ACTUARIAL ANALYSIS OF THE MULTIPLE LIFE ENDOWMENT INSURANCE CONTRACT

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Abstract

We extend the classical analysis of the endowment contract on a single life to multiples lives. The two lives case covering the joint-life and the last-survivorship status is discussed thoroughly. In practice actuarial values of the tariff book are calculated under the simplifying assumption of independent future lifetimes. It is therefore important to measure the impact of this assumption under the observation that independence is not fulfilled in real life. In the two lives case the maximal impact can be measured using the well-known Höffding-Fréchet lower and upper bounds. The independence assumption overestimates the joint-life net single and level premiums and underestimates the last-survivor net single and level premiums. The maximal deviations are obtained by perfect positive dependence. Some formulas illustrate the application to multiple life insurance contracts for more than two lives, which point out to further possible developments.

Key words

endowment life insurance, multiple lives, joint-life, last-survivorship, lifetime dependence

1. Introduction

The purpose of this study is to extend the classical analysis of the endowment contract on a single life to multiples lives. The two lives case covering the joint-life and the last-survivorship status is discussed thoroughly. Some formulas illustrate the application to multiple life insurance contracts for more than two lives, which point out to further possible developments.

2. The Notion of a General Life Status

The extension of life insurance for a single life to multiple lives is based on the notion of *general life status*, for which there are definitions of survival and failure. Consider a group of g lives aged $x_1, x_2, ..., x_g$ and let $T_k = T(x_k)$ denote the random future lifetime of the single life aged $x_k, k = 1,...,g$. Based on these elements a status (u) with random future lifetime $T = T(u)$ will be defined such that ${}_t p_u = P(T(u) > t)$ is the probability that the status will survive to time $t > 0$ and ${}_t q_u = P(T(u) \le t)$ is the probability that the status will fail to time $t > 0$. We will develop models for *life insurances* payable upon the failure of the status and *life annuities* payable as long as the status survives. Denote by $D^{(m)}(u)$ the *net single premium* (NSP) of a life insurance with one unit of benefit payment payable at the end of the m-thly period of a year following failure of the status, where $m \in [0,1]$. For a yearly period $m = 1$ one sets by convention $D^{(1)}(u) = D(u)$ and for a continuous payment mode $m = 0$ one writes $D^{(0)}(u) = \overline{D}(u)$. Similarly, denote by $a^{(c)}(u)$ the NSP of a life annuity of one unit per year payable in instalments of c fractional units at the beginning of each payment cycle of length $c \in [0,1]$ of a year as long as the status survives. For a yearly period $c = 1$ one sets by convention $a^{(1)}(u) = a(u)$ and for the limiting case of continuous payments $c = 0$ one defines $a^{(0)}(u) = \overline{a}(u)$.

The most important instances of a general life status, which will suffice to specify completely the endowment insurance on two lives, are the following four ones:

Single life status

A single life aged x defines a status $u = x$ that survives while (x) lives.

Joint-life status

A status that exists as long as all members of the group are alive and fails upon the first death is called a *joint-life status* and is denoted by $u = x_1 : x_2 : ... : x_g$. Its future lifetime is described by the random variable $T(u) = \min(T_1,...,T_g)$.

Last-survivorship status

A status that exists as long as at least one member of the group is alive and fails upon the last death is called a *last-survivorship status* and is denoted by $u = \overline{x_1 : x_2 : ... : x_g}$. It has the future lifetime $T(u) = \max(T_1,...,T_g)$.

Term certain status

The term certain status, which is denoted by $u = n$, defines a life status surviving for exactly n years and then failing. It has the deterministic future lifetime $T(n) = n$. This particular status is useful when describing temporary life insurances and life annuities. For example, the general life status $u = x : n$ defines the NSP of single life endowment insurances of the type $A^{(m)}(x:n), m \in [0,1]$. It defines also the NSP of temporary life annuities of the type $a^{(c)}(x:n), c \in [0,1]$. In this framework, the *net level premium* (NLP) of single life endowment insurances with benefit payment cycle $m \in [0,1]$ and premium payment cycle $c \in [0,1]$ and one unit of sum insured is determined by the quotient

$$NLP^{(m,c)}(x:n) = \frac{A^{(m)}(x:n)}{a^{(c)}(x:n)}, \quad m \in [0,1], c \in [0,1]. \tag{2.1}$$

3. Probabilities of Survival and Failure

To describe the endowment insurance on two lives one requires probabilities for the joint-life status $u = x : y$ and the last-survivorship status $u = \overline{x:y}$. Let X, Y be the age-at-deaths of the lives $(x), (y)$ and $T(x) = X - x, T(y) = Y - y$ the corresponding future lifetimes. Given the joint survival function $S(x, y) = P(X > x, Y > y)$ of the couple (X, Y), the survival probabilities of the future lifetimes $T(x:y) = \min(T(x), T(y))$ and $T(\overline{x:y}) = \max(T(x), T(y))$ are obtained as follows:

$$\begin{aligned} {}_t p_{x:y} &= P(T(x:y) > t) = P(X > x+t \wedge Y > y+t \mid X > x, Y > y) \\ &= \frac{S(x+t, y+t)}{S(x,y)} \end{aligned} \tag{3.1}$$

$$\begin{aligned} {}_t p_{\overline{x:y}} &= P(T(\overline{x:y}) > t) = P(X > x+t \vee Y > y+t \mid X > x, Y > y) \\ &= \frac{S(x+t, y) + S(x, y+t) - S(x+t, y+t)}{S(x,y)} \end{aligned} \tag{3.2}$$

Though in general the random variables show a non-trivial dependence structure, it is common practice to assume for pricing purposes that the lives (x) and (y) are independent. In this simplified situation, the probabilities of survival depend on the life table of the single lives only and are given by

$$ {}_t p_{x:y} = {}_t p_x \cdot {}_t p_y, \quad {}_t p_{\overline{x:y}} = {}_t p_x + {}_t p_y - {}_t p_{x:y} \tag{3.3}$$

Remark 3.1. In a recent paper, Youn et al.(2002) have made a thorough analysis of the more general assumption of "partial independence" ${}_t p_{x:y} + {}_t p_{\overline{x:y}} = {}_t p_x + {}_t p_y$ (suggested by Bowers et al.(1986)), which simplifies much multiple life calculations. In particular, in case of a married couple, they show that this identity holds under the assumption that the mortality rate of the wife or the husband should not depend on whether they have a surviving spouse or not, nor on the surviving spouse's age. This is generally assumed in practice. Insurance companies do not classify according to whether one has a surviving spouse or not, nor to spouse's age. It can be shown that the partial independence assumption also holds for certain special survival functions:

Common shock survival model: $S(x,y)=S_1(x)\cdot S_2(x)\cdot R(\max(x,y))$, where $S_1(x), S_2(x), R(x)$ are survival distributions (see Denuit et al.(2006) for an application)

Fréchet copula model: $C(u,v)=(1-\theta)\cdot uv+\theta\cdot\min(u,v),\quad \theta\in[0,1]$

4. **Premiums**

Let $m\in[0,1], c\in[0,1]$ be the benefit and premium payment cycles of the endowment insurance on two lives.

Net single premiums of life insurances

$A^{(m)}(x:n)$	: NSP of a n-year endowment insurance for a single life
$D^{(m)}(x:n)$	: NSP of a n-year term insurance payable at failure of single life
$E(x:n)$	: NSP of a n-year pure endowment payable at survival of single life
$A^{(m)}(x:y:n)$	: NSP of a n-year endowment insurance for a joint-life
$D^{(m)}(x:y:n)$	: NSP of a n-year term insurance payable at failure of joint-life
$E(x:y:n)$	: NSP of a n-year pure endowment payable at survival of joint-life
$A^{(m)}(\overline{x:y}:n)$	: NSP of a n-year endowment insurance for a last-survivor
$D^{(m)}(\overline{x:y}:n)$	: NSP of a n-year term insurance payable at failure of last-survivor
$E(\overline{x:y}:n)$	: NSP of a n-year pure endowment payable at survival of last-survivor

The endowment NSP is the sum of the term insurance and pure endowment NSP's:

$$A^{(m)}(x:n)=D^{(m)}(x:n)+E(x:n) \qquad (4.1)$$

$$A^{(m)}(x:y:n)=D^{(m)}(x:y:n)+E(x:y:n) \qquad (4.2)$$

$$A^{(m)}(\overline{x:y}:n)=D^{(m)}(\overline{x:y}:n)+E(\overline{x:y}:n) \qquad (4.3)$$

Denote by $v=1/(1+i)$ the discount factor to the *technical interest rate* i. The NSP's are determined by the formulas:

$$D^{(m)}(x:n)=\begin{cases}\sum_{j=1}^{\frac{n}{m}} v^{j\cdot m}\cdot{}_{(j-1)\cdot m}\,p_x\cdot\left(1-{}_m\,p_{x+(j-1)\cdot m}\right), & m>0\\ \int_0^n v^s\cdot{}_s\,p_x\cdot\mu_{x+s}ds, & m=0\end{cases} \qquad (4.4)$$

$$E(x:n)=v^n\cdot{}_n\,p_x \qquad (4.5)$$

$$D^{(m)}(x:y:n)=\begin{cases}\sum_{j=1}^{\frac{n}{m}} v^{j\cdot m}\cdot{}_{(j-1)\cdot m}\,p_{x:y}\cdot\left(1-{}_m\,p_{x+(j-1)\cdot m:y+(j-1)\cdot m}\right), & m>0\\ \int_0^n v^s\cdot{}_s\,p_{x:y}\cdot\left(\mu_{x+s}+\mu_{y+s}\right)ds, & m=0\end{cases} \tag{4.6}$$

$$E(x:y:n)=v^n\cdot{}_n\,p_{x:y} \tag{4.7}$$

$$D^{(m)}\left(\overline{x:y}:n\right)=D^{(m)}(x:n)+D^{(m)}(y:n)-D^{(m)}(x:y:n) \tag{4.8}$$

$$E\left(\overline{x:y}:n\right)=v^n\cdot{}_n\,p_{\overline{x:y}}=E(x:n)+E(y:n)-E(x:y:n) \tag{4.9}$$

<u>Net single premiums of life annuities</u>

$a^{(c)}(x:n)$: NSP of a n-year life annuity for a single life
$a^{(c)}(x:y:n)$: NSP of a n-year life annuity for a joint-life
$a^{(c)}\left(\overline{x:y}:n\right)$: NSP of a n-year life annuity for a last-survivor

The NSP's are determined by the following formulas:

$$a^{(c)}(x:n)=\begin{cases}\sum_{j=0}^{\frac{n}{c}-1} v^{j\cdot c}\cdot{}_{j\cdot c}p_x, & c>0\\ \int_0^n v^s\cdot{}_s\,p_x ds, & c=0\end{cases} \tag{4.10}$$

$$a^{(c)}(x:y:n)=\begin{cases}\sum_{j=0}^{\frac{n}{c}-1} v^{j\cdot c}\cdot{}_{j\cdot c}p_{x:y}, & c>0\\ \int_0^n v^s\cdot{}_s\,p_{x:y} ds, & c=0\end{cases} \tag{4.11}$$

$$a^{(c)}\left(\overline{x:y}:n\right)=a^{(c)}(x:n)+a^{(c)}(y:n)-a^{(c)}(x:y:n) \tag{4.12}$$

<u>Net level premiums of endowment insurances</u>

$NLP^{(m,c)}(x:n)$: yearly NLP of a n-year endowment insurance for a single life
$NLP^{(m,c)}(x:y:n)$: yearly NLP of a n-year endowment insurance for a joint-life
$NLP^{(m,c)}\left(\overline{x:y}:n\right)$: yearly NLP of a n-year endowment insurance for a last-survivor

The net level premium rates (NLPR) for a unit of sum insured and the NLP's for a sum insured SI and a general life status u are determined by the following formulas:

$$NLPR^{(m,c)}(u:n) = \frac{A^{(m)}(u:n)}{a^{(c)}(u:n)} \tag{4.13}$$

$$NLP^{(m,c)}(u:n) = NLPR^{(m,c)}(u:n) \cdot SI \tag{4.14}$$

Level premiums of endowment insurances

Consider the following level premiums (LP) of endowment insurances with a sum insured SI for a two-life status:

$LP^{(m,c)}(x:n)$: LP of a n-year endowment insurance for a single life
$LP^{(m,c)}(x:y:n)$: LP of a n-year endowment insurance for a joint-life
$LP^{(m,c)}(\overline{x:y}:n)$: LP of a n-year endowment insurance for a last-survivor

Level premiums include the cover for all kind of expenses an insurance company may have. Similarly to Gerber(1986), Chap. 10, four types of expenses are considered:

Acquisition costs

Expenses related to the sale of a new life insurance contract are single costs, which are paid at the beginning of the contract at the rate of α per unit of sum insured.

Premium proportional operating costs

These are variable recurring expenses at the rate of β_v per unit of level premium.

Constant operating costs

These are recurring operating expenses of fixed constant value β_f .

Operating costs proportional to the insurance benefit

These are variable recurring operating expenses at the rate of γ per unit of sum insured.

The level premium suffices to finance in expected value the insurance benefits and costs. The level premium rate (LPR) per unit of sum insured of a n-year endowment insurance with a general life status u satisfies the equation:

$$LPR^{(m,c)}(u:n) \cdot a^{(c)}(u:n) = A^{(m)}(u:n) + \alpha + \left(\beta_v \cdot LPR^{(m,c)}(u:n) + \gamma + \frac{\beta_f}{SI} \right) \cdot a^{(c)}(u:n) \tag{4.15}$$

It follows that the level premium for a general life status u is given by

$$LP^{(m,c)}(u:n) = \frac{NLP^{(m,c)}(u:n) + \left(\frac{\alpha}{a^{(c)}(u:n)} + \gamma \right) \cdot SI + \beta_f}{1 - \beta_v} \tag{4.16}$$

5. Mathematical Reserves, Actuarial Reserves and Premium Reserves

For a life status (u) with random future lifetime $T = T(u)$ one considers the following random variables associated to it:

$K(u) = [T(u)]$: the number of completed future years lived by the status (u), also called curtate-future-lifetime

$S(u) = T(u) - K(u)$: the fractional portion of a year the status lives in the year of failure

$S^{(m)}(u) = m \cdot \left[\frac{1}{m} \cdot S(u) + 1\right]$: the fractional portion $S(u)$ rounded up to the next m-th of a year, $m \in (0,1]$

$T^{(m)}(u) = K(u) + S^{(m)}(u)$: the moment of benefit payment in case the status fails

For a group of g lives aged $x_1, x_2, ..., x_g$ with status (u) and a time $t > 0$, let $(u+t)$ denote the status obtained from (u) with each life aged $x_k + t, k = 1,...,g$. For example, if $(u) = (x : y)$ is the joint-life status on two lives, then $(u+t) = (x+t : y+t)$. The random *prospective loss* at contract time $t > 0$ of a n-year endowment insurance with life status (u) and sum insured SI is the random variable defined by

$$L_t^{(m,c)}(u:n) = v^{\min\left(T^{(m)}(u+t), n-t\right)} \cdot SI - NLP^{(m,c)}(u:n) \cdot \ddot{a}^{(c)}_{\overline{\min\left(T^{(m)}(u+t), n-t\right)}}, \tag{5.1}$$

$$\text{where } \ddot{a}^{(c)}_{\overline{n}} = \frac{1-v^n}{d^{(c)}}, \quad d^{(c)} = \frac{i^{(c)}}{1 + c \cdot i^{(c)}}, \quad i^{(c)} = \frac{(1+i)^c - 1}{c}, \tag{5.2}$$

denotes a n-year *annuity certain* of one unit per year payable in instalments of c fractional units at the beginning of each payment cycle of length $c \in (0,1]$. In the limiting case of continuous payments $c = 0$ one defines and sets $\ddot{a}^{(0)}_{\overline{n}} = \frac{1-v^n}{\delta} = \overline{a}_{\overline{n}}$, where $\delta = \ln(1+i)$ is the force of interest. Obviously one has $\lim_{c \to 0} \ddot{a}^{(c)}_{\overline{n}} = \overline{a}_{\overline{n}}$. To define actuarial reserves properly, it is necessary to consider the possible states a status can take over future time. In the case of a single life aged x at contract time $t = 0$, one observes that with respect to the mortality risk the life can be in two different states $X_t \in \{1,2\}$ at time $t > 0$, which are defined by

$X_t = 1 \Leftrightarrow \left(T^{(m)}(x) > t\right)$ ((x) is alive at time $t > 0$)
$X_t = 2 \Leftrightarrow \left(T^{(m)}(x) \le t\right)$ ((x) is dead at time $t > 0$)

Generalizing to a couple of lives aged x and y at $t = 0$, one observes that with respect to the mortality risk the couple can be in four different states $X_t \in \{1,2,3,4\}$ at time $t > 0$:

$X_t = 1 \Leftrightarrow \left(T^{(m)}(x) > t, T^{(m)}(y) > t\right)$ (both (x) and (y) are alive at time $t > 0$)
$X_t = 2 \Leftrightarrow \left(T^{(m)}(x) > t, T^{(m)}(y) \le t\right)$ ((x) is alive and (y) is dead at time $t > 0$)
$X_t = 3 \Leftrightarrow \left(T^{(m)}(x) \le t, T^{(m)}(y) > t\right)$ ((x) is dead and (y) is alive at time $t > 0$)
$X_t = 4 \Leftrightarrow \left(T^{(m)}(x) \le t, T^{(m)}(y) \le t\right)$ (both (x) and (y) are dead at time $t > 0$)

Obviously, a group of g lives can be in 2^g different states $X_t = i, i \in \{1,2,...,2^g\}$ at a future time $t > 0$. In general, the *mathematical reserve* at time $t > 0$ of a n-year endowment insurance with a life status (u) in state $X_t = i$ at time $t > 0$ is defined to be the conditional expectation of the prospective loss given (u) is in state $X_t = i$ at time $t > 0$, which is denoted and calculated as follows:

$$ {}_tV_i^{(m,c)}(u:n) = E\left[L_t^{(m,c)}(u:n) \middle| X_t = i\right], \quad i \in \{1,2,...,2^g\}. \tag{5.3}$$

For a single life (x) the mathematical reserve in state $X_t = 2$ vanishes and the mathematical reserve in state $X_t = 1$ is given by (dropping as usual the index $i = 1$)

$$ {}_tV^{(m,c)}(x:n) = A^{(m)}(x+t:n-t) \cdot SI - NLP^{(m,c)}(x:n) \cdot a^{(c)}(x+t:n-t) \tag{5.4}$$

Similarly, for a joint-life $(x:y)$ the mathematical reserves in the states $X_t = 2,3,4$ vanish and the mathematical reserve in state $X_t = 1$ is given by

$$ \begin{aligned} &{}_tV^{(m,c)}(x:y:n) \\ &= A^{(m)}(x+t:y+t:n-t) \cdot SI - NLP^{(m,c)}(x:y:n) \cdot a^{(c)}(x+t:y+t:n-t) \end{aligned} \tag{5.5}$$

In contrast to this, for a last-survivor status $(\overline{x:y})$, only the mathematical reserve in the state $X_t = 4$ vanishes and the mathematical reserves in the states $X_t = 1,2,3$ are given by (see e.g. Bowers et al.(1986), p. 501, for the special case $(m,c) = (0,0)$)

$$ \begin{aligned} &{}_tV_1^{(m,c)}(\overline{x:y}:n) = E\left[L_t^{(m,c)}(\overline{x:y}:n) \middle| (T^{(m)}(x) > t, T^{(m)}(y) > t)\right] \\ &= A^{(m)}(\overline{x+t:y+t}:n-t) \cdot SI - NLP^{(m,c)}(\overline{x:y}:n) \cdot a^{(c)}(\overline{x+t:y+t}:n-t) \end{aligned} \tag{5.6}$$

$$ \begin{aligned} &{}_tV_2^{(m,c)}(\overline{x:y}:n) = E\left[L_t^{(m,c)}(\overline{x:y}:n) \middle| (T^{(m)}(x) > t, T^{(m)}(y) \le t)\right] \\ &= A^{(m)}(x+t:n-t) \cdot SI - NLP^{(m,c)}(\overline{x:y}:n) \cdot a^{(c)}(x+t:n-t) \end{aligned} \tag{5.7}$$

$$ \begin{aligned} &{}_tV_3^{(m,c)}(\overline{x:y}:n) = E\left[L_t^{(m,c)}(\overline{x:y}:n) \middle| (T^{(m)}(x) \le t, T^{(m)}(y) > t)\right] \\ &= A^{(m)}(y+t:n-t) \cdot SI - NLP^{(m,c)}(\overline{x:y}:n) \cdot a^{(c)}(y+t:n-t) \end{aligned} \tag{5.8}$$

Besides the mathematical reserves, which depend on the states of a status, one considers the *net premium reserve* at time $t > 0$, which is defined to be the conditional expectation of the prospective loss given survival to time $t > 0$ (e.g. Bowers et al.(1986), Chap.17.7, p. 500):

$$ {}_tV^{(m,c)}(u:n) = E\left[L_t^{(m,c)}(u:n) \middle| T^{(m)}(u) > t\right] = \sum_{i=1}^{2^g} {}_tV_i^{(m,c)}(u:n) \cdot P\left(X_t = i \middle| T^{(m)}(u) > t\right) \tag{5.9}$$

For a single life (x), respectively a joint-life $(x:y)$, the net premium reserve (5.9) coincides with the mathematical reserves (5.4) respectively (5.5). For a last-survivor status $(u) = (\overline{x:y})$, the net premium reserve is a probability weighted sum of the mathematical reserves (5.6) to (5.8) determined as follows (e.g. Bowers et al.(1986), Chap.17.7, p. 502):

$$ {}_tV^{(m,c)}\left(\overline{x:y}:n\right) = \frac{{}_tp_{x:y}\cdot {}_tV_1^{(m,c)}\left(\overline{x:y}:n\right)+{}_tp_x\left(1-{}_tp_y\right)\cdot {}_tV_2^{(m,c)}\left(\overline{x:y}:n\right)+{}_tp_y\left(1-{}_tp_x\right)\cdot {}_tV_3^{(m,c)}\left(\overline{x:y}:n\right)}{{}_tp_{x:y}+{}_tp_x\left(1-{}_tp_y\right)+{}_tp_y\left(1-{}_tp_x\right)} \quad (5.10) $$

Corresponding to the state dependent mathematical reserves, we consider state dependent *deferred acquisition costs*, which for a life status (u) in state $X_t = i$ at time $t > 0$ are denoted and defined by

$$ {}_tVE_i^{(m,c)}(u:n) = -\alpha\cdot\left(SI - {}_tV_i^{(m,c)}(u:n)\right), \quad i\in\left\{1,2,...,2^g\right\}. \quad (5.11) $$

Besides the state independent net premium reserve we consider the state independent *expense reserve*, which for a life status (u) at time $t > 0$ is denoted and defined by

$$ {}_tVE^{(m,c)}(u:n) = -\alpha\cdot\left(SI - {}_tV^{(m,c)}(u:n)\right). \quad (5.12) $$

The state dependent *actuarial reserves* are defined to be the sum of the mathematical reserves and the deferred acquisition costs and are denoted by

$$ {}_tVA_i^{(m,c)}(u:n) = {}_tV_i^{(m,c)}(u:n) + {}_tVE_i^{(m,c)}(u:n), \quad i\in\left\{1,2,...,2^g\right\}. \quad (5.13) $$

The state independent *premium reserve* is defined to be the sum of the net premium reserve and the expense reserve and is denoted by

$$ {}_tVA^{(m,c)}(u:n) = {}_tV^{(m,c)}(u:n) + {}_tVE^{(m,c)}(u:n). \quad (5.14) $$

Remark 5.2. The concept of state independent reserves for the last-survivor status has been introduced by Frasier(1978) (see also “The Actuary(1978)” and Margus(2002)). The choice between state independent and state dependent reserves depends upon loss recognition in the balance sheet (recognition or not of a status change). With state independent reserves, the insurance company administers the contract as if it had no knowledge of any deaths, as long as at least one insured survives.

6. Premium Components

In the following only the most realistic case $m = c > 0$ is discussed. Corresponding to the dual situation of state dependent and state independent reserves, two types of premium decompositions are considered.

6.1. State dependent Premium Components

Quantities are expressed as actuarial functions of the status (u) and the state $X_t = i$ at the discrete times $t = k\cdot c,\ k = 0,1,...,\frac{n}{c}-1$ whenever relevant.

Risk Premium

$$RP_{t,i}^{(m,c)}(u:n) = v^c \cdot {}_c q_{(u+t)} \cdot \left(SI - {}_{t+c}V_i^{(m,c)}(u:n)\right), \quad i \in \{1,2,\ldots,2^g\} \tag{6.1}$$

Saving Premium

$$SP_{t,i}^{(m,c)}(u:n) = v^c \cdot {}_{t+c}V_i^{(m,c)}(u:n) - {}_tV_i^{(m,c)}(u:n), \quad i \in \{1,2,\ldots,2^g\} \tag{6.2}$$

The sum of the risk premium and saving premium is *consistent* with the net level premium at contract issue:

$$RP_{t,i}^{(m,c)}(u:n) + SP_{t,i}^{(m,c)}(u:n) = NLP^{(m,c)}(u:n), \quad i \in \{1,2,\ldots,2^g\} \tag{6.3}$$

Expense Premium

$$EP^{(m,c)}(u:n) = LP^{(m,c)}(u:n) - NLP^{(m,c)}(u:n) \tag{6.4}$$

Risk Component Expense Premium

$$REP_{t,i}^{(m,c)}(u:n) = \alpha \cdot RP_{t,i}^{(m,c)}(u:n), \quad i \in \{1,2,\ldots,2^g\} \tag{6.5}$$

Saving Component Expense Premium

$$SEP_{t,i}^{(m,c)}(u:n) = EP^{(m,c)}(u:n) - REP_{t,i}^{(m,c)}(u:n), \quad i \in \{1,2,\ldots,2^g\} \tag{6.6}$$

Operating Cost Charge

$$OC_{t+c,i}^{(m,c)}(u:n) = v^{-c} \cdot \left({}_tVE_i^{(m,c)}(u:n) + SEP_{t,i}^{(m,c)}(u:n)\right) - {}_{t+c}VE_i^{(m,c)}(u:n), \quad i \in \{1,2,\ldots,2^g\} \tag{6.7}$$

Acquisition Cost Charge

$$AC_{t+c,i}^{(m,c)}(u:n) = v^{-c} \cdot EP^{(m,c)}(u:n) - OC_{t+c,i}^{(m,c)}(u:n), \quad i \in \{1,2,\ldots,2^g\} \tag{6.8}$$

6.2. State independent Premium Components

Quantities are expressed as actuarial functions of the status (u) at the discrete times $t = k \cdot c,\ k = 0,1,\ldots,\frac{n}{c} - 1$ whenever relevant.

Risk Premium

$$RP_t^{(m,c)}(u:n) = v^c \cdot {}_c q_{(u+t)} \cdot \left(SI - {}_{t+c}V^{(m,c)}(u:n)\right) \tag{6.9}$$

Saving Premium

$$SP_t^{(m,c)}(u:n) = v^c \cdot {}_{t+c}V^{(m,c)}(u:n) - {}_tV^{(m,c)}(u:n) \tag{6.10}$$

The sum of the risk premium and saving premium is *consistent* with the net level premium:

$$RP_t^{(m,c)}(u:n) + SP_t^{(m,c)}(u:n) = NLP^{(m,c)}(u:n) \tag{6.11}$$

<u>Expense Premium</u>

$$EP^{(m,c)}(u:n) = LP^{(m,c)}(u:n) - NLP^{(m,c)}(u:n) \tag{6.12}$$

<u>Risk Component Expense Premium</u>

$$REP_t^{(m,c)}(u:n) = \alpha \cdot RP_t^{(m,c)}(u:n) \tag{6.13}$$

<u>Saving Component Expense Premium</u>

$$SEP_t^{(m,c)}(u:n) = EP^{(m,c)}(u:n) - RP_t^{(m,c)}(u:n) \tag{6.14}$$

<u>Operating Cost Charge</u>

$$OC_{t+c}^{(m,c)}(u:n) = v^{-c} \cdot \left({}_tVE^{(m,c)}(u:n) + SEP_t^{(m,c)}(u:n)\right) + {}_{t+c}VE^{(m,c)}(u:n) \tag{6.15}$$

<u>Acquisition Cost Charge</u>

$$AC_{t+c}^{(m,c)}(u:n) = v^{-c} \cdot EP^{(m,c)}(u:n) - OC_{t+c}^{(m,c)}(u:n) \tag{6.16}$$

7. <u>Reduction of Calculation</u>

The preceding formulas show that all actuarial values related to the multiple life endowment insurance depend solely on the actuarial functions $A^{(m)}(u:n)$ and $a^{(c)}(u:n)$. In fact, under the popular uniform distribution of deaths assumption, the formulas can be further reduced to the calculation of $D(u:n)$ and $E(u:n)$ only. Following Gerber(1986), p. 27-28, one shows that

$$D^{(m)}(u:n) = \begin{cases} \frac{i}{i^{(m)}} \cdot D(u:n), & m > 0 \\ \frac{i}{\delta} \cdot D(u:n), & m = 0 \end{cases} \tag{7.1}$$

where $i^{(m)}$ is equal to the m-thly nominal technical interest rate convertible $\frac{1}{m}$ times in a year in case $m > 0$, which is defined by

$$i^{(m)} = \frac{(1+i)^m - 1}{m}, \tag{7.2}$$

and $i^{(0)}$ is the technical force of interest (standard notation δ), which is defined as the limiting value of $i^{(m)}$ as $m \to 0$ and equal to

$$\delta = i^{(0)} = \lim_{m\to 0} i^{(m)} = \lim_{m\to 0} \frac{(1+i)^m - 1}{m} = \frac{d}{dx}(1+i)^x\Big|_{x=0} = \ln(1+i). \tag{7.3}$$

Similarly, $d^{(m)}$ is equal to the m-thly nominal technical interest rate in advance defined in case $m > 0$ by

$$d^{(m)} = \frac{i^{(m)}}{1 + m \cdot i^{(m)}}. \tag{7.4}$$

Since $\frac{1}{d^{(m)}} = m + \frac{1}{i^{(m)}}$ one recovers in the limiting case as $m \to 0$ the technical force of interest

$$d^{(0)} = \lim_{m\to 0} d^{(m)} = \lim_{m\to 0} i^{(m)} = \delta. \tag{7.5}$$

From (7.1) it follows that

$$A^{(m)}(u:n) = A(u:n) + \left(\tfrac{i}{i^{(m)}} - 1\right) \cdot D(u:n). \tag{7.6}$$

Furthermore, following Gerber(1986), p.36-37, and using (7.6) one shows the relationship

$$a^{(c)}(u:n) = \frac{1 - A^{(c)}(u:n)}{d^{(c)}} = \frac{1 - A(u:n) - \left(\tfrac{i}{i^{(c)}} - 1\right) \cdot D(u:n)}{d^{(c)}}. \tag{7.7}$$

Clearly, the relations (7.6) and (7.7) complete the desired reduction of calculation.

8. Impact of Independence Assumption on Premium Calculation

Since actuarial values of the tariff book have been calculated under the simplifying assumption of independent future lifetimes $T(x)$ and $T(y)$, it is important to measure the impact of this assumption under the observation that independence is not fulfilled in real life. In the two lives case the maximal impact can be measured using the Fréchet lower and upper bounds introduced in Höffding(1940) and Fréchet(1951) and first applied to life insurance by Carrière and Chan(1986). Consider the Fréchet class of all bivariate distributions with fixed margins ${}_t q_x = P(T(x) \le t)$ and ${}_t q_y = P(T(y) \le t)$. The *Fréchet upper bound* is the distribution $F^U(s,t) = P(T(x) \le s, T(y) \le t) = \min({}_s q_x, {}_t q_y)$ and the *Fréchet lower bound* is the distribution $F^L(s,t) = P(T(x) \le s, T(y) \le t) = \max({}_s q_x + {}_t q_y - 1, 0)$. Any joint distribution $F(s,t) = P(T(x) \le s, T(y) \le t)$ with fixed margins is constrained from above and below by

$$F^L(s,t) \le F(s,t) \le F^U(s,t). \tag{8.1}$$

The Fréchet bounds generate four different future lifetimes for the joint-life and last-survivor status. Their survival distributions are denoted and determined by

$$ {}_t p^L_{x:y} = P\left(T^L(x:y) > t\right) = \max\left({}_t p_x + {}_t p_y - 1, 0\right), \tag{8.2} $$

$$ {}_t p^U_{x:y} = P\left(T^U(x:y) > t\right) = \min\left({}_t p_x, {}_t p_y\right), \tag{8.3} $$

$$ {}_t p^L_{\overline{x:y}} = P\left(T^L(\overline{x:y}) > t\right) = \min\left({}_t p_x + {}_t p_y, 1\right), \tag{8.4} $$

$$ {}_t p^U_{\overline{x:y}} = P\left(T^U(\overline{x:y}) > t\right) = \max\left({}_t p_x, {}_t p_y\right). \tag{8.5} $$

For comparison purposes, the survival distributions of the future lifetimes for the joint-life and last-survivor status under the independence assumption are denoted and determined by

$$ {}_t p^\perp_{x:y} = P\left(T^\perp(x:y) > t\right) = {}_t p_x \cdot {}_t p_y, \tag{8.6} $$

$$ {}_t p^\perp_{\overline{x:y}} = P\left(T^\perp(\overline{x:y}) > t\right) = {}_t p_x + {}_t p_y - {}_t p^\perp_{x:y}. \tag{8.7} $$

The defined survival distributions satisfy the inequalities

$$ {}_t p^L_{x:y} \le {}_t p^\perp_{x:y} \le {}_t p^U_{x:y}, \quad {}_t p^U_{\overline{x:y}} \le {}_t p^\perp_{\overline{x:y}} \le {}_t p^L_{\overline{x:y}}, \tag{8.8} $$

which imply that the corresponding random future lifetimes are ordered in the stochastic dominance sense such

$$ T^L(x:y) \le_{st} T^\perp(x:y) \le_{st} T^U(x:y), \quad T^U(\overline{x:y}) \le_{st} T^\perp(\overline{x:y}) \le_{st} T^L(\overline{x:y}). \tag{8.9} $$

The reduction of calculation obtained in Section 3.6 shows that it suffices to analyze the maximal impact of the independence assumption on premium calculation for the one-year case $m = c = 1$. For a general life status (u) the NSP and the NLP of the multi-life n-year endowment insurance in this situation are determined by the actuarial functions

$$ NSP(u:n) = 1 - d \cdot a(u:n), \quad NLP(u:n) = \frac{1}{a(u:n)} - d, \tag{8.10} $$

where the multi-life n-year life annuity is calculated from the formula

$$ a(u:n) = \sum_{k=0}^{n-1} v^k \cdot {}_k p_u . \tag{8.11} $$

Inserting the six different life distributions (8.2)-(8.7) into (8.11) and using the stochastic inequalities (8.8)-(8.9), one obtains the following bounding inequalities between the different joint-life and last-survivor n-year life annuities

$$ a^L(x:y:n) \le a^\perp(x:y:n) \le a^U(x:y:n), \tag{8.12} $$

$$ a^U(\overline{x:y}:n) \le a^\perp(\overline{x:y}:n) \le a^L(\overline{x:y}:n), \tag{8.13} $$

which according to (8.10) imply the following inequalities between the NSP's and the NLP's

$$NSP^{U}(x:y:n) \le NSP^{\perp}(x:y:n) \le NSP^{L}(x:y:n), \tag{8.14}$$

$$NLP^{U}(x:y:n) \le NLP^{\perp}(x:y:n) \le NLP^{L}(x:y:n), \tag{8.15}$$

$$NSP^{L}(\overline{x:y}:n) \le NSP^{\perp}(\overline{x:y}:n) \le NSP^{U}(\overline{x:y}:n), \tag{8.16}$$

$$NLP^{L}(\overline{x:y}:n) \le NLP^{\perp}(\overline{x:y}:n) \le NLP^{U}(\overline{x:y}:n). \tag{8.17}$$

To measure the impact of the independence assumption, it is natural to assume that the future lifetimes $T(x)$ and $T(y)$ are positively quadrant dependent and follow a simple Fréchet distribution

$$F^{\theta}(s,t) = (1-\theta)\cdot {}_{s}q_{x} \cdot {}_{t}q_{y} + \theta \cdot \min({}_{s}q_{x}, {}_{t}q_{y}), \quad \theta \in [0,1], \tag{8.18}$$

which satisfies the inequality $F^{\perp}(s,t) \le F^{\theta}(s,t) \le F^{U}(s,t)$. Using (8.10) one obtains the following deviations between independence assumption and Fréchet assumption (8.18) for the NSP's and the NLP's

$$NSP^{\perp}(u:n) - NSP^{\theta}(u:n) = d\cdot\theta\cdot\left(a^{U}(u:n) - a^{\perp}(u:n)\right), \tag{8.19}$$

$$NLP^{\perp}(u:n) - NLP^{\theta}(u:n) = \frac{1}{a^{\perp}(u:n)} - \frac{1}{a^{\perp}(u:n) + \theta\cdot\left(a^{U}(u:n) - a^{\perp}(u:n)\right)}. \tag{8.20}$$

Using (8.12) and (8.13) it is clear that (8.19) and (8.20) are non-negative for the joint-life status $u = x:y$ and non-positive for the last-survivorship status $u = \overline{x:y}$. Therefore the independence assumption overestimates the joint-life NSP's and NLP's and underestimates the last-survivor NSP's and NLP's. The maximal deviations are obtained for a perfect positive dependence $\theta = 1$. Note that the dependence parameter θ can be interpreted as Spearman's grade correlation coefficient (e.g. Hürlimann(2004), Section 3.1). The Table 3.1 below illustrates the maximal deviations numerically. We assume that the marginal future lifetimes follow the distribution of Gompertz(1825) (see also Willemse and Kopelaar(2000)), which is defined by

$${}_{t}p_{x} = \exp\left\{ e^{-\left(\frac{a-x}{b}\right)} \cdot \left(1 - e^{\frac{t}{b}}\right) \right\}. \tag{8.21}$$

The parameters of the Gompertz distribution are set equal to $a = 85, b = 10$, which are close to those listed in the study by Milevsky and Posner(2001), Table 4.

Table 3.1: Maximal deviations for NSP's and NLP's for the two-lives endowment insurance

interest i	male x	female y	term n	maximal deviations in per mill $\Delta SP_{x:y:n}$	$\Delta SP_{\overline{x:y}:n}$	$\Delta LP_{x:y:n}$	$\Delta LP_{\overline{x:y}:n}$
2%	30	30	10	0.4	-0.4	0.3	-0.3
			20	2.4	-2.4	0.5	-0.4
			30	7.4	-7.4	0.8	-0.7
			40	17.8	-17.8	1.3	-1.2
			50	34.5	-34.5	2.1	-1.9
4%	30	30	10	0.8	-0.8	0.3	-0.3
			20	3.7	-3.7	0.5	-0.5
			30	9.7	-9.7	0.8	-0.8
			40	20.0	-20.0	1.3	-1.3
			50	33.7	-33.7	2.0	-1.9
2%	40	40	10	1.2	-1.2	0.7	-0.7
			20	6.4	-6.4	1.2	-1.2
			30	18.5	-18.5	2.1	-1.9
			40	38.5	-38.5	3.3	-2.8
			50	56.7	-56.7	4.5	-3.6
4%	40	40	10	2.1	-2.1	0.8	-0.8
			20	9.6	-9.6	1.3	-1.3
			30	24.2	-24.2	2.2	-2.0
			40	44.3	-44.3	3.3	-3.0
			50	59.5	-59.5	4.3	-3.7
2%	50	50	10	3.2	-3.2	2.0	-2.0
			20	15.9	-15.9	3.4	-3.1
			30	39.6	-39.6	5.5	-4.5
			40	61.9	-61.9	7.5	-5.6
			50	66.9	-66.9	8.0	-5.9
4%	50	50	10	5.5	-5.5	2.1	-2.0
			20	24.1	-24.1	3.6	-3.3
			30	52.9	-52.9	5.7	-4.8
			40	75.5	-75.5	7.4	-5.9
			50	79.8	-79.8	7.8	-6.1
2%	60	60	10	8.1	-8.1	5.7	-5.2
			20	34.2	-34.2	9.6	-7.5
			30	61.7	-61.7	13.5	-9.3
			40	68.1	-68.1	14.6	-9.7
			50	68.2	-68.2	14.6	-9.7
4%	60	60	10	14.0	-14.0	5.9	-5.4
			20	52.4	-52.4	10.0	-8.0
			30	86.2	-86.2	13.7	-9.9
			40	93.0	-93.0	14.5	-10.3
			50	93.0	-93.0	14.5	-10.3

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