Defined Contribution Pension Plans Management And Market Opportunities

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Abstract

DC plan members have been severely impacted by the financial earthquakes and now face a drop of the accumulated assets devoted to their retirement funding. But many financial analysts have argued that the financial crisis represents a great opportunity to carry out new investments, in particular for long horizon investors. In this context, the debate dealing with the pension solution panel offered to DC plan members is now in the limelight. Do the financial crisis and the sharp decline of the equity returns represent an opportunity for defined contribution pension plan participants to reinforce their current exposure towards risky assets? Koijen and al. (2009) develop a tractable continuous portfolio choice model where stock returns exhibit short run momentum and long run mean reversion. We extend this previous framework in three different ways. First we consider a DC plan pension investor during the accumulation phase whose aim is to maximize his terminal wealth. Second, we extend the asset investment possibilities by introducing mean reverting short term interest rate. Third, we consider that the plan member is allowed to invest the pension wealth in three assets - cash bond and stock and study the optimal investment strategy.

Keywords: **Defined contribution pension scheme, Market opportunities, Portfolio asset allocation**

JEL codes: G23, G12

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1 Introduction and motivations

The landscape of funded pension schemes has seen significant changes occur over the past months. The proportion of defined benefit pension plans has significantly been reduced due to a series of well-known shocks (financial crises, accounting rules changes and population aging). All these shocks have involved huge financial imbalances between pension assets and pension scheme commitments, which led numerous DB plan to disclose significant financial deficits. For instance, British DB pension plans displayed a deficit of £200bn in December 2008[3].

Meanwhile, the number of defined contribution pension plans has gained ground, sustained by recent pension reforms, which encourage employees to adopt this type of pension solution to organize their retirement. However, DC plan members have also been impacted by the financial earthquakes and face a drop of the accumulated assets devoted to their retirement funding. Beyond the financial troubles, the losses generated by the financial crisis raise the question of the nature of the pension investment solutions offered to DC plan members. Widespread solutions such as lifecycle funds or constant mix funds do not provide any guarantee to the pensioners against unfavourable and unexpected changes. Moreover, they do not allow the investor to benefit from financial market opportunities due to their “static” nature while more and more financial analysts argue that the financial crisis represents a great opportunity to carry out new investments in particular for long horizon investors. Against this background, the debate dealing with the pension solution panel offered to DC plan members is now in the limelight. Do the financial crisis and the sharp decline of equity returns represent an opportunity for defined contribution pension plan participants to reinforce their current exposure towards risky assets?

Several authors like Fama and French (1986, 1987) or Poterba and Summers (1988) have provided evidence about the mean reverting nature of stock returns. Against this backdrop, the development of market timing strategies makes sense (Koijen and al., 2009). Nevertheless, Poterba and Summers (1988) observe positive autocorrelation in returns at shorter horizon, which attests the existence of returns momentum. Momentum and mean reversion represent the two faces of the same coin. In essence, the existence of a mean reverting process implies a temporary spread from the fundamental valuation which translates into returns continuation in a short period of time or a momentum[4]. Against this backdrop, portfolio strategies have been built to exploit the stock returns characteristics (Lo and MacKinlay (1990), Jegadeesh (1990), Jegadeesh and Titman (1993), Balvers and Wu (2005)). Besides, Koijen and al. (2009) developed a tractable continuous portfolio choice model (Merton, 1973) assuming that stock performances have two key characteristics: equity performances “tend to continue” over a short period of time which corresponds to the so-called momentum properties while over a medium and long term

[3] This amount has been calculated using a panel of 7800 DB plans (source; Pension Protection Funds)

[4] The presence of momentum in stocks returns has been explained by the financial market overreaction. Investors could both be subject to the pessimism and optimism behaviours.

horizon, stock performances are assumed to be mean reverting. In this context, Koijen and al. (2009) predict that under a complete information framework, the allocation to equities is not a linear function of the remaining time horizon, namely, of the remaining years of the investor before his retirement. As the momentum in the stock returns is less persistent than mean reverting, the investor will hedge "the performance variable by reducing the allocation to stock whereas the mean reverting variable will be hedged by increasing the allocation to this asset class" (Koijen and al. (2009))

We extend the framework developed by Koijen and al. in three different ways. On the one hand, we consider a standard DC plan investor during the accumulation phase and whose aim is to maximize his terminal wealth at date *T*. In addition, whereas Koijen and al. consider two asset classes (equity and cash), we extend this framework assuming that the nominal interest rate dynamics is described by a Ornstein Ulhenbeck process and we assume that the term structure of interest rates has the same form as in Vasicek (1977), which allows us to introduce bonds as an investment asset classes. The investor can then invest in three securities: cash, stocks and bonds.

The outline of this paper is as follow. In section 2, we focus on the theoretical and empirical evidences of stock momentum and mean reversion existence. In section 3, we present the different blocks of the model focusing our attention on the financial one. In particular, we explain with great care the equity market modelling and we derive the optimal portfolio. The optimal investment strategy will be discussed and we finally conclude.

2 Questioning the stock market efficiency

2.1 A brief review of the literature

Dealing with the stock market prices properties refers to the burning question of the financial markets efficiency[5]. In a way the efficiency hypothesis assumes that stock market prices follow random walk processes implying thus the absence of all forms of persistence in stock returns or abnormal returns. The immediate consequence of such assumption is that stock market evolutions are unpredictable. Despite the lack of clear cut positions concerning this issue[6] numerous contributions reject this assertion using both time series data and cross sectional data questioning the random walk behaviour of the financial asset prices and thus market efficiency. Fama and French (1988) found negative autocorrelation in returns for a long horizon leading them to conclude on the predictability of stock prices variations. Against this backdrop, they put forward a decomposition of stock prices into a predictable permanent component and

[5] Fama (1970) distinguished three forms of market efficiency: weak efficiency, semi strong efficiency and strong efficiency.

[6]. Kim and al (1991) demonstrate that the mean reversion phenomenon occurred only before the World War II. In addition, Cochran and Defina (1994), from database made up with 18 countries, showed that stock indexes changes are inconsistent with a mean reverting process.

a temporary component that swings away from the fundamental values. The Fama and French approach's give support to the presence of a mean reverting process in stock indexes. Besides, using both American corporate stocks and international equity indexes, Poterba and Summers (1987) sustained the existence of a transitory component in the stock prices with returns exhibiting positive autocorrelation in short term and negative serial correlation in the long term[7]. Lo and MacKinlay (1988) have focused their attention on testing the random walk assumptions on the American financial markets considering different sub-periods (from 1962 to 1985) and several aggregate stock index prices taken on a weekly basis. From a specification test based on variance estimator and applied for different frequencies, the authors "strongly reject" the random walk assumption. However they signal that this result does not give support to the mean reverting assumption. Jegadeesh (1990) also posed the question of stock returns predictability. From monthly individual American stocks prices, he found that the first order serial correlation is significant and negative in the short term. The author underlined furthermore the positive serial correlations at longer horizon. Despite the controversy surrounding the stock market return predictability, data tend to invalidate the pure random behaviour assumption.

More recently, Balvers, Gilliland and Wu (2000) address this issue trough a panel data approach build from the MSCI data for 18 countries (16 OECD plus Hong Kong and Singapore) from 1969. Testing first the stationarity of each stock index in the panel, and estimating for all the countries together an *AR(p)* process, the authors find significant evidence of mean reversion in the national stock indexes. The value of the autoregressive root is statistically significant and lower to the unity validating the mean reverting stock return property. This approach has been extended to the emerging stock markets by Chaudhuri and Wu (2004). From a panel made up of 17 countries, over a period going from 1985 to 2002, the authors are led to reject the random walk assumptions. However, the estimation of the autoregressive value in the *AR(p)* shows that the mean reversion phenomenon is much slower for emerging countries than developed economies.

Finally, Koijen and al. (2009) summed up this set of finding sustaining that: "Equity returns tend to continue over short horizons and revert over long horizon". In this context, they conjecture that financial strategies can be built benefiting from stock market properties such as momentum or returns mean reversion. To shed light on these issues, we start our investigations using simple statistic tools and tests widely detailed in the literature to verify the properties quoted above.

[7] Performing variance ratios tests (see section 2.2) the authors rejected the random walk assertion giving hence additional support to their findings.

2.2 Empirical analysis

2.2.1 The random walk hypothesis

We propose to begin this section focusing our attention on to the random walk[8] process detection applying variance ratio tests. If a stock price index behaves as a random walk, then the variance of increments index increases linearly in the observation interval. Against this background, Cochrane (1988) and Lo and MacKinlay (1988) proposed an individual test, for a given holding period. If the variance of the q^{th} differenced variable is q times as large as the first differenced variable, then the return series follows a random walk. Let s_t be the log of a stock price index:

$$Var\left[s_t - s_{t-q}\right] = qVar\left[s_t - s_1\right] \qquad [1]$$

The variance ratio is hence given by the following expression

$$V_{ratio}\left[q\right] = \frac{\frac{1}{q}\left[Var\left[s_t - s_{t-q}\right]\right]}{Var\left[s_t - s_1\right]} = \frac{\sigma^2(q)}{\sigma^2(1)} \qquad [2]$$

The variance ratio allows testing simultaneously two assumptions relative to the residuals behaviour. Under the null hypothesis:

- Residuals are i.i.d and follow a Gaussian distribution which corresponds to random walk behaviour (i.e. $V_{ratio}\left[q\right] = 1$). This case corresponds to the homoscedastic random walk
- Residual are non auto-correlated but they are not normally distributed and the variance may vary through the time horizon. This case corresponds to the heterocedastic random walk

For both cases, Lo and MacKinlay provide two statistics *Z(q)* and *Z*(q)* asymptotically distributed as a Gaussian variable. The former statistic is derived under the assumption that stock prices residuals are homoscedastic while *Z*(q)* treat them as heteroscedastic (Smith and Ryoo, 2003).

Chow and Denning (1993) enhanced this approach suggesting a multiple variance ratio test. In a way the Chow and Denning test corresponds to a Lo and MacKinlay test for several holding periods taken simultaneously[9]. Chow and Denning formulate the null hypothesis as $H_0 : VR(k_i) = 1$ for $i = 1, 2, \ldots, m$ and define their statistic as:

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[8] The literature proposes several type of test to diagnose random walks. Among the random walk detection family of tests, we find the runs test (or Wald-Wolfowitz test) which is a non parametric test used to signal whether the elements of a chronic are mutually independent.

[9] See Colletaz (2003) for a critical presentation of the Lo and Mackinlay and the Chow and Denning test.

$$Z_1(m) = \max_{1 \le i \le m} |Z_1(q_i)|$$

$$Z_2(m) = \max_{1 \le i \le m} |Z_2(q_i)|$$

These tests are performed over a sample of 14 stock indexes of developed countries, from January 1970 until July 2009 (or 260 observations), and data are taken on a monthly basis[10]. Our frequency choice is motivated by a wider sample range and the willingness not to introduce noisy information (as outliers) in the database. Note furthermore that this database will be used for all the tests implemented in this study. Stock market returns are calculated taking the log difference of the stock prices[11] (the descriptive statistics of the stock index are reported in appendix 1).

The analysis begins with the calculation of the variance ratios for each stock index for different holding periods going from 1 month to 20 months. The results of the variance ratio are reported in figure 3 in the second appendix. Most of the time, the variance ratio are different to unity whatever the stock market considered. In addition, we can observe besides that most of variance ratios strongly move away from the unity in particular for short holding periods (for 6 months and more) and tend to move back for wider horizons (see the chart below). Beyond the magnitude of the phenomena, the variance ratio behaviour tends to reject the random walk hypothesis on the one hand, and on the other part to validate the Koijen and al (2009) conjecture quoted before on the other.

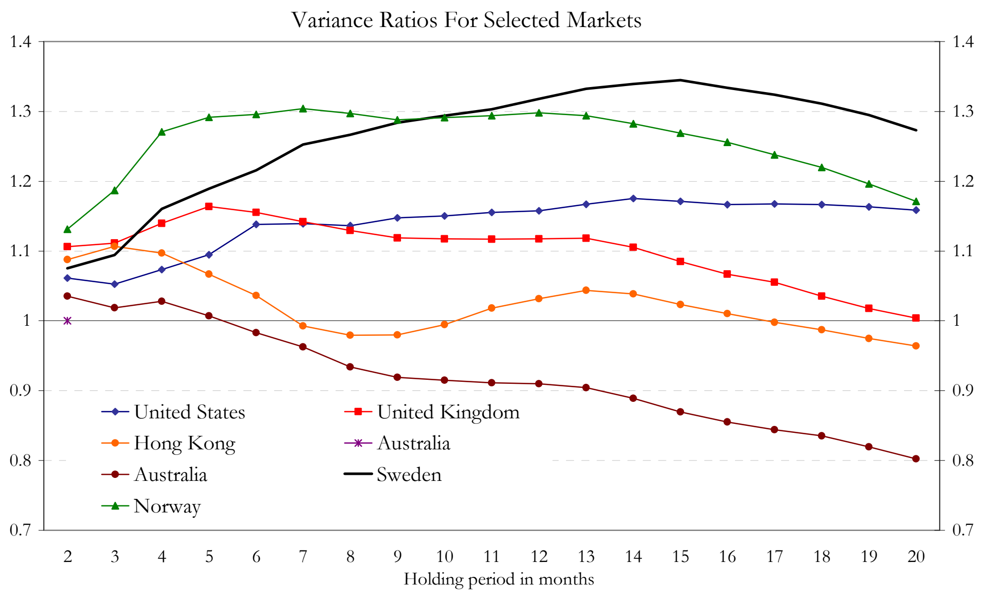

Figure 1: Variance ratio calculation

[10] Data source: Morgan Stanley Capital Investment (MSCI). The database comprises the United States, the United Kingdom, France, Germany, Spain, Hong Kong, Japan, Italy, Canada, Australia, Sweden, Norway, Belgium, Netherlands

[11] Note that returns calculated on a year on year basis introduce an artificial smoothing, which would corrupt the results of the empirical analysis.

We turn to the Lo and MacKynlay test, applied for 3 selected intervals, (2 months, 10 months and 20 moths) which provides both statistics *Z(q)* and *Z*(q)* (table 4, appendix 2). Comparing the critical values for different holding periods to the normal distribution at a 5% threshold, the random walk assumption is rejected in most of the case. Stock markets in Belgium, Italy, the United Kingdom, Spain and Japan do behave as a random walk according to the Lo and Mackinlay test at 5% threshold. Note that H_0 is also rejected at a 10% threshold for markets as France, Canada, Hong Kong, and Sweden. However, the American, the Australian and the German stock market prices follow a random walk process as the critical values of the Lo and Mackinlay test are below the threshold.

We then apply the Chow and Denning test providing again the *Z(m)* and *Z*(m)* statistics for each countries. These statistics are then compared to the one computed by Chow and Denning (2003) (table 4, appendix 2). The results appear more contrasted as the random walk hypothesis is "accepted" for eight stock markets over fourteen. The alternative assumption is only accepted for Japan, Italy, Norway, Spain and the United Kingdom. The lack of clear cut results does not allow us to accept or reject the random walk assertion. With this output in mind, we decide to continue our examination of stock markets indexes focusing on the momentum and the mean reversion issues.

2.2.2 The detection of stock returns momentum

As we mentioned it above equity return momentum is defined as the continuation within a short period of time of the past returns trend. Several methods can be used to detect momentum in the stock returns. We consider three kinds of test: we firstly pay attention to the autocorrelation function of the stock indexes returns for short holding period (from 1 to 6 months). For instance, significant autocorrelations mean that returns are temporally linked each other. The second level of our analysis consists in performing the Breusch Godfrey serial correlation test[12] to test the autocorrelation significance. We end this section with the ARCH effect test[13] which aim is to detect the presence of heteroscedascity in the perturbation parameters. The detection of a conditional variance process effect will drive us to invalidate the random

[12] The rejection of the assumption H_0 indicates the existence of an autocorrelation in the time series whereas the acceptance of H_0leads to conclude to the absence of autocorrelation. The Breush Godfrey test is based on the Lagrangian multiplier statistics (LM) which follows a $\chi^2_\alpha(q)$ where q is the number of degree of freedom. For a given threshold α:

- The assumption H_0 is rejected if LM > $\chi^2_\alpha(q)$
- The assumption H_0 is accepted if LM < $\chi^2_\alpha(q)$

[13] The ARCH effect test consists in testing the H_0 i.e. resides are homoscedastic versus H_1 i.e. resides are heteroscedastic. To do so, the estimated resides are regressed on their past squared values. As for the Breush-Godfrey test, a TR statistics is calculated and if:

- If TR > $\chi^2_\alpha(q)$ then H_0 is rejected involving thus the rejection of the homoscedasticity assumption
- If TR < $\chi^2_\alpha(q)$ then H_0 is accepted involving thus the acceptance of the homoscedasticity assumption

walk assumptions[14]. We decide to exclude emerging markets from the analysis for two main reasons: on the one hand emerging stock indexes have only existed over a recent period and on the other hand pension funds of developed countries are not exposed in such markets.

Calculating the autocorrelations for the different stock indexes and for short frequencies (see appendix 3), it can be noticed that the first autocorrelation is positive and significant for 9 stock indexes giving support to the presence of momentum in stock returns. While the first order serial correlation is not significant for countries like Germany or United States, it can be observed that the higher order autocorrelations are positive and significant. Despite the lack of a solid consensus, results tend to give support to the momentum existence some stock markets. We perform additional tests to strengthen these findings.

In line with the autocorrelation calculations, we implement the Breush-Godfrey test (the results are recorded in the tables in appendix 3). The serial correlation tests give us little evidence about momentum existence. Actually, comparing the LM statistics to the $\chi^2(q)$ for $q=1...6$ we reject the serial correlation in the most cases except for the Italy, Norway and the United States at a 5% level and Spain and the United Kingdom at a 10% level. Despite this mitigated result, we continue our investigations implementing ARCH effect test. The intuition is the following: the absence of serial correlation returns it does not imply the independence[15] of the returns. In the light, of the results of the test, at a 5% level we are led to accept the alternative assertion namely that stock markets returns are heteroscedastics (see appendix for the corresponding *p-values*). This result implies on the one hand that stock past volatility returns explain current volatility returns thus making market stock volatility predictable. The random walk assertion has hence to be reconsidered. On the other hand, returns heteroscedasticity in the short run reinforces the momentum assumptions.

2.2.3 Do stock returns mean revert?

Balvers, Gilliland and Wu (2000) point out the difficulty related to this issue "due to the absence of reliable long time series". A frequent method used to detect the mean reversion character of time series consist in performing non stationnarity tests (Balvers and al (2000), Chadhuri and Wu (2004), Higgs, Worthington (2003)). The rejection of the unit root hypothesis translates into the fact that stock indexes tend to mean revert which also means that a shock should have a temporary effect on the chronic. Different unit root tests are available to detect the presence of a unit root in the stock returns: Augmented Dickey Fuller test, the Phillips and Perron test and the Kwiatkowski, Phillips, Schmidt, Shin test. We propose to implement the three tests over the sample used before. The individual unit root tests results are recorded in Appendix 4. It can be noted that, for all the countries, the unit root assumption is rejected at a 1% level.

[14] Alexander (1992) notes that focusing only on the firs moment could lead to spurious conclusions Applying ARCH test, Alexander tries to detect "pseudo random walk process" analysing the conditional variance.

[15] In other words $cov\left(\varepsilon_t^2 - \varepsilon_{t-1}^2\right) \neq 0$ which is equivalent to a conditional variance assumption

However, it is well known that non stationary tests suffer from several weaknesses, in particular for small samples. Although, the PP and the KPSS tests are more powerful than the ADF test, we aim to avoid the risks of spurious conclusion by implementing panel unit root tests in order to improve the power of the test. Narayan and Prasad (2007) have adopted this approach and tested whether European stock indexes are mean reverting. Considering seventeen European countries, their finding are consistent with the efficient market hypothesis i.e. they accept the random walk assumption. To reinforce our analysis we implement five panel unit root tests. The Levin Lin and Chu (2002), the Breitung test (2000) and the Hadri[16] test (2000), where the autoregressive parameter is assumed to be common to all the countries of the panel. Due to this restrictive assumption we also apply the Im, Pesaran, Shin (2003) and Maddala Wu (1999) which allow heterogeneity in the autoregressive parameter. The results of the unit root panel tests are reported in appendix 4. Comparing the statistics with a normal distribution, all the tests performed strongly reject the unit root hypothesis. These results have strong implications: whatever the magnitude of the random shock for instance, the stationary properties of returns imply no persistence in the return chronic.

3 Portfolio choice with market time opportunities – A simple approach

3.1 The literature

In line with these results, numerous studies have proved the benefits of stock markets properties (also called anomalies) in the construction of the investment policies. Adopting the stock markets overreaction hypothesis, De Bondt and Thaler (1985) sustained and demonstrated the possibility to structure portfolios that displayed exceptional returns[17]. Lo and MacKinlay (1990) showed that benefits can be withdrawn from stock market overreactions allowing thus the construction of contrarian strategies which consists in "selling winners and buying losers". Jegadeesh (1990) found significant negative serial correlation over the short term and positive autocorrelation over longer horizon. In this context they show that strategies based on the past monthly stock performances generate "abnormal" returns. Jegadeesh and Titman (1993) provide additional evidence about the performance of "buying winner and selling losers". Although this asset allocation rule generates positive return over a 3 - 12 months horizon, they contrast their results noticing that this excess return is not permanent and abnormal negative returns beyond this horizon could be experienced.

Considering a panel of 18 stock markets, Balvers Gilliland and Wu (2000) detected mean reversion existence and determined the speed of the mean reversion process. In this context, an accumulated return deficit over a specific stock market can thus be offset within a limited period of time. Based on their

[16] Although the Levin Lin Shin test and the Breitung test set the unit root as the null hypothesis, the Hadri adopts a strategy where the null hypothesis corresponds to the absence of unit root.

[17] These abnormal returns occurred in January (January effect) and Jegadeesh (1990) identifies abnormal returns as the result of seasonal effects.

findings, they compared a parametric contrarian strategy to a buy and hold strategy and random walk strategy and found that the former strategy strongly outperforms the other strategies. Extending Balvers and al (2000) Balvers and Wu (2005) point out that stock returns can be persistent in the short term, which enhances market opportunity perspectives. They show in this context that the combination of momentum and mean reverting properties in building financial strategies outperforms pure momentum or pure mean reverting strategies. In what follows, we continue our analysis showing first the existence of stock returns continuation and mean reverting phenomena.

Taking into account the pension fund institutions objectives - i.e. to pay pension benefit to their participants - their financial importance[18] and the progressive reduction of pay-as-you-go structures, the investment policies issues have caught both the researchers and the asset managers' attention. More or less sophisticated solutions, as life cycle fund strategies (see Bodie and al, 2007) or portfolio insurance strategies (see Prigent (2006) and Ma, 2008), have already been analyzed and proposed to the pension funds. Nevertheless, pension fund management has received a considerable echo with the financial crisis and became an important issue. While analysts argue that the low stock markets price should encourage investors to increase the portfolio exposure toward those markets, pension fund managers and institutional investors remain traumatized by the huge losses recorded after the successive financial downturns. Total pension fund losses induced by the last financial crisis would have reached $ 5,000 bln according to the OECD estimations. In this context, the determination of the defined contribution pension funds optimal investment strategies comes into the limelight.

This topic has been widely addressed in the financial and the economical literature and numerous studies solved the asset allocation problem derived from the expected utility maximization considering different frameworks as a starting point. The foundations of this literature have been laid by Merton (1969, 1971) who settled within a simple continuous time model that long term investors should hedge unexpected changes of the investment portfolio. A natural extension to the Merton model has been considered with the introduction of other sources of risks as stochastic interest rates. Assuming that nominal short term interest rates behave as Ornstein Uhlenbeck process and considering that the yield curve is driven by only one factor, as in Vasicek framework[19] (1977), explicit bond pricing formulae can be deduced and included in the determination of the optimal financial strategy.

From that point, the literature has explored several ways insisting in particular the source of risks. Bodie and al (1992) introduced human capital as a financial asset in the optimization problem and showed how the labour market flexibility influences the optimal portfolio building over the life cycle. Brennan and Xia (2000) studied the optimal investment issue considering long term investors exposed to inflation risks.

[18] The total financial asset held by pension funds represent $ xx bln (source OECD)

[19] Enhancing the standard Vasicek framework assuming that short interest rate dynamics behaves as a CIR framework, Deelstra and al (2000) provide a closed form formulae of the investment problem within a complete framework.

Assuming that inflation changes cannot be perfectly hedged with nominal financial assets and under a set of additional constraints, the optimal portfolio is made up of three separated portfolios as in the Merton framework: a nominal mean variance tangency portfolio which inversely depends on the investor relative risk aversion, and a nominal interest rate risk hedging portfolio and a price level risk hedging portfolio[20].

In a similar vein, Battochio and al (2003), Devolder and al (2006), Blake and al (2007) showed a deep interest in the mortality risk management modeling the time of death as a stochastic process. Studying both the accumulation phase and the decumulation phase, Battochio (2003) showed that the fraction of equities decrease until the retirement date and then increase when the investor is in retirement due to the randomness of time of death. Defining the mortality bonds as fixed income assets which pay coupons proportional to the survival rate (supposed to be stochastic) of a given population, Menoncin (2006) demonstrated that longevity risk can be hedge on the financial markets. Hence the presence of such asset reduces the fraction of standard bond products in the optimal portfolio. However, it does not affect risky asset portfolio weight nor risk free asset portfolio weight.

One of the first attempts to include stock market properties consisted in the introduction of a mean reverting market price of risk, or the Sharp ratio (Kim and Omberg, 1996). Taking the Kim and Omberg model as a starting point, Watcher (2002) examined the portfolio choice problem within a complete market framework assuming that stock market returns are mean reverting. Using the martingale method, the paper provides an exact form of the solution where the portfolio allocation takes the form of a weighted average. Munk and Sorensen (2007) develop a more global framework gathering at once stochastic income process and both stochastic mean reverting interest rates and risk premium. It is assumed that wages are instantaneously correlated to the stock prices, the interest rate and hence to the bond prices. The authors show how and how much the uncertainty of wage dynamics affects the financial asset breakdown in the portfolio.

In line with this literature we propose a simple model, inspired from the Korn (2001), Ma (2008) and Koijen and al (2009) frameworks. We consider a closed defined contribution pension scheme during its accumulation phase and which has a positive initial wealth. The aim of the pension fund is to optimize its terminal financial wealth to pay benefit to the pension members. Note however that conversely to the DB case there is no commitment from the DC plans to pay pension benefits to the plan members[21]. Our attention will be focused on the stock markets structure in line with Kojein and al. (2009) which considers simultaneously momentum and mean reverting stock returns. We propose an extension of this framework allowing the pension to be exposed to the interest rate risk. On the other hand, we will consider the

[20] The combination of two these last portfolios is interpreted as the minimum risk portfolio which closely replicates the indexed bond return and which the maturity is equivalent to the remaining investor horizon. Besides, Berkelaar and Kowemberg (2007) consider a mean reverting inflation rate modelling and derived a closed form solution to the portfolio optimization problem.

[21] The maximisation of the pension fund wealth does not necessarily correspond to the one of the plan members

possibility to manage these risks allowing the fund manager to invest in bonds and stocks. Finally, we extend this framework considering a more global framework made up of three financial assets: cash bonds and stock.

3.2 The model structure

Financial markets are assumed to be arbitrage free, frictionless and continuously open between *0* and *T*, where *T* is a strictly positive real number and represents the terminal date of the investment horizon and interpreted as the retirement date of the plan member. In this first attempt we consider a complete market framework and consider that market uncertainty is only based on two dimensional Brownian motions $\{Z_S(t), Z_r(t), \Im_t\}_{t\geq 0}$ with *t* comprises between *0* and *T* and defined in a complete probability space $(\Omega, \Im, P)$ where *P* denotes the historical probability and the filtration $\{\Im_t\}_{t\geq 0}$ represents the information structure generated by the Brownian motion. Note besides that only self financed strategies are considered in this first attempt as we do not introduce either contributions nor salary income process (see Henderson (2004), Battochio and Menoncin (2002, 2004) Blake and al (2006), and Munk and Sorensen (2007) for further details about this topics)

Turning to the financial market structure, we firstly consider a standard framework where the financial pension wealth can be invested in two asset classes: cash and stocks. We suppose that cash assets dynamics is governed by the following equation:

$$\frac{dB_t}{B_t} = r_t dt \qquad [3]$$

where r_t is the nominal short term interest rate and B_t denotes the cash asset price. At this stage, we assume no interest rate risk as the cash asset evolves in a deterministic way.

As it was showed in the previous section long term returns exhibit a mean reversion process whereas positive persistence has been founded in the short run supporting the momentum existence. Against this backdrop, a particular attention will be given the stock market modelling. Our objective is to present a structure in which both short and long term equity returns will be taken into account. Letting S_t be the stock market price, we start by writing the stock market index as a standard geometrical Brownian motion as in the following relation

$$\frac{dS_t}{S_t} = (\mu_t + am_t)dt + \sigma_S dZ_{S,t}$$

where μ_t is the drift of the process (or the average stock return), σ_S is the diffusion of the process and both parameters are assumed to be constant. It can be noticed that the drift is augmented with the parameter

am_t. We propose to dwell upon the "augmented drift parameter" of the stochastic process. In particular, we suggest to decompose the parameter $\mu_{S,t}$ into three parts and propose the following form for the drift:

$$\begin{cases} \mu_t \equiv r_t + \mu_t - r_t \\ m_t \equiv M_t - \mu_t \end{cases} \qquad [4]$$

where M_t is the "momentum" variable which refers to the existence of a short term returns persistence (further details will be provided about this variable in the coming paragraphs). The first element of [4] is composed of a liquidity premium (r_t) and the second part $(\mu_t - r_t)$ corresponds to the standard excess stock returns[22]. As a first step, the excess stock return will be considered as a deterministic and continuous process. This assumption will be relaxed thereafter.

The third part of the instantaneous stock return is the difference between momentum and the stock return be interpreted as "abnormal stock returns" coming from market efficiency default the constant parameter *a*, which values are between *0* and *1*, could be seen as an inefficiency market indicator. A closer interpretation is provided by Koijen and al (2009). Considering the case where $a > 0$ the authors signal that past returns can help to foresee the expected stock returns questioning thus the market efficiency assumption[23]. Let λ_t be the excess stock return and assumed to be constant.

$$\frac{d\lambda_t}{\lambda_t} = \lambda_t dt \qquad [5]$$

This leads to the following stock market dynamics:

$$dS_t = S_t\left(r_t + \mu_t - r_t + a\left(M_t - \mu_t\right)\right)dt + S_t\sigma_S dZ_{S,t}$$

$$\frac{dS_t}{S_t} = \left(r_t + \lambda_t + am_t\right)dt + \sigma_S dZ_{S,t} \qquad [6]$$

We provide now more details about the momentum variables M_t. To understand the interactions between the momentum and long stock returns, we assume that the variable M_t is the sum of the past stock returns. For $t > u$, the momentum variable is given by the following equation:

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[22] The risk premium (also known as the Sharpe ratio) is usually calculated as follow $rp_t = \frac{\mu_t - r_t}{\sigma_t} \Leftrightarrow rp_t = \frac{\lambda_t}{\sigma_t}$

[23] The two extreme cases can be considered. Assuming that $a = 0$ i.e. there are no abnormal returns, we meet up with the framework developed by Merton. However, assuming that $a = 1$ which means that past returns are the best forecast of the expected stock return, only momentum will be considered in the stock market dynamics.

$$M_t = \int_0^t e^{-(t-u)} \frac{dS_u}{S_u} \qquad [7]$$

where dS_u / S_u is the past stock index prices. Using the Itô lemma (see appendix 5 for the calculations), the dynamics of the momentum variable can be written as follow

$$dM_t = -M_t dt + \frac{dS_t}{S_t}$$

$$dM_t = (1-a)(\mu_t - M_t)dt + \sigma_S dZ_{S,t}$$

Looking at the dynamics of the momentum, we can observe that "the performance variable fluctuate around the average stock return μ_t". Besides Koijen and al (2009) relax the constant average stock return assumption and suppose that μ_t follows a mean reverting process.

The financial pension wealth results from the combination of a fraction of stocks and cash assets plus the flow of contributions paid by the pensioners. We let X_t be the financial pension wealth with $X_0 > 0$ and let π_t be the fraction of the pension wealth invested in stock assets. In this framework the dynamic of X_t is given by:

$$dX_t = X_t \left[\pi_t \left[\frac{dS_t}{S_t} \right] + (1-\pi_t) \left[\frac{dB_t}{B_t} \right] \right] \qquad [8]$$

$$\frac{dX_t}{X_t} = \left[\pi_t \left((r_t + \lambda_t + am_t) - r_t \right) + r_t \right] dt + \pi_t \sigma_S dZ_{St}$$

$$\frac{dX_t}{X_t} = \left[\pi_t \left(\lambda_t + a(M_t - \mu_t) \right) + r_t \right] dt + \pi_t \sigma_S dZ_{St} \qquad [9]$$

As we mentioned it before, we adopt a progressive approach presenting first the optimisation problem within the simplest framework and increment progressively this basis introducing new markets or relaxing assumptions. In a first extension, we will relax the deterministic excess stock return assumption. We will continue by introducing bond assets and study the solution considering a two risky assets framework. To finish we will allow the pension to be invested in three assets: cash, bonds, and stocks.

3.3 Resolution of the problem - A cash asset and stock portfolio

3.3.1 Constant excess return

The pension manager objective is to optimize the terminal value of the financial pension fund. Under the constraint given by the equation [9] the pension fund manager problem can be expressed in these terms:

$$J\left(X(t),M(t),t,T\right)\equiv \underset{\pi_S,s\in[t,T]}{Max}\left\{E_t\left[U\left(X_T\right)\right]\right\} \qquad [10]$$

where $J(.,.)$ is the indirect utility function and $U\left(X_T\right)$ the utility function describing the investor preferences. Most of the studies specify a functional form of the utility function. Without any loss for the results and to simplify the presentation we consider a general form of the utility function which respects the Inada conditions. To solve this problem, the Dynkin calculation (or the infinitesimal generator) is required. The use of the Dynkin allows us to simplify the previous problem. Let *D* being the Dynkin operator. Hence we have:

$$DJ = J_t + J_X dX + J_M dM + \frac{1}{2}J_{XX}\left\langle dX^2\right\rangle + \frac{1}{2}J_{MM}\left\langle dM^2\right\rangle + \frac{1}{2}J_{XM}\left\langle dXdM\right\rangle \qquad [11]$$

where $J_t, J_X, J_{XX}, J_{XM}, J_{MM}$ are respectively the partial derivatives with respect to the denoted indices and the $\langle . \rangle$ is the covariance between the state variables. Thus the first order condition takes the following form:

$$\underset{\pi_S,s\in[t,T]}{Max}\left\{E\left[DJ\right]\right\}=0 \qquad [12]$$

Replacing the infinitesimal operator in the equation [11] by the dynamics of the pension wealth, we get

$$\underset{\pi_t}{Max}\left\{E_t\left[\begin{array}{r} J_t + J_X X\left[\left(\pi_t\left(\lambda_t + a\left(M_t-\mu_t\right)\right)+r\right)dt + \pi_t\sigma_S dZ_S\right] + J_M\left(\left(1-a\right)\left(\mu_t - M_t\right)dt + \sigma_S dZ_S\right) \\ +\frac{1}{2}J_{XX}X^2\left(\pi_t\sigma_S\right)^2 + \frac{1}{2}J_{MM}\sigma_S^2 dt + \pi_t\sigma_S^2 dtXJ_{XM} \end{array}\right]\right\}=0$$

$$\underset{\pi_t}{Max}\left\{\begin{array}{r} J_t + J_X X\left[\left(\pi_t\left(\lambda_t + a\left(M_t-\mu_t\right)\right)+r\right)\right] + J_M\left(\left(1-a\right)\left(\mu_t - M_t\right)\right) \\ +\frac{1}{2}J_{XX}X^2\left(\pi_t\sigma_S\right)^2 + \frac{1}{2}J_{MM}\sigma_S^2 + \pi_t\sigma_S^2 XJ_{XX} \end{array}\right\}=0 \qquad [13]$$

The solution of the problem, i.e. the optimal weight of each asset class in the pension fund portfolio can easily be deduced from equation (11) deriving it with respects to the variable π_t. Let π^* being the optimal fraction of stock in the portfolio:

$$\pi^* = -\frac{J_X}{J_{Xx}X}\left(\frac{\lambda_t + a\left(M_t-\mu_t\right)}{\sigma_S^2}\right) - \frac{J_{XM}}{J_{XX}X}$$

$$\pi^* = \underbrace{-\frac{J_X}{J_{XX}X}\frac{\mu_t - r_t}{\sigma_S^2}}_{Proportion\ I} \underbrace{-\frac{J_X}{J_{XX}X}\frac{a\left(M_t-\mu_t\right)}{\sigma_S^2}}_{Proportion\ II} \underbrace{-\frac{J_{XM}}{J_{XX}X}}_{Prop\ III} \qquad [14]$$

The partial derivative indirect utility function ratio denotes the absolute risk tolerance within the CRRA utility functions class[24]. The optimal fraction of stocks held in the portfolio is separated into three components which is consistent with the fund separation theorem. As expected, the first two element of the solution correspond to the static portfolio (as in Merton, 1969 and 1971) and shows how the portfolio equity exposure varies according to the risk tolerance and the stock index excess return. The second components exhibits the role played by the market timing opportunities in the risky asset investor acquisition behaviour. Two observations can be made:

- Depending on the dynamic of the recent stock market returns the equity exposure will be higher or lower than in the standard framework. Assuming for instance a positive innovation on M_t , and as momentum and stock returns are positively correlated, the share of stocks will increase which is in line with the "buy winners and sell loser" strategy (cf. literature)
- In addition, stock market exposure should increase in line with the value of our efficiency indicator (the parameter a). Assuming that the value of a is important (close but lesser 1 for instance) this will reinforce equity investment as past stock returns help to forecast current stock returns.

The third fraction of the solution is the hedging demand against unexpected changes in recent stock market returns as they are modelled in a stochastic way. It can be noticed that this hedging demand has a particular form as the momentum and stock market dynamics are perfectly correlated. In this context this third part could be seen as a correction term related to the investor aversion to take more risks to withdraw additional benefits from market opportunities within a short period of time.

3.3.2 Stochastic excess return

Taking account the restrictive aspect of the constant stock market return, this assertion can easily be relaxed. The main studies dealing with a close issue consider a framework where the market price of risk reverts to its mean (Omberg, 1996 and Watcher, 2003). In line with our initial objective, we consider here that excess stock return is described by the following process:

$$d\lambda_S = \kappa\left(\overline{\lambda} - \lambda_S\right)dt + \sigma_\lambda dZ_{S,t} \qquad [15]$$

where $\overline{\lambda}$ is the long term stock return average, κ represents the degree of the mean reversion, σ_λ is the volatility associated to the process and supposed to be constant. Note besides that the source of

[24] CRRA (Constant Relative Risk Aversion) is comprised within the HARA (Hyperbolic Absolute Risk Aversion) utility function. Several functional forms are usually taken in the literature. The most widespread utility function is $U(X_T) = \frac{1}{1-\gamma} X_T^{1-\gamma}$ and $\frac{U'(X_T)}{U''(X_T)X_T} = \frac{1}{\gamma}$.

uncertainty comes directly from the stock market which seems to be a consistent assumption with the intuition. In this context, let $\sigma_{S\lambda} = \sigma_S \sigma_\lambda dt$ be the covariance linking both processes as $dZ_S dZ_S = dt$

The introduction of stochastic long stock return leads to changes in the pension fund financial wealth dynamics. In this case, the Dynkin operator becomes:

$$\begin{aligned} DJ = J_t + J_X dX + J_M dM + J_\lambda d\lambda + \frac{1}{2} J_{XX} \left\langle dX^2 \right\rangle + \frac{1}{2} J_{MM} \left\langle dM^2 \right\rangle + \frac{1}{2} J_{\lambda\lambda} \left\langle d\lambda^2 \right\rangle \\ + J_{XM} \left\langle dXdM \right\rangle + J_{X\lambda} \left\langle dXd\lambda \right\rangle + J_{M\lambda} \left\langle dMd\lambda \right\rangle \end{aligned} \quad [16]$$

where J_λ and $J_{\lambda\lambda}$ are the first and the second order partial derivatives of the indirect utility function with respect to the variable λ. Substituting in the equation [15] by the appropriate equations, we get:

$$\max_{\pi_t} E\left[DJ\right] = \max \left\{ \begin{array}{c} J_t + XJ_X \left(\pi_t \left(\lambda_S + a\left(M_t - \mu_{S,t}\right)\right) + r\right) + J_M \left(\left(1-a\right)\left(\mu_S - M_t\right)\right) + J_\lambda \left(\kappa\left(\overline{\lambda} - \lambda_S\right)\right) \\ + \frac{1}{2} J_{XX} X^2 \left(\pi_t \sigma_S\right)^2 + \frac{1}{2} J_{MM} \sigma_S^2 dt + \frac{1}{2} J_{\lambda\lambda} \sigma_\lambda^2 dt + J_{XM} X \pi_t \sigma_S^2 dt + J_{W\lambda} X \pi_t \sigma_{S\lambda} dt + J_{M\lambda} \sigma_{S\lambda} dt \end{array} \right\}$$

We apply the same strategy as above i.e. the optimal solution is founded solving at $\max\left\{E\left[DJ\right]\right\} = 0$ (see the appendix for the calculations)

$$\pi_t^* = \underbrace{-\frac{J_X}{J_{XX} X} \frac{\mu_S - r_t}{\sigma_S^2}}_{Proportion\ I} \underbrace{- \frac{J_X}{J_{XX} X} \frac{a\left(M_t - \mu_t\right)}{\sigma_S^2} - \frac{J_{XM}}{XJ_{XX}}}_{Proportion\ II} \underbrace{- \frac{J_{X\lambda}}{J_{XX} X} \frac{\sigma_\lambda}{\sigma_S}}_{Proportion\ III} \quad [17]$$

As we introduce a new source of randomness a fourth element appeared in the solution [16]. In this framework, the optimal proportion of stock held in the portfolio is made up of the three components described in the previous section a new hedging demand destined to hedge the unfavourable changes in the excess returns. As expected, this new hedging demand is a positive function of the volatility the excess return process (cf. equation [15]). These results are in line with the findings of the studies dealing with these concerns.

4 Assuming mean reverting interest rates dynamics

4.1 A two risky assets portfolio

As in the previous section, we relax the deterministic interest rate dynamics assumption. We assume in turn that the short term nominal interest rate dynamics is described by an Ornstein - Uhlenbeck process as in Boulier (2001), Munk and Sorensen (2003 and 2007), Menoncin (2002 and 2004):

$$dr_t = \beta\left(r_t - \overline{r}\right)dt + \sigma_r dZ_r \quad [18]$$

where β corresponds to the degree of mean reversion, $\overline{r}$ is the long term mean interest rate, σ_r is the constant diffusion of the process and Z_r is a Brownian motion. As the short term interest rate satisfies the dynamics given by [16], the solution for r_t can explicitly be written as follow:

$$r(u)=(r_0-\overline{r})e^{-\beta(u-t)}+\overline{r}+\sigma e^{-\beta u}\int_t^T e^{\beta t}dZ_r \quad [19]$$

Considering that the interest rate term structure is governed by only one factor, Vasicek (1977) give the exact form a zero coupon bond price with time to maturity τ. For all maturities $\tau>t$, let $P(r,t,\tau)$ be the zero coupon bond price:

$$\frac{dP(r_t,t,\tau)}{P(r_t,t,\tau)}=\mu_P dt-\sigma_P dZ_r \quad [20]$$

where $\mu_P=r_t+\xi_P\sigma_P$ which leads to the following expression $\mu_P-r_t=\xi_P\sigma_P$ or $\mu_P-r_t=\lambda_P$. Hence the bond price is given by:

$$\frac{dP(r_t,t,\tau)}{P(r_t,t,\tau)}=(r_t+\sigma_P\xi_P)dt-\sigma_P dZ_{r,t}\Leftrightarrow\frac{dP(r_t,t,\tau)}{P(r_t,t,\tau)}=(r_t+\lambda_P)dt-\sigma_P dZ_{r,t} \quad [21]$$

where

$$\sigma_P=\sigma_r a(T,\tau) \text{ and } a(T,\tau)=\frac{1-e^{-\beta(T-\tau)}}{\beta}$$ [25]

The parameter ξ_P can be interpreted as the constant market price of risk. At this stage, the bond market comprised an infinite zero coupon bonds. Battochio and Menoncin (2002) signal that the bond market dimension can be reduced under a set of certain conditions using only one zero coupons bond (one source of risk, a specified bond price process and thus a specified market price of risk). In the same vein as Boulier and al. (2001), a constant maturity K zero coupon bond price denoted $P_K(r,t)$ is given by[26]:

$$\frac{dP_K(r_t,t)}{P_K(r_t,t)}=(r_t+a_K\sigma_r\xi_P)dt+a_K\sigma_r dZ_{r,t} \quad [22]$$

[25] See Vasicek (1977) for a complete presentation.

[26] Boulier and al (2001) suggest a way to connect the bond asset to the risk free asset and the constant K maturity zero coupon through the following linear combination $\frac{dP(r_t,t,\tau)}{P(r_t,t,\tau)}=\left(1-\frac{a(t,\tau)}{a_K}\right)\frac{dR_t}{R_t}+\frac{a(t,\tau)}{a_K}\frac{dP_K(r_t,t)}{P_K(r_t,t)}$

where

$$a_K = \frac{e^{-\beta K} - 1}{\beta}$$

Note that $\left(\frac{e^{-\beta K} - 1}{\beta}\right)\sigma_r = \sigma_{P,K}$, hence the K maturity zero coupon bond price is:

$$\frac{dP_K\left(r_t, t\right)}{P_K\left(r_t, t\right)} = \left(r_t + \sigma_{P,K}\xi_P\right)dt + \sigma_{P,K}dZ_{r,t} \qquad [23]$$

As nominal interest rates are characterized by a stochastic process, it is expected that stock price and bond price interact within the pension wealth dynamics. In this context, we let $\rho_{Sr} = dZ_{S,t}dZ_{r,t}$ be the constant correlation coefficient between the stock market and the interest rate. In the same vein, we let $\sigma_{Sr} = \sigma_S\sigma_r\rho_{Sr}$ be the covariance vector between the stock market and the interest rate. Note furthermore the intuitive correlation coefficient $\rho_{SP} = -\rho_{Sr}$ and $\rho_{Br} = -1$. The financial asset prices dynamics is then described by the following equations[27]

$$\begin{cases} \dfrac{dP_K}{P_K} = \left(r_t + a_K\sigma_r\xi_P\right)dt + a_K\sigma_r dZ_{r,t} \\ \dfrac{dS_t}{S_t} = \left(r_t + \lambda_S + am_S\right)dt + \sigma_S dZ_{S,t} \end{cases} \qquad [24]$$

where $dr_t = \beta\left(r_t - \overline{r}\right)dt + \sigma_r dZ_{r,t}$, $d\lambda_S = \kappa\left(\overline{\lambda} - \lambda_S\right)dt + \sigma_\lambda dZ_{S,t}$ and $dM_t = \left(1-a\right)\left(\mu_S - M_t\right)dt + \sigma_S dZ_{S,t}$ Taking into account all the elements presented until now, we are now able to determine the dynamics of the pension wealth:

$$dX_t = X_t\left[\pi_t\left(\left(r_t + \lambda_S + am_S\right)dt + \sigma_S dZ_{S,t}\right) + \left(1-\pi_t\right)\left(\left(r_t + a_K\sigma_r\xi_P\right)dt + a_K\sigma_r dZ_{r,t}\right)\right] \qquad [25]$$

and the Dynkin takes the following forms

$$\begin{aligned} DJ = {} & J_t + J_X dX + J_M dM + J_\lambda d\lambda + J_r dr + J_{XM}\left\langle dXdM\right\rangle + J_{X\lambda}\left\langle dXd\lambda\right\rangle + J_{Xr}\left\langle dXdr\right\rangle + J_{M\lambda}\left\langle dMd\lambda\right\rangle \\ & + J_{\lambda r}\left\langle d\lambda dr\right\rangle + J_{Mr}\left\langle dMdr\right\rangle + \frac{1}{2}J_{WW}\left\langle dX^2\right\rangle + \frac{1}{2}J_{MM}\left\langle dM^2\right\rangle + \frac{1}{2}J_{\lambda\lambda}\left\langle d\lambda^2\right\rangle + \frac{1}{2}J_{rr}\left\langle dr^2\right\rangle + \end{aligned} \qquad [26]$$

[27] It can easily be assumed that interest rate changes have a direct influence on the stock market dynamics which matches even more with the current financial market situation. Authors like Korn and Kraft (2001), Munk and Sorensen (2007) propose to include such interactions in the stock price dynamics.

The maximisation problem can be presented as follow:

$$J\left(X(t),M(t),\lambda(t),r(t),t,T\right)\equiv \underset{\pi_S,s\in[t,T]}{Max}\left\{E_t\left[U\left(X_T\right)\right]\right\} \quad [27]$$

To lighten the presentation, all the calculations are reported in the annex. The optimal asset weight is:

$$\pi_t=\Omega_{Sr}^{-1}\frac{J_X}{XJ_{XX}}\left(\lambda_S+am_S-a_K\sigma_r\xi_P\right)+\Omega_{Sr}^{-1}\frac{J_{XM}}{XJ_{XX}}\left(\left(\sigma_S^2+a_K\sigma_{Sr}\right)\right)+\Omega_{Sr}^{-1}\frac{J_{X\lambda}}{XJ_{XX}}\left(\sigma_S\sigma_\lambda+a_K\sigma_r\sigma_\lambda\rho_{Sr}\right)$$
$$+\Omega_{Sr}^{-1}\left(\sigma_{Sr}+a_K\sigma_r^2\right)\left(\frac{J_{Xr}}{XJ_{XX}}+a_K\right)$$

where with $\Omega_{Sr}=\left(\sigma_S dZ_{S,t}+a_K\sigma_r dZ_{r,t}\right)^2$ is the covariance of $\left(\sigma_S dZ_{S,t}+a_K\sigma_r dZ_{r,t}\right)$. Rearranging the previous equation the solution can be written as follow:

$$\pi_t=\underbrace{\Omega_{Sr}^{-1}\frac{J_X}{XJ_{XX}}\left(\left(\mu_t-\mu_P\right)+a\left(M_t-\mu_t\right)\right)}_{Proportion\ I}+\underbrace{\Omega_{Sr}^{-1}\frac{J_{XM}}{XJ_{XX}}\left(\left(\sigma_S^2+\sigma_{SP}\right)\right)}_{Proportion\ II}$$
$$+\underbrace{\Omega_{Sr}^{-1}\frac{J_{X\lambda}}{XJ_{XX}}\left(\sigma_S\sigma_\lambda+\sigma_P\sigma_\lambda\rho_{Sr}\right)}_{Proportion\ III}+\underbrace{\Omega_{Sr}^{-1}\left(\sigma_{Sr}+a_K\sigma_r^2\right)\left(\frac{J_{Xr}}{XJ_{XX}}+a_K\right)}_{Proportion\ IV} \quad [28]$$

As in the previous cases, the optimal solution is split into several funds and each of them is acting as a hedge against a specific risk. Note that the substitution of the cash asset with a bond asset exposes the pension wealth to the interest rate risk. Note furthermore, that the covariance between the two sources of noise of the two financial markets appears and affects all the hedging demand of the solution through Ω_{Sr}^{-1}.

In the first fund, we can notice that the cash asset return (r_t) has been substituted by instantaneous bond return. This modification can be interpreted as the new definition of the market price of risk as there is no more risk free asset in our framework. The second fund represents the hedging demand toward unexpected changes in momentum variables. The third fund aim is to hedge unfavorable changes in the stock market excess return and the fourth is intended to hedge unexpected changes in the interest rate.

4.2 A three assets portfolio

The pension investor is now allowed to invest the financial wealth into cash asset, bonds and stock. We amend the risky asset dynamics assuming that the bond prices influence the stock market price trough the dynamics of the short term interest rate. Hence, in line with Korn and al. (2001) the dynamics of S_t becomes[28]:

$$\frac{dS_t}{S_t}=\left(r_t+\lambda_S+am_S\right)dt+\sigma_S dZ_S+\sigma_{SP}dZ_r \qquad [29]$$

This new framework requires little changes in the way in which the financial wealth is invested. Let π_S, π_P and π_C be the proportion of the pension wealth invested respectively in stocks, bonds and cash. In this context the following constraint holds: $\pi_S+\pi_P+\pi_C=1$ which implies

$$dX_t=X_t\left[\pi_{S,t}\left[\frac{dS_t}{S_t}\right]+\pi_{P,t}\left[\frac{dP_t}{P_t}\right]+\left(1-\pi_{S,t}-\pi_{P,t}\right)\left[\frac{dB_t}{B_t}\right]\right] \qquad [30]$$

$$\frac{dX_t}{X_t}=\left[\pi_S\left(\lambda_S+am_S\right)+\pi_P\left(a_K\sigma_r\xi_P\right)+r\right]dt+\pi_S\sigma_S dZ_S+\left(\pi_S\sigma_{SP}+\pi_P\sigma_P\right)dZ_r$$

Using the same method as above, the optimal weight π_S and π_B can be deduced and thus the fraction of the pension wealth invested in the cash asset π_C. The infinitesimal operator remains unchanged – we use therefore thus the equation [23] as the Dynkin operator. The two first order conditions with respect to π_S and π_P lead to the following system

$$\pi_P^*=-\frac{J_X\left(a_K\sigma_r\xi_P\right)}{XJ_{XX}}-\frac{J_{XM}\sigma_P\sigma_S\rho_{sr}}{XJ_{XX}}-\frac{J_{X\lambda}\sigma_P\sigma_S\rho_{sr}}{XJ_{XX}}-\frac{J_{Xr}\sigma_P\sigma_r}{XJ_{XX}}-\pi_S\frac{\sigma_{SP}\sigma_P}{\sigma_P^2} \qquad [31]$$

$$\pi_S^*\left(\sigma_S^2+\sigma_{SP}^2\right)=-\frac{J_X\left(\lambda_S+am_S\right)}{XJ_{XX}}-\frac{J_{XM}\left(\sigma_S^2+\sigma_{SP}\sigma_S\rho_{sr}\right)}{XJ_{XX}}-\frac{J_{X\lambda}\left(\sigma_S\sigma_\lambda+\sigma_{SB}\sigma_S\rho_{sr}\right)}{XJ_{XX}}+\pi_P\sigma_P\sigma_{SP}+\frac{J_{Xr}\left(\sigma_{Sr}+\sigma_{SP}\sigma_r\right)}{XJ_{XX}} \qquad [32]$$

Using [29] and [30], we can deduce the exact form of π_S and π_P

$$\pi_S^*=-\frac{J_X}{XJ_{XX}}\left[\frac{\left(\lambda_S+am_S\right)}{\sigma_S^2}-\frac{\xi_P\sigma_{SP}}{\sigma_S^2}\right]-\frac{J_{XM}}{XJ_{XX}}-\frac{J_{X\lambda}}{XJ_{XX}}\frac{\sigma_\lambda}{\sigma_S}-\frac{J_{Xr}}{XJ_{XX}}\frac{\sigma_{Sr}}{\sigma_S^2} \qquad [33]$$

[28] Munk and Sorensen (2007) use Cholevski decomposition to link both markets

$$\pi_P^* = -\frac{J_X}{XJ_{XX}}\left[-\frac{\xi_P}{\sigma_P}\left(1-\sigma_{SP}\right)-\frac{\lambda_S\sigma_{SP}}{\sigma_P}-\frac{am_S\sigma_{SP}}{\sigma_P}\right]-\frac{J_{XM}}{XJ_{XX}}\left[\frac{\sigma_S\rho_{sr}}{\sigma_P}-\frac{\sigma_{SP}}{\sigma_P}\right] - \frac{J_{Xr}}{XJ_{XX}}\left[\frac{\sigma_r}{\sigma_P}-\frac{\sigma_{SP}}{\sigma_P}\frac{\sigma_{Sr}}{\sigma_S^2}\right]-\frac{J_{X\lambda}}{XJ_{XX}}\left[\frac{\sigma_S\rho_{sr}}{\sigma_P}-\frac{\sigma_\lambda}{\sigma_S}\frac{\sigma_{SP}}{\sigma_P}\right] \quad [34]$$

Knowing both π_S and π_P we can thus determine the wealth amount invested into cash asset. Turning to the equation [31] and [32] we observe some changes in particular on the mean variance portfolio side. Indeed we can notice how the bond market price of risk is modified due to covariance included in the solution. We can notice that the fraction of bond is, as expected, negatively impacted by the momentum variable. It can be noted in [32] that all the hedging demands are influenced by both bond market and stock market volatility. Nevertheless, due to the presence of covariance terms in the final solution, it becomes difficult to capture all the effects and the sensitivity. To fully understand all nuances of the solution, simulation exercises are required. These simulations will allow to evaluation the benefits or the losses coming from strategies based on market opportunities.

Conclusion and Extensions

In this first attempt, we tried to put forward how stock returns properties can be used to withdraw additional performances. Our empirical investigations showed that some equity markets present both momentum (serial autocorrelation tests and ARCH effect tests) and mean reversion (unit root and panel unit root test) in their returns. Based on these observations, we have introduced and extended the framework developed by Koijen and al (2009) which takes into account both momentum and mean reversion of stock return. Firstly, we have considered plan members with a finite investment horizon (the retirement date) and we allowed them to be invested into stock, bond and cash assets. The derivation of the optimal solution exhibits several separated funds in line with the number of source risks considered. We have observed that the stock momentum existence impacts the fractions of equity and bond held by the investors. Note moreover that this phenomenon is magnified as past stock returns help to predict current and future returns $(a \neq 0)$. The optimal solution also exhibits a component intended to hedge unfavourable change in the momentum variable.

Several extensions can be introduced to enhance this framework. Another source of abnormal stock returns comes from the existence of jumps which occurs after to unexpected news of figures. Against this backdrop, the stock market dynamics could be extended through the introduction of Poisson processes (see for instance Wu (2003), Longshaft, Lui, Pan (2006) and Chan and Purcal (2006)). In addition, one could include a regular contributions stream which feeds the pension wealth considering on the one hand a deterministic salary process and on the other hand a stochastic wage as Blake an al (2006). This assumption implies on the one hand to move away from the self financing strategy framework (see El Karoui and Jeanblean-Picqué, 1998) and on the other hand to be able to replicate the flow of contributions through financial markets. In the similar vein, other background risk could be considered as the mortality risk or the inflation risk as in Brennan and Xia (2000), Battochio and al. (2004) or Zhang and al. (2007). Furthermore, the simulation of the solution is best way to appreciate how financial market opportunities may influence the plan member asset holding behaviour and its pension wealth. We will explore this issue to reinforce the results of our study.

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Appendix 1 – Stock markets prices descriptive statistics

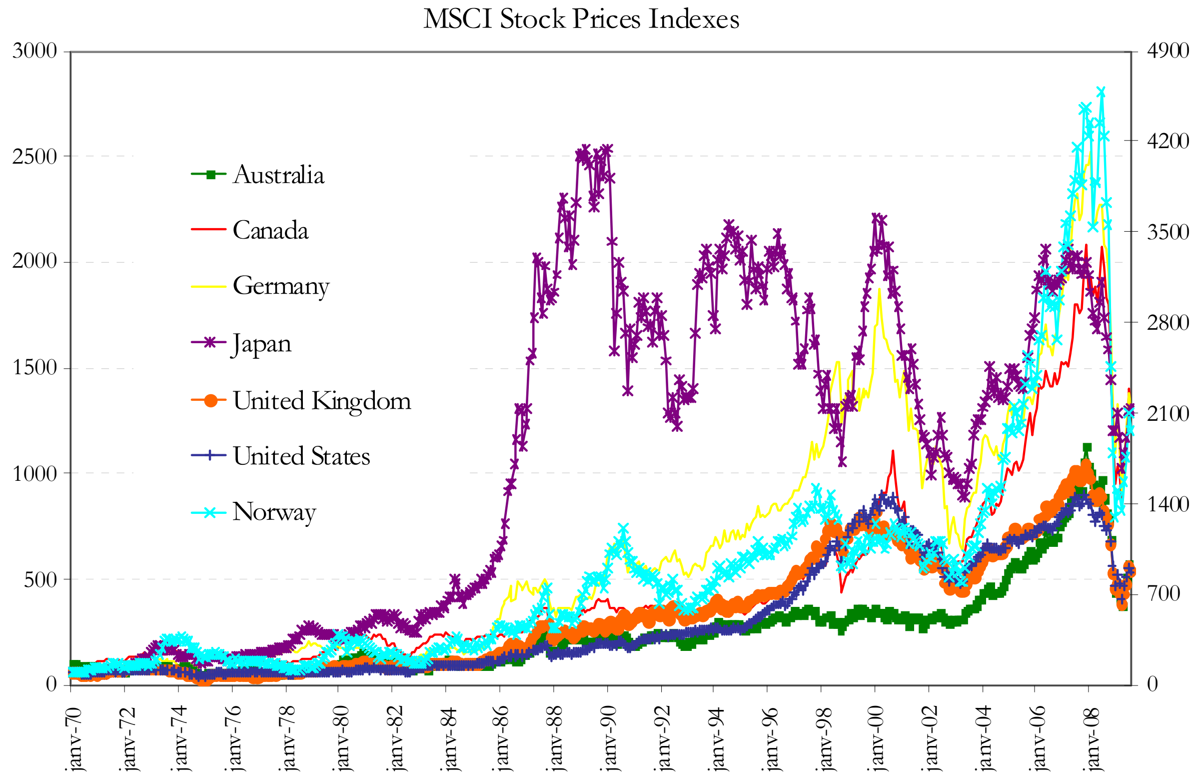

Figure 2: MSCI Stock index prices for a set of countries

Summary Statistics of National Stock Index Returns

	Average monthly returns	Annualised return	Standard Deviation	Annualised Std Deviation
Australia	0,63%	7,50%	7,05%	24,41%
Belguim	0,65%	7,77%	6,18%	21,40%
Canada	0,72%	8,58%	5,84%	20,25%
France	0,76%	9,18%	6,69%	23,17%
Germany	0,75%	8,99%	6,36%	22,04%
Hong Kong	1,42%	17,06%	10,58%	36,65%
Italy	0,52%	6,22%	7,41%	25,67%
Japan	0,86%	10,33%	6,57%	22,77%
Netherlands	0,74%	8,86%	5,62%	19,49%
Norway	0,96%	11,53%	8,06%	27,92%
Spain	0,58%	7,02%	6,73%	23,32%
Sweden	1,04%	12,51%	7,06%	24,47%
United Kingdom	0,67%	8,01%	6,55%	22,69%
United States	0,56%	6,76%	4,53%	15,68%
World	0,58%	6,94%	4,40%	15,22%

MSCI data : Monthly returns from 1970 to 2009

Figure 3: Descriptive statistics

Appendix 2 – Random walk tests

Variance ratio

Holding period in months	Belgium	Canada	France	Germany	HK	Italy	Japon	NH	Norway	Spain	UK	US
2	**1.15853**	**1.0879948**	**1.083697**	**1.048912**	**1.0878189**	**1.08385**	**1.101141**	**1.041341**	**1.131481**	**1.104552**	**1.075149**	**1.10637**
3	1.235961	1.0907203	1.09187	1.05361	1.1068223	1.076381	1.116514	1.052658	1.186962	1.097643	1.094323	1.111598
4	1.290752	1.1117472	1.142915	1.083931	1.0971671	1.11618	1.162463	1.065342	1.270538	1.095213	1.160463	1.139482
5	1.359497	1.115299	1.181818	1.112537	1.0668919	1.165921	1.214236	1.06355	1.29162	1.097797	1.189281	1.163892
6	1.41529	1.1343948	1.224279	1.13078	1.0365283	1.211803	1.259505	1.077521	1.295625	1.11873	1.215441	1.155266
7	1.451137	1.1432696	1.252533	1.153884	0.992968	1.270406	1.291851	1.074889	1.303896	1.172182	1.252474	1.14203
8	1.475349	1.1316308	1.266623	1.1786	0.9794583	1.318293	1.325482	1.063044	1.2971	1.21505	1.266451	1.129554
9	1.506043	1.1271677	1.275399	1.203735	0.9797146	1.358944	1.372024	1.062675	1.287805	1.256436	1.283693	1.118986
10	**1.537524**	**1.1272408**	**1.280464**	**1.207019**	**0.9945741**	**1.39978**	**1.424025**	**1.063765**	**1.291031**	**1.301576**	**1.293902**	**1.117587**
11	1.554903	1.1189447	1.290042	1.205121	1.0182672	1.428162	1.469442	1.066054	1.293744	1.360062	1.303311	1.116875
12	1.570243	1.115113	1.296445	1.207662	1.031837	1.463044	1.520358	1.065809	1.297897	1.413364	1.318058	1.117506
13	1.585816	1.1031797	1.301903	1.20848	1.0439674	1.501784	1.56419	1.077866	1.293804	1.455164	1.332269	1.118346
14	1.590303	1.0844064	1.311636	1.210553	1.0387776	1.54017	1.595949	1.081992	1.282301	1.492602	1.339272	1.105306
15	1.592532	1.0608642	1.322836	1.209775	1.0232668	1.56848	1.627138	1.086809	1.268801	1.531726	1.344906	1.084848
16	1.593199	1.0370349	1.325939	1.196899	1.0103879	1.587852	1.652547	1.076673	1.255972	1.563788	1.333664	1.066924
17	1.604658	1.0148826	1.330159	1.186573	0.9978763	1.60229	1.669379	1.075946	1.237766	1.604155	1.324048	1.055535
18	1.611431	0.9941539	1.321868	1.170764	0.9870918	1.606629	1.677717	1.071732	1.21971	1.638762	1.311283	1.035564
19	1.614591	0.9754684	1.306798	1.155701	0.9747115	1.604366	1.688988	1.062711	1.196203	1.67108	1.294779	1.017832
20	**1.611877**	**0.9542427**	**1.29397**	**1.143516**	**0.9641546**	**1.59754**	**1.699014**	**1.046158**	**1.17118**	**1.702956**	**1.272989**	**1.003772**

Figure 4: Variance ratio Calculation

Lo and MacKinlay Tests according three Holding Periods *(q)*

	Holding Period (q)	2	10	20		Holding Period (q)	2	10	20
Germany	VR(q)	1.048912	1.207019	1.143516	**Hong Kong**	VR(q)	1.0878189	0.9945741	0.9641546
	Z(q)	1.0649	1.3349	0.6287		Z(q)	**1.9119***	-0.0350	-0.1570
	Z*(q)	0.7479	1.0139	0.5200		Z*(q)	1.3608	-0.0242	-0.1184
Belgium	VR(q)	1.15853	1.537524	1.611877	**Italy**	VR(q)	1.08385	1.39978	1.59754
	Z(q)	**3.4514**	**3.4661**	**2.6804**		Z(q)	1.8255	**2.5778**	**2.6176**
	Z*(q)	1.7053	**2.1421**	1.8910		Z*(q)	1.3876	**2.1967**	**2.3465**
Canada	VR(q)	1.0879948	1.1272408	0.9542427	**Japan**	VR(q)	1.101141	1.424025	1.699014
	Z(q)	**1.915782***	0.8205	-0.2004		Z(q)	**2.2020**	**2.7341**	**3.0625**
	Z*(q)	1.2752	0.6107	-0.1619		Z*(q)	1.8792	**2.4644**	**3.0621**
France	VR(q)	1.083697	1.280464	1.29397	**Nether-lands**	VR(q)	1.131481	1.291031	1.17118
	Z(q)	**1.822213***	1.8085	1.2878		Z(q)	0.9001	0.4112	0.2022
	Z*(q)	1.4510	1.5347	1.1498		Z*(q)	0.6176	0.3161	0.1678
United States	VR(q)	1.061315	1.150496	1.158532	**United Kingdom**	VR(q)	1.10637	1.117587	1.003772
	Z(q)	1.3349	0.9704	0.6944		Z(q)	**2.3158**	0.7582	0.0165
	Z*(q)	0.8228	0.6988	0.5560		Z*(q)	1.4624	0.5205	0.0122
Spain	VR(q)	1.104552	1.301576	1.702956	**Norway**	VR(q)	1.131481	1.291031	1.17118
	Z(q)	**2.276256**	1.944615	**3.079421**		Z(q)	**2.8225**	1.8766	0.7988
	Z*(q)	1.797124	1.590642	2.716869		Z*(q)	1.8240	1.4424	0.6329
Sweeden	VR(q)	1.075149	1.293902	1.272989	**Australia**	VR(q)	1.0355078	0.9147659	0.8023503
	Z(q)	1.6361	**1.895128***	1.1959		Z(q)	0.7730	-0.5496	-0.8658
	Z*(q)	1.1788	1.5122	1.0207		Z*(q)	0.6444	-0.4549	-0.7588

* Random Walk hypothesis is rejected for a 10 % thershold

Figure 5: Lo and MacKinlay test

Chow and Denning Tests for the 2, 10 and 20 months holding periods

Belgium		**Hong Kong**	
Z(n)	**3.466051**	Z(n)	1.911954
Z*(n)	2.142063	Z*(n)	1.36082
Canada		**Italy**	
Z(n)	1.915782	Z(n)	**2.617628**
Z*(n)	1.27519	Z*(n)	**2.34654**
France		**Japan**	
Z(n)	1.822213	Z(n)	**3.0621**
Z*(n)	1.534671	Z*(n)	**2.8271**
Germany		**Netherlands**	
Z(n)	1.334892	Z(n)	0.900058
Z*(n)	1.013949	Z*(n)	0.6176293
Sweden		**Australia**	
Z(n)	1.8951	Z(n)	0.8658385
Z*(n)	1.5122	Z*(n)	0.7588312
Norway		**United Kingdom**	
Z(n)	**2.862553**	Z(n)	**2.3158**
Z*(n)	1.824094	Z*(n)	1.4624
Spain		**United States**	
Z(n)	**3.079421**	Z(n)	1.334922
Z*(n)	**2.716869**	Z*(n)	0.8228615

Figure 6: Chow and Denning test

Critical Values for the 1%, 5% and 10% thersholds

10%	5%	1%
2,114054	2,387738	2,934161

Chow and Denning (1993)

Figure 7: Chow Denning Critical Values

Appendix 3 – Momentum tests results

Stock Returns* Autocorrelation function for diffrent time horizons

	Autocorrelation orders				
	1 Month	**2 Months**	**3 Months**	**4 Months**	**6 Months**
Australia	0.036	-0.056	0.053	-0.014	-0.015
	0.6096	2.0941	3.4501	3.5401	3.7309
Belguim	**0.137**	0.025	0.038	**0.104**	0.031
	8.9728	9.2727	9.9528	15.191	15.714
Canada	**0.095**	-0.05	**0.084**	0.009	0.034
	4.3149	5.5121	8.8798	8.9228	10.166
France	**0.077**	-0.027	**0.118**	0.037	0.024
	2.8124	3.1604	9.794	10.463	11.202
Germany	0.045	-0.033	**0.09**	0.055	0.07
	0.9549	1.4747	5.32	6.776	9.3551
Hong Kong	0.075	-0.009	-0.021	-0.049	-0.077
	2.7128	2.7511	2.962	4.1318	7.4563
Italy	**0.096**	-0.051	**0.116**	**0.088**	**0.114**
	4.4049	5.6486	12.111	15.816	22.528
Japan	**0.096**	-0.035	**0.093**	0.074	0.001
	4.4223	5.0119	9.1425	11.756	12.027
Netherlands	0.03	-0.015	0.039	-0.012	-0.016
	0.4175	0.5214	1.2634	1.3273	1.6198
Norway	**0.123**	-0.001	**0.126**	-0.056	0.068
	7.2238	7.2242	14.805	16.297	18.823
Spain	**0.107**	-0.07	0.027	0.039	**0.136**
	5.4365	7.7765	8.1289	8.8431	19.045
Sweden	0.068	-0.019	**0.135**	0.019	0.073
	2.1737	2.3547	11.053	11.228	14.392
United Kingdom	**0.12**	-0.063	0.075	0.041	-0.015
	6.8594	8.7427	11.438	12.238	17.015
United States	0.053	-0.059	**0.077**	0.059	-0.051
	1.3513	3.0406	5.8935	7.5707	13.042
World	**0.124**	-0.06	**0.088**	0.054	-0.019
	7.3374	9.0304	12.71	14.122	17.096

* We considere arithmetics returns for the autocorelation function calculation. For each serial correlation the Ljung-Box Q statistics is provided. The Ljung-Box Q stat follows a $\chi^2(r)$ distribution. Under H_0 the data are randomwhile the alternative assumption asserts taht data are nor random

Figure 8: Autocorrelation function[29]

[29] The test statistics is $Q=n(n+2)\sum_{j=1}^{h}\frac{\hat{\rho}_j^2}{n-j}$ and if $Q>\chi^2_{1-\alpha,h}$ the randomness assumption is rejected.

Test of serial correlation - Test of Breush Godfrey based on a LM test*

	Lag orders				
	1 Month	2 Months	3 Months	4 Months	6 Months
Australia	0.415839	1.107209	2.020735	2.814826	3.979617
Belguim	0.161364	0.354418	3.98637	4.376018	4.625876
Canada	0.412423	2.509757	3.242475	3.569196	4.233621
France	0.424569	0.483875	6.010116	6.112063	6.652313
Germany	8.33E-05	0.795852	2.725026	3.4776	4.635895
Hong Kong	0.024872	0.132197	0.456871	1.558107	5.830354
Italy	1.469127	2.052435	6.314956	8.487739	12.28602
Japan	0.810149	1.739485	4.233336	5.815307	6.257848
Netherlands	0.005119	0.025248	0.154713	0.444493	2.143659
Norway	0.031548	2.695135	8.985135	12.15464	12.54738
Spain	2.093151	2.929268	2.929282	3.141317	10.0084
Sweden	0.044843	2.050991	8.297657	8.311014	8.771024
United Kingdom	1.934542	5.690287	5.716904	5.874051	10.94116
United States	2.516795	2.696529	4.194688	4.374874	12.3175
World					

The Lagrange Multiplier statistics follows a $\chi^2(r)$ distrisbution with r degrees of freedom

Figure 9: Serial correlation tests – Breush and Godfrey

Stock Returns ARCH test effects

	Lag orders				
	1 Month	2 Months	3 Months	4 Months	6 Months
Australia	3.525716	7.957291	12.75882	16.03891	17.2966
Belguim	20.35957	23.47471	25.34111	39.23106	40.41236
Canada	20.74349	43.94258	44.7169	47.89538	59.21512
France	8.480475	10.99303	12.80748	26.13383	26.19923
Germany	35.26215	46.9464	58.14226	59.46912	59.47553
Hong Kong	7.007997	33.79224	34.53335	36.00727	35.86041
Italy	25.69891	25.59263	26.95716	28.27686	34.3901
Japan	5.660533	6.830947	7.187574	7.391117	18.93619
Netherlands	18.7159	22.63028	24.1024	33.1642	34.29005
Norway	26.17814	32.19243	32.66894	32.69976	34.15794
Spain	8.883312	27.62507	28.89939	28.9138	29.37956
Sweden	32.20523	41.80325	48.47078	48.45719	56.91598
United Kingdom	5.541116	13.18601	5.716904	14.49158	21.35025
United States	46.63781	51.25146	60.07631	62.96126	63.68958
World	44.38476	62.35389	69.63954	73.99416	81.5079

The ARCH test follows a $\chi^2(q)$ distrisbution with q degrees of freedom

Figure 10: ARCH effect tests

Appendix 4 – Mean reversion tests

Individual Stock Returns Unit Root test

	Augmented Dickey Fuller Test		Philipps and Perron test ‡		KPSS test based on the LM tests	
	t-statistics	*p-value*	*t-statistics*	*p-value*	*t-statistics*	*p-value*
Australia	-20.82358	*0.00000*	-20.81642	*0.00000*	0.066263	
Belguim	-18.74245	*0.00000*	-18.9657	*0.00000*	0.119767	
Canada*	-19.90793	*0.00000*	-19.89734	*0.00000*	0.033214	
France	-19.93525	*0.00000*	-20.14016	*0.00000*	0.079751	
Germany*	-20.82119	*0.00000*	-20.82032	*0.00000*	0.060207	
Hong Kong**	-20.15868	*0.00000*	-20.10034	*0.00000*	0.227343	
Italy	-19.75971	*0.00000*	-19.94414	*0.00000*	0.113972	
Japan**	-19.70694	*0.00000*	-19.83627	*0.00000*	0.406579	
Netherlands**	-21.1342	*0.00000*	-21.13401	*0.00000*	0.137045	
Norway	-19.03286	*0.00000*	-19.30144	*0.00000*	0.042674	
Spain	-19.34976	*0.00000*	-19.30565	*0.00000*	0.24734	
Sweden **	-20.3142	*0.00000*	-20.42299	*0.00000*	0.070988	
United Kingdom	-19.13644	*0.00000*	-19.11589	*0.00000*	0.11451	
United States*	-20.67517	*0.00000*	-20.69496	*0.00000*	0.15578	
World Index*	-19.25279	0.00000	-19.25406	0.00000	0.136232	

* The Constant is significant at a 10% thershold. *** The Constant is significant at a 5% thershold
‡ The Newey-West

Figure 11: Individual Unit Root tests (ADF, PP, KPSS)

Common Unit root test

	Levin Lin Chu test		Breitung test		Hadri test	
	t-statistics	*p-value*	*t-statistics*	*p-value*	*t-statistics*	*p-value*
With constant and drift	-33.0112	0.00000	-20.2616	0.00000	-0.67694	0.7508
With Contant only	-23.0855	0.00000	-19.1308	0.00000	0.33561	0.3686
Neither constant nor drift	-33.8061	0.00000	-28.5319	0.00000	-	-

Individual Unit root test

	Im Perasan Shin test		Maddala and Wu test - ADF Fisher Chi-square		Maddala and Wu test - ADF Choi Z-stat	
	W-statistics	*p-value*	*t-statistics*	*p-value*	*t-statistics*	*p-value*
With constant and drift	-33.8638	0.00000	963.399	0.00000	-29.4884	0.00000
With Contant only	-33.2626	0.00000	997.237	0.00000	-30.0666	0.00000
Neither constant nor drift	-	-	1027.45	0.00000	-30.5662	0.00000

Figure 12: Panel Unit root test

Appendix 5 – Model Calculations

- **Dynamics of M_t**

We derive the dynamics of the past cumulated return variable M_t . We first introduce some notations

$$M_t = f(t,U_t) \text{ where } U_t = \int_0^t e^{u} \frac{dS_u}{S_u}$$

$$\text{and } f(t,X) = e^{-t} X$$

Using the Itô lemma, we derive the dynamics of M_t

$$dM_t = \frac{\partial f}{\partial t}(t,U_t)dt + \frac{\partial f}{\partial X}(t,U_t)dU_t + \frac{1}{2}\frac{\partial^2 f}{\partial X^2}\langle dU_t \rangle$$

$$\text{where } \frac{\partial f}{\partial t} = e^{-t} X, \ \frac{\partial f}{\partial X} = e^{-t} \text{ and } \frac{\partial^2 f}{\partial X^2} = 0 \text{ and } dU_t = e^t \times \frac{dS_t}{S_t}$$

Substituting the previous derivative into the dynamics of M_t we get the following expression:

$$dM_t = \underbrace{-e^{-t}\partial(U_t)}_{-M_t} dt + e^{-t} \times e^t \frac{dS_t}{S_t}$$

$$dM_t = -M_t dt + \frac{dS_t}{S_t}$$

$$dM_t = -M_t dt + \left(r_t + \mu_{S,t} - r_t + a\left(M_t - \mu_{S,t}\right)\right)dt + \sigma_S dZ_t$$

$$dM_t = (1-a)\left(M_t - \mu_{S,t}\right)dt + \sigma_S dZ_t$$

- **Description of the dynamics of W_t in the simple case**

$$\frac{dX_t}{X_t} = \left[\pi_t \left[\frac{dS_t}{S_t}\right] + (1\text{-}\pi_t)\left[\frac{dB_t}{B_t}\right]\right]$$

$$\frac{dX_t}{X_t} = \pi_t\left(r_t + \lambda_S + am_t\right)dt + \pi_t\sigma_S dZ_{S,t} + (1\text{-}\pi_t)rdt$$

$$\frac{dX_t}{X_t} = \pi_t\left(\lambda_S + am_t\right)dt + \pi_t\sigma_S dZ_{S,t} + rdt$$

Portfolio solution within the simplest framework

In a first step, we calculate the infinitesimal operator *DJ* which will be used later in the calculation of the first order condition.

$$DJ = J_t + J_W dW + J_M dM + \frac{1}{2} J_{WW} \left\langle dW^2 \right\rangle + \frac{1}{2} J_{MM} \left\langle dM^2 \right\rangle + \frac{1}{2} J_{WM} \left\langle dWdM \right\rangle$$

Replacing on the HJB equation the dynamics of the pension wealth, we get:

$$\underset{\pi_t}{Max} \left\{ E_t \begin{bmatrix} J_t + J_X X\left[\left(\pi_t\left(\lambda_S + a\left(M_t - \mu_{S,t}\right)\right) + r\right)dt + \pi_t \sigma_S dZ_{S,t}\right] + J_M\left(\left(1-a\right)\left(\mu_S - M_t\right)dt + \sigma_S dZ_{S,t}\right) \\ + \frac{1}{2} J_{XX} X^2 \left(\pi_t \sigma_S\right)^2 + \frac{1}{2} J_{MM} \sigma_S^2 dt + \pi_t \sigma_S^2 dt X J_{XM} \end{bmatrix} \right\} = 0$$

With the following variance - covariance structure

$$\left\langle dXdM \right\rangle = X\pi_t \sigma_S^2 dt$$

$$\left\langle dX^2 \right\rangle = X^2 \pi_t^2 \sigma_S^2 dt$$

$$\left\langle dM^2 \right\rangle = \sigma_S^2 dt$$

Hence

$$\underset{\pi_t}{Max} \left\{ \begin{matrix} J_t + J_X X\left[\left(\pi_t\left(\lambda_S + a\left(M_t - \mu_{S,t}\right)\right) + r\right)\right] + J_M\left(\left(1-a\right)\left(\mu_S - M_t\right)\right) \\ + \frac{1}{2} J_{XX} X^2 \left(\pi_t \sigma_S\right)^2 + \frac{1}{2} J_{MM} \sigma_S^2 + \pi_t \sigma_S^2 X J_{XX} \end{matrix} \right\} = 0$$

$$\underset{\pi_t}{Max}\, E\left[DJ\right] = 0 \Leftrightarrow J_X X\left(\lambda_S + a\left(M_t - \mu_{S,t}\right)\right) + \pi_t \sigma_S^2 J_{XX} X^2 + \sigma_S^2 X J_{XX} = 0$$

$$\pi^* = \left[\frac{1}{\sigma_S^2 J_{XX} X^2}\right]\left[-J_X X\left(\lambda_S + a\left(M_t - \mu_{S,t}\right)\right) - \sigma_S^2 X J_{XX}\right]$$

Thus the optimal portfolio π^* is the solution of the following relation

$$\pi^* = \underbrace{-\frac{J_W}{J_{WW} X} \frac{\mu_S - r_t}{\sigma_S^2}}_{Proportion\ I} \underbrace{- \frac{J_W}{J_{WW} X} \frac{a\left(M_t - \mu_S\right)}{\sigma_S^2}}_{Proportion\ II} \underbrace{- \frac{J_{WM}}{J_{WW} X}}_{Proportion\ III}$$

Optimal portfolio with stochastic excess returns

As before the first step is devoted the Dynkin operator calculation:

$$DJ = J_t + J_X dX + J_M dM + J_\lambda d\lambda + \frac{1}{2} J_{WW} \left\langle dX^2 \right\rangle + \frac{1}{2} J_{MM} \left\langle dM^2 \right\rangle + \frac{1}{2} J_{\lambda\lambda} \left\langle d\lambda^2 \right\rangle$$
$$+ J_{WM} \left\langle dXdM \right\rangle + J_{X\lambda} \left\langle dXd\lambda \right\rangle + J_{M\lambda} \left\langle dMd\lambda \right\rangle$$

With the following variance - covariance structure

$$\left\langle dXdM \right\rangle = X\pi_t \sigma_S^2 dt \qquad \left\langle dXd\lambda \right\rangle = X\pi_t \sigma_S \sigma_\lambda dt$$

$$\left\langle dM^2 \right\rangle = \sigma_S^2 dt \qquad \left\langle dMd\lambda \right\rangle = X\pi_t \sigma_\lambda \sigma_S dt$$

$$\left\langle dX^2 \right\rangle = X^2 \pi_t^2 \sigma_S^2 dt \qquad \left\langle d\lambda^2 \right\rangle = \sigma_\lambda^2 dt$$

We are now able to write $E_t[DJ]$:

$$\max_{\pi_t} E[DJ] = \max \left\{ \begin{array}{r} J_t + XJ_X \left(\pi_t \left(\lambda_S + a\left(M_t - \mu_{S,t} \right) \right) + r \right) + J_M \left((1-a)(\mu_S - M_t) \right) + J_\lambda \left(\kappa \left(\overline{\lambda} - \lambda_S \right) \right) \\ + \frac{1}{2} J_{XX} X^2 \left(\pi_t \sigma_S \right)^2 + \frac{1}{2} J_{MM} \sigma_S^2 dt + \frac{1}{2} J_{\lambda\lambda} \sigma_\lambda^2 dt + J_{XM} X\pi_t \sigma_S^2 dt + J_{X\lambda} X\pi_t \sigma_S \sigma_\lambda dt + J_{M\lambda} \sigma_S \sigma_\lambda dt \end{array} \right\}$$

$$\max_{\pi_t} E[DJ] = 0 \Leftrightarrow XJ_X \left(\lambda_S + a\left(M_t - \mu_{S,t} \right) \right) + \pi_t \sigma_S^2 J_{XX} X^2 + J_{XM} X\sigma_S^2 + J_{X\lambda} X\sigma_{S\lambda} = 0$$

$$\pi^* = \left[\frac{1}{\sigma_S^2 J_{XX} X^2} \right] \left[-XJ_X \left(\lambda_S + a\left(M_t - \mu_{S,t} \right) \right) - J_{XM} X\sigma_S^2 - J_{X\lambda} X\sigma_{S\lambda} \right]$$

Which finally leads to the definition of π^*

$$\pi_t^* = \underbrace{-\frac{J_X}{J_{XX} X_t} \frac{\mu_S - r_t}{\sigma_S^2}}_{Proportion\ I} \underbrace{- \frac{J_X}{J_{XX} X_t} \left(\frac{a\left(M_t - \mu_t \right)}{\sigma_S^2} - \frac{J_{XM}}{J_X} \right)}_{Proportion\ II} \underbrace{- \frac{J_{X\lambda}}{J_{XX} X_t} \frac{\sigma_\lambda}{\sigma_S}}_{Proportion\ III}$$

Optimal portfolio with stochastic short term interest rate

$$dX_t = X_t\left[\left(\pi_t\left(\lambda_S + am_S - a_K\sigma_r\xi_P\right) + \left(r_t + \sigma_{P,K}\xi_P\right)\right)dt + \pi_t\left(\sigma_S dZ_{S,t} + a_K\sigma_r dZ_{r,t}\right) - a_K\sigma_r dZ_{r,t}\right]$$

The next step consists in calculating the infinitesimal operator where

$$\begin{aligned} DJ = {} & J_t + XJ_X\left[\left(\pi_t\left(\lambda_S + am_S - a_K\sigma_r\xi_P\right) + \left(r_t + \sigma_{P,K}\xi_P\right)\right)dt + \pi_t\left(\sigma_S dZ_{S,t} + a_K\sigma_r dZ_{r,t}\right) - a_K\sigma_r dZ_{r,t}\right] \\ & + J_M\left(\left(1-a\right)\left(\mu_S - M_t\right)dt + \sigma_S dZ_{S,t}\right) + J_\lambda\left(\kappa\left(\overline{\lambda} - \lambda_S\right)dt + \sigma_\lambda dZ_{S,t}\right) + J_r\left[\beta\left(r_t - \overline{r}\right)dt + \sigma_r dZ_{r,t}\right] \\ & + XJ_{XM}\left(\pi_t\left(\sigma_S^2 dt + a_K\sigma_r\sigma_S\rho_{Sr}\right) - a_K\sigma_r\sigma_S\rho_{Sr}\right) + XJ_{X\lambda}\left(\pi_t\left(\sigma_S\sigma_\lambda dt + a_K\sigma_r\sigma_\lambda\rho_{Sr}\right) - a_K\sigma_r\sigma_\lambda\rho_{Sr}\right) \\ & + XJ_{Xr}\left(\pi_t\left(\sigma_r\sigma_S\rho_{Sr} + a_K\sigma_r^2 dt\right) - a_K\sigma_r^2 dt\right) + J_{M\lambda}\sigma_S\sigma_\lambda dt + J_{Mr}\sigma_r\sigma_S\rho_{Sr} + J_{\lambda r}\sigma_r\sigma_S\rho_{Sr} \\ & + \frac{1}{2}X^2 J_{XX}\left[\pi_t^2\left(\sigma_S dZ_{S,t} + a_K\sigma_r dZ_{r,t}\right)^2 - \pi_t\left(a_K\sigma_r\sigma_S\rho_{Sr} + a_K^2\sigma_r^2 dt\right)\right] + \frac{1}{2}J_{MM}\sigma_S^2 dt + \frac{1}{2}J_{\lambda\lambda}\sigma_\lambda^2 dt + \frac{1}{2}J_{rr}\sigma_r^2 dt \end{aligned}$$

Before concentrating on the optimization study we first provide the mathematical notation of $E_t\left[DJ\right]$

$$\begin{aligned} E_t\left[DJ\right] = {} & J_t + XJ_X\left[\left(\pi_t\left(\lambda_S + am_S - a_K\sigma_r\xi_P\right) + \left(r_t + \sigma_{P,K}\xi_P\right)\right)\right] + J_M\left(\left(1-a\right)\left(\mu_S - M_t\right)\right) \\ & + J_\lambda\left(\kappa\left(\overline{\lambda} - \lambda_S\right)\right) + J_r\left[\beta\left(r_t - \overline{r}\right)\right] + XJ_{XM}\left(\pi_t\left(\sigma_S^2 + a_K\sigma_r\sigma_S\rho_{Sr}\right) - a_K\sigma_r\sigma_S\rho_{Sr}\right) \\ & + XJ_{X\lambda}\left(\pi_t\left(\sigma_S\sigma_\lambda + a_K\sigma_r\sigma_\lambda\rho_{Sr}\right) - a_K\sigma_r\sigma_\lambda\rho_{Sr}\right) + XJ_{Xr}\left(\pi_t\left(\sigma_r\sigma_S\rho_{Sr} + a_K\sigma_r^2\right) - a_K\sigma_r^2\right) + J_{\lambda r}\sigma_r\sigma_S\rho_{Sr} \\ & + J_{M\lambda}\sigma_S\sigma_\lambda + J_{Mr}\sigma_r\sigma_S\rho_{Sr} + \frac{1}{2}X^2 J_{XX}\left[\pi_t^2\underbrace{\left(\sigma_S dZ_{S,t} + a_K\sigma_r dZ_{r,t}\right)^2}_{\Omega_{Sr}} - \pi_t\left(a_K\sigma_r\sigma_S\rho_{Sr} + a_K^2\sigma_r^2\right)\right] \\ & + \frac{1}{2}J_{MM}\sigma_S^2 + \frac{1}{2}J_{\lambda\lambda}\sigma_\lambda^2 + \frac{1}{2}J_{rr}\sigma_r^2 \end{aligned}$$

Let Ω_{Sr} the covariance of $\left(\sigma_S dZ_{S,t} + a_K\sigma_r dZ_{r,t}\right)$. We turn now to the calculation of the first order condition

$$\begin{aligned} \max_{\pi_S, s\in[t,T]}\left\{E_t\left[DJ\right]\right\} = 0 \Leftrightarrow {} & XJ_X\left(\lambda_S + am_S - a_K\sigma_r\xi_P\right) + XJ_{XM}\left(\left(\sigma_S^2 + a_K\sigma_r\sigma_S\rho_{Sr}\right)\right) + XJ_{X\lambda}\left(\sigma_S\sigma_\lambda + a_K\sigma_r\sigma_\lambda\rho_{Sr}\right) \\ & + XJ_{Xr}\left(\sigma_r\sigma_S\rho_{Sr} + a_K\sigma_r^2\right) + X^2 J_{XX}\left[\pi_t\underbrace{\left(\sigma_S dZ_{S,t} + a_K\sigma_r dZ_{r,t}\right)^2}_{\Omega_{Sr}} - \left(a_K\sigma_r\sigma_S\rho_{Sr} + a_K^2\sigma_r^2\right)\right] = 0 \end{aligned}$$

$$\begin{aligned} -\pi_t X^2 J_{XX}\left[\Omega_{Sr}\right] = {} & -X^2 J_{XX}\left(a_K\left(\sigma_{Sr} + a_K\sigma_r^2\right)\right) + XJ_X\left(\lambda_S + am_S - a_K\sigma_r\xi_P\right) + XJ_{XM}\left(\left(\sigma_S^2 + a_K\sigma_r\sigma_S\rho_{Sr}\right)\right) \\ & + XJ_{X\lambda}\left(\sigma_S\sigma_\lambda + a_K\sigma_r\sigma_\lambda\rho_{Sr}\right) + XJ_{Xr}\left(\sigma_r\sigma_S\rho_{Sr} + a_K\sigma_r^2\right) \end{aligned}$$

$$-\pi_t\left[\Omega_{Sr}\right]=-\left(a_K\left(\sigma_{Sr}+a_K\sigma_r^2\right)\right)+\frac{J_X}{XJ_{XX}}\left(\lambda_S+am_S-a_K\sigma_r\xi_P\right)+\frac{J_{XM}}{XJ_{XX}}\left(\left(\sigma_S^2+a_K\sigma_r\sigma_S\rho_{Sr}\right)\right)$$

$$+\frac{J_{X\lambda}}{XJ_{XX}}\left(\sigma_S\sigma_\lambda+a_K\sigma_r\sigma_\lambda\rho_{Sr}\right)+\frac{J_{Xr}}{XJ_{XX}}\left(\sigma_r\sigma_S\rho_{Sr}+a_K\sigma_r^2\right)$$

which leads with our notations to the following solution:

$$\pi_t=\Omega_{Sr}^{-1}\frac{J_X}{XJ_{XX}}\left(\lambda_S+am_S-a_K\sigma_r\xi_P\right)+\Omega_{Sr}^{-1}\frac{J_{XM}}{XJ_{XX}}\left(\sigma_S^2+a_K\sigma_{Sr}\right)+\Omega_{Sr}^{-1}\frac{J_{X\lambda}}{XJ_{XX}}\left(\sigma_S\sigma_\lambda+a_K\sigma_r\sigma_\lambda\rho_{Sr}\right)$$

$$+\Omega_{Sr}^{-1}\left(\sigma_{Sr}+a_K\sigma_r^2\right)\left(\frac{J_{Xr}}{XJ_{XX}}+a_K\right)$$

using the appropriate notations:

$$\pi_t=\underbrace{\Omega_{Sr}^{-1}\frac{J_X}{XJ_{XX}}\left(\left(\mu_t-\mu_P\right)+a\left(M_t-\mu_t\right)\right)}_{Proportion\ I}+\underbrace{\Omega_{Sr}^{-1}\frac{J_{XM}}{XJ_{XX}}\left(\left(\sigma_S^2+\sigma_{SP}\right)\right)}_{Proportion\ II}$$

$$+\underbrace{\Omega_{Sr}^{-1}\frac{J_{X\lambda}}{XJ_{XX}}\left(\sigma_S\sigma_\lambda+\sigma_P\sigma_\lambda\rho_{Sr}\right)}_{Proportion\ III}+\underbrace{\Omega_{Sr}^{-1}\left(\sigma_{Sr}+a_K\sigma_r^2\right)\left(\frac{J_{Xr}}{XJ_{XX}}+a_K\right)}_{Proportion\ IV}$$

Optimal portfolio with three assets and new stock market dynamics

$$dX_t = X_t\left[\pi_S\left[\frac{dS_t}{S_t}\right] + \pi_P\left[\frac{dP_t}{P_t}\right] + (1-\pi_S-\pi_P)\left[\frac{dB_t}{B_t}\right]\right]$$

$$\frac{dX_t}{X_t} = \left[\pi_S(\lambda_S + am_S) + \pi_P(a_K\sigma_r\xi_P) + r\right]dt + \pi_S\sigma_S dZ_S + \pi_S\sigma_{SP}dZ_r + \pi_P\sigma_P dZ_r$$

$$\frac{dX_t}{X_t} = \left[\pi_S(\lambda_S + am_S) + \pi_P(a_K\sigma_r\xi_P) + r\right]dt + \pi_S\sigma_S dZ_S + (\pi_S\sigma_{SP} + \pi_P\sigma_P)dZ_r$$

$$\begin{aligned}
DJ = J_t + XJ_X &\left[\left(\pi_S(\lambda_S + am_S) + \pi_P(a_K\sigma_r\xi_P) + r\right)dt + \pi_S\sigma_S dZ_S + (\pi_S\sigma_{SP} + \pi_P\sigma_P)dZ_r\right] \\
&+ J_M\left((1-a)(\mu_S - M_t)dt + \sigma_S dZ_{S,t}\right) + J_\lambda\left(\kappa(\overline{\lambda} - \lambda_S)dt + \sigma_\lambda dZ_{S,t}\right) \\
&+ J_r\left[\beta(r_t - \overline{r})dt + \sigma_r dZ_{r,t}\right] + XJ_{XM}\left(\pi_S\sigma_S^2 dt + (\pi_S\sigma_{SP}\sigma_S + \pi_P\sigma_P\sigma_S)\rho_{sr}\right)dt \\
&+ XJ_{X\lambda}\left(\pi_S\sigma_S\sigma_\lambda dt + (\pi_S\sigma_{SB}\sigma_S + \pi_P\sigma_P\sigma_S)\rho_{sr}\right)dt + XJ_{Xr}\left(\pi_S\sigma_{Sr} + \pi_S\sigma_{SP}\sigma_r + \pi_P\sigma_P\sigma_r\right)dt \\
&+ J_{M\lambda}\sigma_S\sigma_\lambda dt + J_{Mr}\sigma_{Sr}dt + J_{\lambda r}\sigma_{\lambda r}dt + \frac{1}{2}X^2 J_{XX}\left(\pi_S^2\sigma_S^2 dt + (\pi_S\sigma_{SP} + \pi_P\sigma_P)^2\right)dt \\
&+ \frac{1}{2}J_{MM}\sigma_S^2 dt + \frac{1}{2}J_{\lambda\lambda}\sigma_\lambda^2 dt + \frac{1}{2}J_{rr}\sigma_r^2 dt
\end{aligned}$$

Taking the E_t operator allows some simplifications:

$$\begin{aligned}
E_t[DJ] = J_t + XJ_X &\left[\left(\pi_S(\lambda_S + am_S) + \pi_P(a_K\sigma_r\xi_P) + r\right)\right] + J_M\left((1-a)(\mu_S - M_t)\right) + J_\lambda\left(\kappa(\overline{\lambda} - \lambda_S)\right) \\
&+ J_r\left[\beta(r_t - \overline{r})\right] + XJ_{XM}\left(\pi_S\sigma_S^2 + (\pi_S\sigma_{SP}\sigma_S + \pi_P\sigma_P\sigma_S)\rho_{sr}\right) + XJ_{X\lambda}\left(\pi_S\sigma_S\sigma_\lambda + (\pi_S\sigma_{SP}\sigma_S + \pi_P\sigma_P\sigma_S)\rho_{sr}\right) \\
&+ XJ_{Xr}\left(\pi_S\sigma_{Sr} + \pi_S\sigma_{SP}\sigma_r + \pi_P\sigma_P\sigma_r\right) + J_{M\lambda}\sigma_S\sigma_\lambda + J_{Mr}\sigma_{Sr} + J_{\lambda r}\sigma_{\lambda r} + \frac{1}{2}X^2 J_{XX}\left(\pi_S^2\sigma_S^2 + (\pi_S\sigma_{SP} + \pi_P\sigma_P)^2\right) \\
&+ \frac{1}{2}J_{MM}\sigma_S^2 + \frac{1}{2}J_{\lambda\lambda}\sigma_\lambda^2 + \frac{1}{2}J_{rr}\sigma_r^2
\end{aligned}$$

We turn now to the calculation of the optimal weight π_S and π_P differentiating the previous relation with respect to the controls π_S and π_P. The first order condition with respect to π_S is:

$$\begin{aligned}
\max_{\pi_S, s\in[t,T]}\left\{E_t[DJ]\right\} = 0 \Leftrightarrow XJ_X(\lambda_S + am_S) &+ XJ_{XM}\left(\sigma_S^2 + \sigma_{SP}\sigma_S\rho_{sr}\right) + XJ_{X\lambda}\left(\sigma_S\sigma_\lambda + \sigma_{SB}\sigma_S\rho_{sr}\right) \\
&+ XJ_{Xr}\left(\sigma_{Sr} + \sigma_{SP}\sigma_r\right) + X^2 J_{XX}\left(\pi_S\sigma_S^2 + \left(\pi_S\sigma_{SP}^2 + \pi_P\sigma_P\sigma_{SP}\right)\right) = 0
\end{aligned}$$

$$-X^2 J_{XX}\pi_S\left(\sigma_S^2+\sigma_{SP}^2\right)=XJ_X\left(\lambda_S+am_S\right)+XJ_{XM}\left(\sigma_S^2+\sigma_{SB}\sigma_S\rho_{sr}\right)+XJ_{X\lambda}\left(\sigma_S\sigma_\lambda+\sigma_{SB}\sigma_S\rho_{sr}\right)$$

$$+XJ_{Xr}\left(\sigma_{Sr}+\sigma_{SP}\sigma_r\right)+X^2 J_{XX}\pi_P\sigma_P\sigma_{SP}$$

$$\pi_S\left(\sigma_S^2+\sigma_{SP}^2\right)=-\frac{J_X\left(\lambda_S+am_S\right)}{XJ_{XX}}-\frac{J_{XM}\left(\sigma_S^2+\sigma_{SP}\sigma_S\rho_{sr}\right)}{XJ_{XX}}-\frac{J_{X\lambda}\left(\sigma_S\sigma_\lambda+\sigma_{SB}\sigma_S\rho_{sr}\right)}{XJ_{XX}}$$

$$+\frac{J_{Xr}\left(\sigma_{Sr}+\sigma_{SP}\sigma_r\right)}{XJ_{XX}}+\pi_P\sigma_P\sigma_{SP}$$

and with respect to π_P is

$$XJ_X\left(a_K\sigma_r\xi_P\right)+XJ_{XM}\sigma_P\sigma_S\rho_{sr}+XJ_{X\lambda}\sigma_P\sigma_S\rho_{sr}+XJ_{Xr}\sigma_P\sigma_r+X^2 J_{XX}\left(\pi_S\sigma_{SP}\sigma_P+\pi_P\sigma_P^2\right)=0$$

$$\pi_P=-\frac{J_X\left(a_K\sigma_r\xi_P\right)}{XJ_{XX}}-\frac{J_{XM}\sigma_P\sigma_S\rho_{sr}}{XJ_{XX}}-\frac{J_{X\lambda}\sigma_P\sigma_S\rho_{sr}}{XJ_{XX}}-\frac{J_{Xr}\sigma_P\sigma_r}{XJ_{XX}}-\pi_S\frac{\sigma_{SP}\sigma_P}{\sigma_P^2}$$

We substitute the value of π_p in the expression of π_S

$$-\frac{J_X\left(\lambda_S+am_S\right)}{XJ_{XX}}-\frac{J_{XM}\left(\sigma_S^2+\sigma_{SB}\sigma_S\rho_{sr}\right)}{XJ_{XX}}-\frac{J_{X\lambda}\left(\sigma_S\sigma_\lambda+\sigma_{SB}\sigma_S\rho_{sr}\right)}{XJ_{XX}}+\frac{J_{Xr}\left(\sigma_{Sr}+\sigma_{SP}\sigma_r\right)}{XJ_{XX}}$$

$$+\left(\sigma_P\sigma_{SP}\right)\left(\frac{J_X\left(a_K\sigma_r\xi_P\right)}{XJ_{XX}\sigma_P^2}-\frac{J_{XM}\sigma_P\sigma_S\rho_{sr}}{XJ_{XX}\sigma_P^2}-\frac{J_{X\lambda}\sigma_P\sigma_S\rho_{sr}}{XJ_{XX}\sigma_P^2}-\frac{J_{Xr}\sigma_P\sigma_r}{XJ_{XX}\sigma_P^2}-\frac{\pi_S\sigma_{SP}\sigma_P}{\sigma_P^2}\right)=\pi_S\left(\sigma_S^2+\sigma_{SP}^2\right)$$

$$-\frac{J_X\left(\lambda_S+am_S\right)}{XJ_{XX}}-\frac{J_{XM}\left(\sigma_S^2+\sigma_{SB}\sigma_S\rho_{sr}\right)}{XJ_{XX}}-\frac{J_{X\lambda}\left(\sigma_S\sigma_\lambda+\sigma_{SB}\sigma_S\rho_{sr}\right)}{XJ_{XX}}+\frac{J_{Xr}\left(\sigma_{Sr}+\sigma_{SP}\sigma_r\right)}{XJ_{XX}}$$

$$+\left(\frac{J_X\left(a_K\sigma_r\xi_P\right)\sigma_{SP}}{XJ_{XX}\sigma_P}-\frac{J_{XM}\sigma_S\rho_{sr}\sigma_{SP}}{XJ_{XX}}-\frac{J_{X\lambda}\sigma_S\rho_{sr}\sigma_{SP}}{XJ_{XX}}-\frac{J_{Xr}\sigma_r\sigma_{SP}}{XJ_{XX}}\right)=\sigma_S^2\pi_S$$

$$-\frac{J_X\left(\lambda_S+am_S\right)}{\sigma_S^2 XJ_{XX}}-\frac{J_{XM}\left(\sigma_S^2+\sigma_{SB}\sigma_S\rho_{sr}\right)}{\sigma_S^2 XJ_{XX}}-\frac{J_{X\lambda}\left(\sigma_S\sigma_\lambda+\sigma_{SB}\sigma_S\rho_{sr}\right)}{\sigma_S^2 XJ_{XX}}+\frac{J_{Xr}\left(\sigma_{Sr}+\sigma_{SP}\sigma_r\right)}{\sigma_S^2 XJ_{XX}}$$

$$+\left(-\frac{J_X\left(a_K\sigma_r\xi_P\right)\sigma_{SP}}{\sigma_S^2 XJ_{XX}\sigma_P}-\frac{J_{XM}\sigma_S\rho_{sr}\sigma_{SP}}{\sigma_S^2 XJ_{XX}}-\frac{J_{X\lambda}\sigma_S\rho_{sr}\sigma_{SP}}{\sigma_S^2 XJ_{XX}}-\frac{J_{Xr}\sigma_r\sigma_{SP}}{\sigma_S^2 XJ_{XX}}\right)=\pi_S$$

$$\pi_S^*=-\frac{J_X}{\sigma_S^2 XJ_{XX}}\left[\left(\lambda_S+am_S\right)-\frac{\left(a_K\sigma_r\xi_P\right)\sigma_{SP}}{\sigma_P}\right]-\frac{J_{XM}}{\sigma_S^2 XJ_{XX}}\left[\left(\sigma_S^2+\sigma_{SB}\sigma_S\rho_{sr}\right)-\sigma_S\rho_{sr}\sigma_{SP}\right]$$

$$-\frac{J_{X\lambda}}{\sigma_S^2 XJ_{XX}}\left[\left(\sigma_S\sigma_\lambda+\sigma_{SB}\sigma_S\rho_{sr}\right)-\sigma_S\rho_{sr}\sigma_{SP}\right]+\frac{J_{Xr}}{\sigma_S^2 XJ_{XX}}\sigma_{Sr}$$

$$\pi_S^* = -\frac{J_X}{XJ_{XX}}\left[\frac{\left(\lambda_S + am_S\right)}{\sigma_S^2} - \frac{\xi_P\sigma_{SP}}{\sigma_S^2}\right] - \frac{J_{XM}}{XJ_{XX}} - \frac{J_{X\lambda}}{XJ_{XX}}\frac{\sigma_\lambda}{\sigma_S} - \frac{J_{Xr}}{XJ_{XX}}\frac{\sigma_{Sr}}{\sigma_S^2}$$

and we deduce the value of π_p taking into account the value of π_S

$$\pi_P = -\frac{J_X\left(a_K\sigma_r\xi_P\right)}{XJ_{XX}} - \frac{J_{XM}\sigma_P\sigma_S\rho_{sr}}{XJ_{XX}} - \frac{J_{X\lambda}\sigma_P\sigma_S\rho_{sr}}{XJ_{XX}} - \frac{J_{Xr}\sigma_P\sigma_r}{XJ_{XX}}$$

$$-\left[-\frac{J_X}{XJ_{XX}}\left[\frac{\left(\lambda_S + am_S\right)}{\sigma_S^2} - \frac{\xi_P\sigma_{SP}}{\sigma_S^2}\right] - \frac{J_{XM}}{\sigma_S^2 XJ_{XX}} - \frac{J_{X\lambda}}{XJ_{XX}}\frac{\sigma_\lambda}{\sigma_S} - \frac{J_{Xr}}{XJ_{XX}}\frac{\sigma_{Sr}}{\sigma_S^2}\right]\frac{\sigma_{SP}\sigma_P}{\sigma_P^2}$$

$$-\frac{J_X\left(a_K\sigma_r\xi_P\right)}{XJ_{XX}} - \frac{J_{XM}\sigma_P\sigma_S\rho_{sr}}{XJ_{XX}} - \frac{J_{X\lambda}\sigma_P\sigma_S\rho_{sr}}{XJ_{XX}} - \frac{J_{Xr}\sigma_P\sigma_r}{XJ_{XX}} - \pi_S\frac{\sigma_{SP}\sigma_P}{\sigma_P^2} = \pi_P$$

$$\pi_P^* = -\frac{J_X}{XJ_{XX}}\left[-\frac{\xi_P}{\sigma_P}\left(1-\sigma_{SP}\right) - \left(\lambda_S + am_S\right)\frac{\sigma_{SP}}{\sigma_P}\right] - \frac{J_{XM}}{XJ_{XX}}\left[\frac{\sigma_S\rho_{sr}}{\sigma_P} - \frac{\sigma_{SP}}{\sigma_P}\right] - \frac{J_{Xr}}{XJ_{XX}}\left[\frac{\sigma_r}{\sigma_P} - \frac{\sigma_{SP}}{\sigma_P}\frac{\sigma_{Sr}}{\sigma_S^2}\right]$$

$$-\frac{J_{X\lambda}}{XJ_{XX}}\left[\frac{\sigma_S\rho_{sr}}{\sigma_P} - \frac{\sigma_\lambda}{\sigma_S}\frac{\sigma_{SP}}{\sigma_P}\right]$$

List of figures and tables