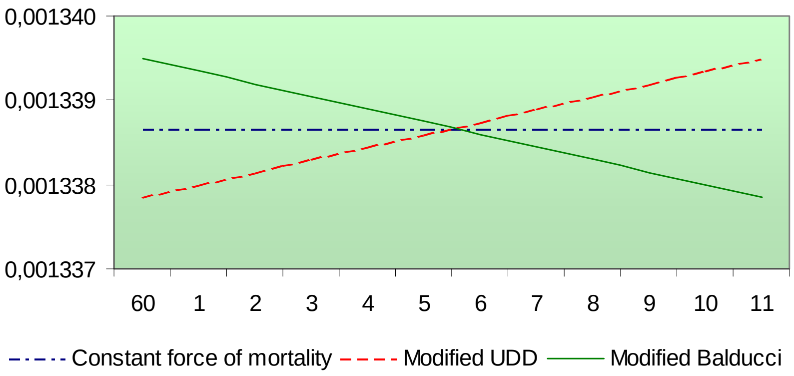
Above I have defined mortalities for several models. Traditional Balducci and UDD models are not discount factor preserving, but the others are.

In picture 6.1 the mortalities for a man between 60 and 62 years using the Finnish force of mortality $\mu_{x+h}=1{,}15\cdot(0{,}00048\cdot+10^{0{,}055\cdot(x+h-94{,}5)-0{,}02\cdot(x+h-72)^{+}})$ (see Pesonen et al. [10] p. 47) and interest rate 3,5 % are shown. The scale is such that the deviations between constant force of mortality, modified UDD and modified Balducci models are not easily seen in picture 6.1 but are clear in picture 6.2.

I have showed below the mortality curves instead of forces of mortality curves because the peak in the shift of years and scaling would have caused that the differences of the models would have not been clearly visible.



Picture 6.1. Monthly mortalities with different mortality assumptions (deviations between constant force of mortality, UDD and modified Balducci models may not be seen from this picture, but from picture 6.2)



Picture 6.2. Monthly mortalities with different mortality assumptions for constant force of mortality, modified UDD and modified Balducci models

In conversion I propose to use one of the discount preserving models. In accordance with this it can be seen from picture 6.1 that the models that give smallest and largest values, i.e. the not-modified UDD and Balducci models, should not be used.

The Balducci model has been sometimes criticized because the mortality is decreasing (see Gerber [3] p. 22 and Forfar [2] p. 1007). So also the linear discount factor model, as shown in picture 6.1. This is for the reason that the shorter interest rate accumulation period is compensated for by lower mortality.

Also Jones and Mereu criticize the models: "In specifying the FAA for each age, we wish to achieve a well-behaved force of mortality over all ages that is consistent with the life table being used." (See Jones et al. [4], p. 363 – 370.)

However, as I mentioned, my point of view is the conversion. It should be considered what is cost-effective and what the other goals are. When choosing the model we may take into account the following factors:

1) What universal life models does the undertaking support currently? – If the other products support e.g. UDD or Balducci model, then it is cost-effective to choose similar model also for the converted products.

2) How large is the portfolio that should be converted? – For small portfolios it is not cost-effective to create new customized mortality models.

3) What are the future plans related to the portfolio? – If the plan were to offer the possibility to change the policy from non-flexible to flexible policy, then a model that best suits for the flexible model would be preferred.

4) Should the universal file formulas match exactly the conventional formulas? – If e.g. the conventional formulas have linear approximation during the year, then linear discount factor preserving model should be chosen.

However, I admit that all models do not behave nicely if we look at them only from the mortality point of view. However, mortality charge is only one small element in payment and reserve structures and its importance should not be exaggerated.

6.3 Mortality charges at non-integer ages

6.3.1 Risk premium preserving method

In previous chapter I described some mortality assumptions for discount factors. In this chapter I consider what should be charged from the policyholder or, in other words, what should be charged from the reserves. By charge in this chapter I mean risk premiums, loadings and other components charged from the reserve.

If we charge reserves monthly using the monthly interest rate and mortalities defined above, this does not preserve the reserves at the end of each insurance year.

The goal in this chapter is that the charge would be the same as in the policy with conventional formulas in integer years. This goal is obvious if discount factor preserving method has been used. I call this **"risk premium preserving method".**

6.3.2 Different options

I consider later the following options:
1) annual charge at the end of each policy year
2) annual charge in the beginning of the insurance year
3) level premium
4) monthly charges resulting in linear reserve changes

If the charge depends on the reserve, then it should be considered what is the monthly sum insured. It is also possible to let the sum insured change due to this reason monthly, but in some cases it is reasonable not to let the sum insured change. One argument for this approach is that in the old policy the sum insured does not change monthly.

The monthly D_x - and N_x -numbers that I denote by $D_{x+t+k/12}$ and $N^{(12)}_{x+t+k/12}$ (k=0,…,12) are not the same as the annual D_x - and N_x -numbers that I denote by D_{x+t} and N_{x+t} (see also chapter 4.3). Only in case of discount factor preserving model $D_{x+t} = D_{x+t+0/12}$. Otherwise one should limit the calculations with D_x - and N_x -numbers to one insurance year and define $D_{x+t+1} = D_{x+t+12/12}$ (actually there is one D_x - and N_x -number series for each age year).

6.3.3 Annual charge at the end of an insurance year

The annual charge can always be charged at the end of each policy year as defined in the previous chapters. This does not affect the reserve compared to the conventional methods.

However, in this case there is no charge for the ongoing insurance year in case of surrender. So, I do not recommend this option.

When deriving the other charge formulas, this is a good starting point. Let us denote this charge as P^A_{x+t+1}.

6.3.4 Annual charge in the beginning of an insurance year

The charge P^{A}_{x+t+1} can be discounted to the beginning of an insurance year. This option can be chosen if the argument is that the policyholder has committed to pay at least annual charges.

Let us denote this value by P^{A+}_{x+t}. In this case the value is

$$P^{A+}_{x+t} = \frac{D_{x+t+1}}{D_{x+t+0/12}} \cdot P^{A}_{x+t+1} = \prod_{m=0}^{11} \frac{D_{x+t+(m+1)/12}}{D_{x+t+m/12}} \cdot P^{A}_{x+t+1} = \frac{\prod_{m=0}^{11}(1-q_{x+t+m/12})}{1+i} \cdot P^{A}_{x+t+1}$$

If the discount factor preserving model has been used, then we may use the annual mortalities:

$$P^{A+}_{x+t} = \frac{D_{x+t+1}}{D_{x+t+0/12}} \cdot P^{A}_{x+t+1} = \frac{D_{x+t+1}}{D_{x+t}} \cdot P^{A}_{x+t+1} = \frac{1-q_{x+t}}{1+i} \cdot P^{A}_{x+t+1}$$

If the calculation period before the insurance year is not 12 but k (k=1,2,….,11) months, then use the following formula:

$$P^{A+}_{x+t} = \frac{D_{x+t+1}}{D_{x+t+k/12}} \cdot P^{A}_{x+1} = \prod_{m=12-k}^{11} \frac{D_{x+t+(m+1)/12}}{D_{x+t+m/12}} \cdot P^{A}_{x+1} = \frac{\prod_{m=12-k}^{11}(1-q_{x+t+m/12})}{1+i} \cdot P^{A}_{x+t+1}$$

At the end of the policy period the formula is

$$P^{A+}_{x+t} = \frac{D_{x+t+k/12}}{D_{x+t+0/12}} \cdot P^{A}_{x+t+1} = \prod_{m=0}^{k-1} \frac{D_{x+t+(m+1)/12}}{D_{x+t+0/12}} \cdot P^{A}_{x+1} = \frac{\prod_{m=0}^{k-1}(1-q_{x+t+m/12})}{1+i} \cdot P^{A}_{x+t+1}$$

6.3.5 Level premium

In this case the premium is charged as a level premium during the year. The monthly level premium $P_{x+m/12}$ for any m = 0,…,11 is found by dividing the annual charge P^{A}_{x+1} by the annuity.

$$P_{x+t+m/12} = \frac{D_{x+t+0/12}}{N^{(12)}_{x+t+0/12} - N^{(12)}_{x+t+12/12}} \cdot P^{A}_{x+t+1}$$

In case of discount factor preserving model this is equal to the following:

$$P_{x+t+m/12} = \frac{D_{x+t}}{N^{(12)}_{x+t+0/12} - N^{(12)}_{x+t+12/12}} \cdot P^{A}_{x+t+1}$$

Change of sum insured during the year changes also the monthly charge. The new payment is equal to

$$P_{x+t+(m+1)/12} = P_{x+t+m/12} + \frac{D_{x+t}}{N^{(12)}_{x+t+(m+1)/12} - N^{(12)}_{x+t+12/12}} \cdot (S_{x+t+1} - S_{x+t})$$

Of course, in the similar way as in annual calculations, instead of commutation numbers, summation can be used as follows:

$$\frac{N^{(12)}_{x+t+m/12} - N^{(12)}_{x+t+12/12}}{D_{x+t}} = \sum_{j=m}^{11} \left[\left(\sqrt[12]{1+i} - 1 \right)^{m+1} \cdot \prod_{n=1}^{m} \left(\frac{1}{1-q_{x+t+n/12}} \right) \right]$$

The same formula can be applied for any k month period (k=1,2,…,11) before the end of an insurance year. For a k month period (k=1,2,…,11 and m<k) after the end of an insurance year use the following formula:

$$\frac{N^{(12)}_{x+t+m/12} - N^{(12)}_{x+t+k/12}}{D_{x+t}} = \sum_{j=m}^{k} \left[\left(\sqrt[12]{1+i} - 1 \right)^{m+1} \cdot \prod_{n=1}^{m} \left(\frac{1}{1-q_{x+t+n/12}} \right) \right]$$

6.3.6 Monthly charge resulting in linear reserve changes

If the linear discount factor preserving model has been used, then it is natural to require also that the reserves change linearly during the insurance year.

Let us consider only the charge part of the reserve. The goal is to find for m = 0,…,11 a charge $P_{x+t+m/12}$ such that the monthly change of reserve is equal to ΔV .

So, the charge at the end of the first month is $P_{x+t+1/12} = \Delta V = \frac{1}{12} \cdot P^{A}_{x+t+1}$.

Each month the reserve of the previous month equal to $(m-1)\cdot\Delta V$ is corrected by interest rate and compensation. Thus, each month the following equation is valid:

$$P_{x+t+m/12} = \Delta V - \left(\frac{D_{x+t+(m-1)/12}}{D_{x+t+m/12}} - 1 \right) \cdot (m-1) \cdot \Delta V = \left[1 - \left(\frac{D_{x+t+(m-1)/12}}{D_{x+t+m/12}} - 1 \right) \cdot (m-1) \right] \cdot \Delta V$$

$$= \left(m - \frac{D_{x+t+(m-1)/12}}{D_{x+t+m/12}} \cdot (m-1) \right) \cdot \Delta V = \left(m - \frac{\sqrt[12]{1+i}}{1-q_{x+t+m/12}} \cdot (m-1) \right) \cdot \frac{1}{12} \cdot P^{A}_{x+t+1}$$

Especially for mortality the following is valid:

$$P_{x+t+m/12} = \frac{q_{x+t+m/12}}{1-q_{x+t+m/12}} \cdot \frac{1-q_{x+t+m/12}}{q_{x+t+m/12}} \cdot \left(m - \frac{\sqrt[12]{1+i}}{1-q_{x+t+m/12}} \cdot (m-1) \right) \cdot \frac{1}{12} \cdot P^{A}_{x+t+1}$$

$$= \frac{q_{x+t+m/12}}{1-q_{x+t+m/12}} \cdot \frac{1}{12 \cdot q_{x+t+m/12}} \cdot \left[\left(1 - q_{x+t+m/12} \right) \cdot m - \sqrt[12]{1+i} \cdot (m-1) \right] \cdot P^{A}_{x+t+1}$$

This means that we may use the same monthly mortality functions if we multiply the sum insured $\frac{1}{12} \cdot P^{A}_{x+t+1}$ by:

$$\frac{1}{q_{x+t+m/12}} \cdot \left[\left(1-q_{x+t+m/12}\right) \cdot m - \sqrt[12]{1+i} \cdot (m-1)\right]$$

In table 6.1 there is an example based on the linear discount factor preserving model example presented in chapter 6.2.8. Premium (without loading) is 9000, risk premium related loading is premium multiplied by $(\sqrt{1+i}-1)$ and sum insured 50000. As a result the reserve decreases by 29,86 each month. The mortality premium decrease is between 0,27 and 0,30. Total risk charge for the year is 824,29 and the premium of the first month 824,29/12 = 68,69. During the first year the risk premium decreases month by month, but later as the initial reserve is greater also the monthly premium increases.

month	premium	risk premium	compensation	interest	reserve
1	9000,00	-68,69	12,95	25,88	8970,14
2	0,00	-68,40	12,74	25,79	8940,28
3	0,00	-68,10	12,54	25,70	8910,41
4	0,00	-67,81	12,33	25,62	8880,55
5	0,00	-67,53	12,13	25,53	8850,69
6	0,00	-67,24	11,93	25,44	8820,83
7	0,00	-66,96	11,74	25,36	8790,97
8	0,00	-66,68	11,54	25,27	8761,10
9	0,00	-66,40	11,35	25,18	8731,24
10	0,00	-66,12	11,16	25,10	8701,38
11	0,00	-65,85	10,98	25,01	8671,52
12	0,00	-65,58	10,79	24,93	8641,65

Table 6.1. Monthly reserve change components in case the reserve changes linearly during the year

For k month period (k=1,…,11) the annual P_{x+1}^{A} is calibrated to $\frac{12}{k} \cdot P_{x+1}^{A}$.

7 SUMMARY

In this paper I have shown several methods for converting traditional life insurance policies into universal life policies.

This paper concentrated on the problem of finding precisely-fitting conversion formulas.

In some cases it is wise to consider the possibility of changing the technical bases of the product in order to get policies that can be managed in an easier manner. When designing these simpler approximation models understanding these exact models is vital.

In this paper I have not covered loadings. Conversion of loadings charged from premiums paid several times a year is rather complex. However, mostly the conversion is very easy.

I have sometimes expressed my belief that a good actuary understands how the products behave but takes reasonable steps to simplify the model in order to get cost-effective systems. I encourage such simplifications.

Hans U. Gerber writes about commutation numbers: "It may be … taken for granted that the days of the glory for the commutation numbers now belong to the past". His argument for this is the "advent of powerful computers" and "growing acceptance of models based on probability theory, which allows a more complete understanding of the essentials of the insurance". (See Gerber [3], p. 119.)

This is for the most part true. I still might see where in some cases use of conventional tools may be reasonable. For example, during the pension period the flexibility given by the universal life methods is not always needed. This is especially the case in statutory pension schemes. However, it is also the case that nowadays, during the pension period the investment risk is more and more often transferred to the policyholder by allowing unit-linked pensions.

This paper has provided tools for converting existing conventional products into universal life products. If general actuarial principles are not followed, then the solution may be found.

In this paper I have also derived new tools to manage conversions. I have derived new concepts such as "discount factor preserving method" and "risk premium preserving method".

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