Cash-flow based valuation of pension liabilities

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Abstract

This paper presents a computational framework for market consistent valuation of insurance claims in incomplete markets. It accounts for the risks associated with both assets and liabilities until maturity in accordance with the modern principles of asset liability management. The framework is easily adapted to different lines of insurance and to different market conditions and it can effectively employ advanced tools for strategic portfolio management. We apply the valuation procedure to the claims associated with the current insurance portfolio of the Finnish private sector occupational pension system where the claims extend over 82 years.

1 Introduction

In the absence of liquid markets for insurance obligations their pricing should be based on the cash flows associated with the settlement of the obligations until maturity; see [International Association of Insurance Supervisors, 2007, Structure element 5]. When valuing long terms pension liabilities, significant risks are associated both with the claims as well as with investment returns that affect the sufficiency of the capital reserved for covering the claims. According to modern risk management principles the value of insurance liabilities should reflect both the underwriting risks as well as investment (market and credit) risks until the amortization of the liabilities; see e.g. International Association of Insurance Supervisors [2007].

Due to significant uncertainties and the incompleteness of financial markets, pension insurance liabilities cannot be fully hedged or full hedging may amount to unreasonable costs. A more pragmatic approach is to define the value of pension

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liabilities as the minimal capital required to cover the claims at an acceptable level of risk when the capital is prudently invested in financial instruments available in the market. Such a value depends essentially on the following subjective factors

1. **risk factors**: the description of future development of investment returns and the insurance claims, both of which involve significant uncertainty that should be appropriately quantified.

2. **risk preferences**: the level of risk at which the investment returns should cover the insurance claims. Instead of simple confidence levels, one could use risk measures that better support risk management.

3. **hedging strategy**: the strategy according to which the given capital is invested in financial markets. Adjusting the investment strategy to the liabilities may allow for a reduction of the required initial capital.

The significance of market expectations is usually well understood but risks are often ignored or they are accounted for by heuristic adjustments (see e.g. the drafts of Solvency II framework). When valuing risky cash flows the effects of (more or less subjective) risk tolerances cannot be avoided. Some studies have suggested the use of so called "risk neutral measures" but in incomplete markets the choice of the measure also comes down (often implicitly) to a specification of risk preferences; see e.g. Wüthrich et al. [2008]. The choice of the investment strategy reflects the insurer's expertise in producing the cash flows associated with its insurance portfolio. Coming up with an appropriate investment strategy is one of the most important functions of an insurance company. A valuation framework should be market consistent by taking into account the investment opportunities available to the insurer at the time of valuation. The interplay between capital requirements and asset management has been recently studied in a simplified setting in Artzner et al. [2009].

The main contribution of this paper is to present a computational framework based on the above principles for market consistent valuation of insurance claims in incomplete markets. The framework is easily adapted to different lines of insurance and different market conditions and it can effectively employ advanced tools for strategic portfolio management. We apply the valuation procedure to a problem coming from the Finnish pension insurance industry, where the liabilities are taken as the claim process associated with current insurance portfolio of the private sector occupational pension system where the claims extend over 82 years. We illustrate the importance of appropriate recognition of both the investment and underwriting risks as well as the specification of risk preferences in the valuation. The effects of the investment strategy are demonstrated by applying different well known and widely studied parametric investment strategies as insurance portfolio hedges and by using an adaptive optimization technique developed for long term risk management. In our case study, the optimization results in over 15% decrease in capital requirements when compared to the best found parametric hedging strategy.

Interaction of the essential risk factors, the dynamic investment strategy and the risk preferences highlight the need for an integrated valuation framework, that allows incorporating all the relevant valuation factors in a flexible way and that can be easily adapted to reflect the views of the insurer. The developed valuation framework can be easily adapted and applied in the valuation of diverse insurance claims, or more generally, in the market consistent pricing of (non-marketable) financial instruments.

The paper is organized as follows. Section 2 reviews the valuation of insurance claims in a deterministic environment and outlines our case study coming from the Finnish pension insurance industry. Section 3 introduces uncertainties into the framework and illustrates how risk preferences affect the capital requirement when investment and underwriting risks are appropriately recognized. Section 4 presents the fully fledged asset-liability management (ALM) framework for the valuation of insurance liabilities. We illustrate the significance of dynamic portfolio rebalancing on the minimum capital requirement for hedging an insurance portfolio.

2 Valuation of liabilities in a deterministic world

Throughout this paper c_t will denote the aggregate claims associated with an existing insurance portfolio during period $[t-1,t]$. We assume that the liabilities will amortize after a finite time so that the last claim will be paid at time T.

For purposes of comparison, we begin with a simplified framework where both the claims and investment returns are deterministic. More realistic models will be developed in the following sections. Assume that there is one perfectly liquid financial instrument that returns R_t over the period $[t-1,t]$ for $t = 1,\dots,T$. In this unrealistically simple setting the valuation of the insurance portfolio is straightforward. The initial capital required to cover the insurance payments c_t is obtained by solving V_0 from the system of equations

$$\begin{aligned} V_t &= R_t V_{t-1} - c_t \quad t = 1,\dots,T, \\ V_T &= 0. \end{aligned}$$

It is easily checked that the solution is given by

$$V_0 = \sum_{t=1}^{T} \frac{c_t}{\prod_{s=1}^{t} R_s}. \tag{1}$$

This corresponds to the traditional actuarial present value of insurance liabilities; see e.g. Bowers et al. [1997], Wüthrich et al. [2008]. The deterministic valuation of insurance claims thus requires point estimates of the periodic aggregate claims and investment returns until the amortization of the claims. The resulting capital requirement depends heavily on the estimates. This will be illustrated in the case study below.

It has been suggested e.g. in the drafts of the Solvency II framework, that instead of a point estimate of future investment returns, discounting should be based on the current term structure of interest rates. This would be appropriate when valuing fixed income instruments but it is poorly suited for valuing uncertain insurance claims. Substituting a point estimate (e.g. the "best estimate") for the uncertain insurance claims misses the main point, the underwriting risk associated with uncertain insurance claims. Instead of resorting to ad hoc risk margins to correct such estimates, we will propose in the following sections a valuation scheme that explicitly accounts for the risks associated with both assets and liabilities of the insurer.

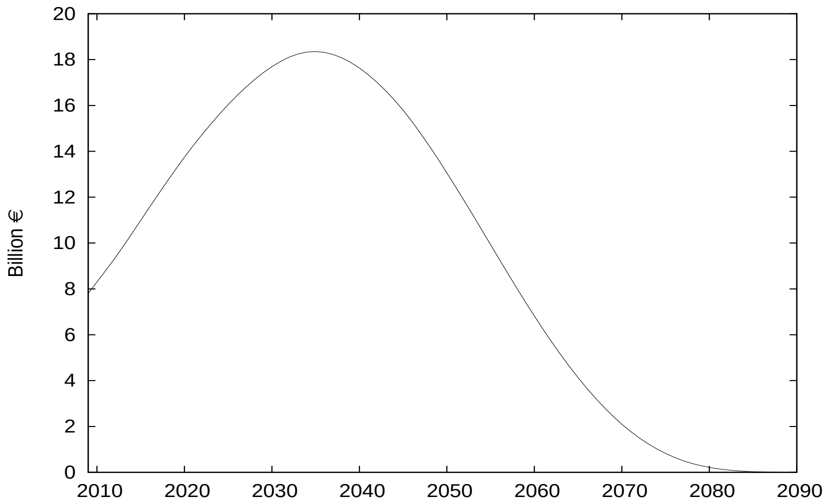
Case study: Finnish private sector pension liabilities

We will demonstrate the valuation on a problem coming from the Finnish private sector occupational pension system. The liabilities consist of the current insurance portfolio of the Finnish private sector occupational pension system. The yearly claims c_t consist of aggregate old age, disability and unemployment pension benefits that have accrued by the end of 2008 and become payable during year t. These payments form the majority (80%) of the total pension expenditure of the Finnish private sector occupational pension system. The liabilities are of the defined benefit type and they depend on the development of wage and consumer price indices. Figure 1 depicts the forecasted annual total pension expenditure. The forecast is based on the assumption of constant annual wage increases of 3.8%, annual inflation of 2.0% and the current accrued pension rights and Finnish mortality tables according to which all the liabilities will be amortized in $T = 82$ years.

Setting the annual investment return to $R_t = 6\%$, as in Biström et al. [2007], the capital requirement given by (1) is $V_0 = 207.7$ billion euros. This is the minimum capital that would, in a deterministic world, suffice to cover the future pension payments associated with the accrued pension rights. The erosion of capital over time $(V_t)_{t=0}^T$ is depicted in Figure 3.

According to Savela [2008] the Finnish pension providers (pension insurance companies and pension funds) have approximately $W_0 = 72.1$ billion euros capital as of the end of 2008. The funds have accumulated from the insurance contributions and asset returns and they will be used to pay part of the accrued pension

Figure 1: Evolution of the projected aggregate claim payments associated with the accrued old age, disability and unemployment pension benefits.



rights. The rest will be covered on the pay-as-you-go basis. In this deterministic setting the *solvency ratio* of the Finnish private sector occupational pension system equals

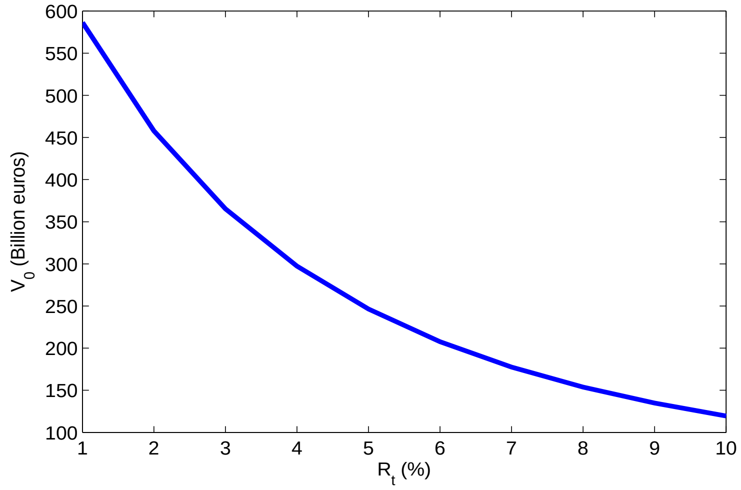
$$\frac{W_0}{V_0} = \frac{72.1}{207.7} = 0.35.$$

The capital requirement based on (1) is very sensitive with respect to the assumed parameter values for the inflation, wage growth and especially the investment return. The sensitivity of V_0 with respect to the anticipated annual investment return is depicted in Figure 2. The capital requirement depends nonlinearly of the expected return and a mere percentage point reduction in the expected annual return of 6%, increases the capital requirement by roughly 20%, or 40 billion euros. This highlights the need for a risk sensitive valuation framework that can accommodate all the relevant risk factors in the determination of adequate capital requirements.

3 Valuation under uncertainty

In this section we extend the liability valuation methodology of the previous section to allow uncertainties in both the claims c_t and investment returns R_t. In an uncertain environment and in the absence of liquid markets for the insurance claims, there is always a risk that in some future scenarios the return on invested capital is insufficient to cover the liabilities. Thus, the valuation of insurance

Figure 2: V_0 as a function of the deterministic annual investment return R_t.



claims must in some way reflect the risk preferences of the insurer. In what follows, the claims c_t and the investment returns R_t are modeled as random variables on a probability space $(\Omega, \mathcal{F}, P)$. The specified probability distribution should reflect the subjective views of the decision maker concerning the future values of the underlying riskfactors.

Following the above principles, the value of the insurance claims can be defined as the solution to the optimization problem

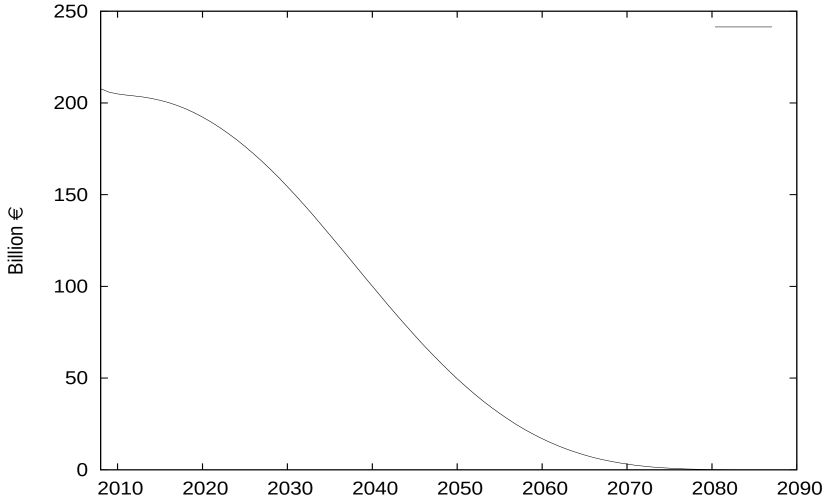
$$\begin{array}{ll} \text{minimize} & V_0 \\ \text{subject to} & V_t = R_t V_{t-1} - c_t \quad t = 1, \ldots, T, \quad P\text{-a.s.} \\ & \rho(V_T) \leq 0, \end{array} \tag{2}$$

where the first equation is required to hold almost surely and ρ is a *risk measure* that quantifies the decision maker's preferences over the random terminal wealth distributions; see e.g. Föllmer and Schied [2004] or Rockafellar [2007] for general treatments of risk measures. Modeling the claims c_t and returns R_t as random variables, implies that the corresponding terminal wealth V_T is indeed random. The minimum initial capital V_0 covers the claim cash flows in a sense of the risk measure ρ, i.e. at a specified level of risk.

The most significant factors affecting V_0 are

1. **risk factors**: Decision maker's views regarding the uncertain future development of the claims $c = (c_t)_{t=0}^T$ and the investment returns $R = (R_t)_{t=0}^T$.

Figure 3: Development of wealth.



2. **risk preferences**: The risk measure that determines the acceptable distributions of the random terminal wealth.

The definition of the capital requirement can be based on various choices of risk measures. Well studied examples include the *Value at Risk* [Jorion, 1997], *Conditional Value at Risk* [Rockafellar and Uryasev, 2000] and the *zero utility principle* [Bühlmann, 1970]. Given a random variable V, the Value at Risk at confidence level $\delta \in [0,1]$, $V@R_\delta(V)$ is defined as the negative of the $1-\delta$-quantile of V. The corresponding initial capital V_0 would be sufficient to cover the pension claims until full amortization with a probability of δ. The Conditional Value at Risk at confidence level δ is defined as the conditional expectation $CV@R_\delta(V_T) = -E[V_T \,|\, V_T \leq -V@R_\delta(V_T)]$. The zero utility principle is based on a concave utility function $u : \mathbb{R} \to \mathbb{R}$. The associated risk measure is defined as

$$\rho_u(V) = \inf\{\alpha \,|\, Eu(V+\alpha) \geq u(0)\}.$$

In the case of exponential utility, this becomes the well-known *entropic risk measure*; see e.g. [Föllmer and Schied, 2004, Chapter 4].

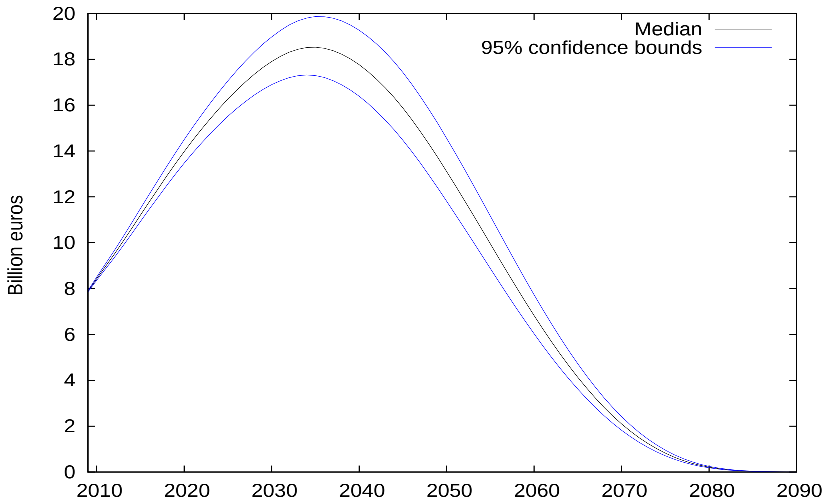
Case study: Finnish private sector pension liabilities

We modify the case study outlined in Section 2 by allowing the wages, the inflation and the investment returns to be stochastic. It follows that the claims c_t are also stochastic. A detailed description of the stochastic model for the wage and inflation indices can be found in Hilli et al. [2007] and for the claims in Hilli et al. [2008]. Figure 4 displays the median and 95% confidence interval of the cash flows c_t associated with the pension rights accrued by the end of 2008. The investment return is modelled as a log-normal process

$$\ln(R_t) = \mu + \sigma\varepsilon, \quad \varepsilon \sim N(0,1),$$

where the parameters μ and σ are chosen so that the annualized logarithmic returns have a mean and standard deviation of 6%.

Figure 4: Evolution of the claims associated with the accrued old age, disability and unemployment pension benefits.



In the calculation of the capital requirement (2) we will quantify the risk preferences with Value at Risk as well as with Conditional Value at Risk with varying confidence levels in order to illustrate the effect of risk tolerances on the capital requirement.

After the specification of the risk measure and the probability distribution of the relevant risk factors, the valuation of the insurance claims can be carried out numerically by generating N scenarios of asset returns R_t and claims c_t over $t = 1,\dots,T$ and by solving the corresponding discretized (2) by a simple line search.

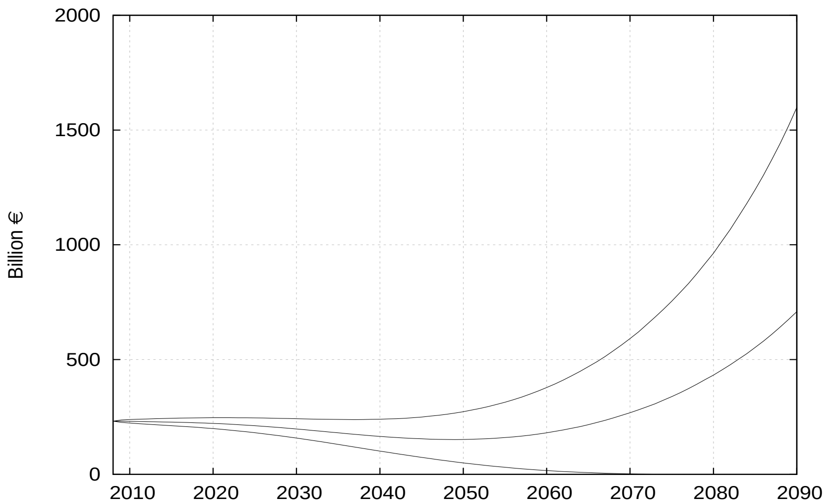
Table 1 displays the capital requirements at various confidence levels using the $CV@R$ and $V@R$ risk measures. The results in Table 1 were obtained with $N = 200000$ scenarios. The 66% confidence level corresponds to 99.5% annual solvency probability (required e.g. in Solvency II) until the full amortization of the pension claims in $T = 82$ years. Figure 5 displays the evolution of the 34%-quantile together with the median and the 66%-quantile of $(V_t)_{t=0}^T$ when the initial wealth is set according to the risk measure $\rho(V_T) = V@R_{66\%}(V_T)$. The figure shows that the 34%:th percentile of the distribution of V_T equals zero after all the pension claims have been paid off.

Table 2 gives the solvency ratios W_0/V_0 corresponding to the 72.1 billion of the Finnish pension system at the beginning of 2008 and the capital requirements V_0 of Table 1. The results reveal the significance of incorporating uncertainties and risk preferences in the valuation which in our example increase the capital requirement by up to 50%.

		Confidence level				
		95%	90%	85%	80%	66%
Risk measure	$V@R$	289	271	259	250	232
	$CV@R$	305	288	276	268	252

Table 1: Pension liability (billion euros) with varying confidence levels.

Figure 5: The development of the 34%, 50%- and 66%-quantiles of $(V_t)_{t=0}^T$ when the initial capital corresponds to $V@R_{66\%}$.



4 Valuation and asset management

The construction of an appropriate hedging strategy for an insurance portfolio is one of the most important tasks of an insurance company.

		Confidence level				
		95%	90%	85%	80%	66%
Risk measure	$V@R$	24.9	26.6	27.9	28.9	31.1
	$CV@R$	23.6	25.1	26.1	26.7	28.7

Table 2: Solvency ratios with varying confidence levels.

In real markets where an insurer has multiple investment opportunities, the chosen investment strategy plays a significant role in the determination of the risk based capital requirement. By choosing an investment strategy whose dynamic risk and return profile conforms to the cash flow profile of the liabilities it may be possible to lower the initial capital requirement.

The same principle is one of the cornerstones of modern finance theory. For example, the classical Black-Scholes option pricing formula gives the capital requirement for exact replication of the option's cash flows in complete markets when the capital is invested according to the so called delta-hedging strategy [Black and Scholes, 1973]. Contrary to the Black-Scholes model, the exact replication of financial instruments is often impossible in real markets or it may become prohibitively expensive. Because of this, the hedging of insurance contracts practically always involves the risk that the return on invested capital might be insufficient to cover the insurance claims. On the other hand, the earned investment returns may well exceed the insurance claims. Given the inherent uncertainties in the resulting net cash flow, the choice of a hedging/investment strategy must be based on the risk preferences of the insurer.

The most important factors affecting the capital requirement are

1. **risk factors**; see Section 3,
2. **risk preferences**; see Section 3,
3. **hedging strategy**: the investment strategy, according to which the given capital is invested in financial markets.

The choice of an asset management strategy plays an important role in competitive insurance markets. The lower the initial capital an insurer needs for hedging its liabilities, the lower its costs of producing insurance contracts, or alternatively, the more it can distribute capital back to its shareholders.

Consider a strategic ALM problem, where the assets under management can be diversified each year among a finite set J of available asset classes. Denote by $R_{t,j}$ the (total) return on class $j \in J$ during period $[t-1,t]$. The amount of wealth $h_{t,j}$ invested in class $j \in J$ in the beginning of year t can react to all available information at time t, but the decision is not allowed to depend on information that will be revealed after time t. The dynamic investment strategy $h = (h_t)_{t=0}^T$, where h_t is $\mathbb{R}^J$-valued random vector, is thus adapted to the available information. Mathematically, for each t, the portfolio h_t is $\mathcal{F}_t$-measurable, where $\mathcal{F}_t \subset \mathcal{F}$ is the sigma-field generated by the information observable by time t; see e.g. Föllmer and Schied [2004].

An insurer may also have investments in illiquid financial assets without well functioning secondary markets (e.g. reinsurance or private placement bonds). Even when liquid secondary markets do exist, the market values of some instruments may not reflect their value in hedging an insurance portfolio. In other words, the market values of some financial instruments may deviate substantially from their true value to the insurer. This may be the case with some hedging instruments such as mortality linked bonds or equity- and interest rate derivatives.

Assume that the insurer's portfolio contains some *held to maturity* investments $\bar{J}$. The investment strategy thus consists of a static allocation $\bar{h} \in \mathbb{R}^{\bar{J}}$ in $\bar{J}$ and a dynamic trading strategy $h = (h_t)_{t=0}^T$ in the liquid assets J. If an investment strategy $(h, \bar{h})$ satisfies

$$
\begin{aligned}
\sum_{j\in J} h_{t,j} + c_t \le \sum_{j\in J} R_{t,j} h_{t-1,j} + \sum_{j\in\bar{J}} \bar{R}_{t,j}\bar{h}_j \quad t = 1,\dots,T,\\
h_{t,j}, \bar{h}_l \ge 0 \quad j \in J\setminus\{0\}, l \in \bar{J},\\
\rho\Big(\sum_{j\in J} h_{T,j}\Big) \le 0,
\end{aligned}
$$

then the initial capital

$$V_0 = \sum_{j\in J} h_{0,j} + \sum_{j\in\bar{J}} \bar{h}_j$$

covers the pension claims $c = (c_t)_{t=1}^T$ at the specified level of risk. Here $\bar{R}_{t,j}$ denotes the annual cash return (e.g. coupon payments or option payouts) per invested capital in asset class $j \in \bar{J}$ and ρ is the risk measure which specifies the acceptable level of risk.

In the above formulation, we have prohibited short-selling of all except the money market account which is indexed by $j = 0$. Due to the uncertain nature of the liabilities and asset returns, it is generally impossible to guarantee that the total wealth remains nonnegative in all scenarios. If we would prohibit short-selling all the asset classes, the above system would become infeasible in general. If wealth becomes negative, it is assumed that the insurer (the owners of the insurance company) is required to provide the missing funds which is accounted for by the risk measure ρ.

The dependence structure between the random claims c and returns R and $\bar{R}$ is an important factor in the determination of the capital requirement. It largely determines how well an investment strategy can be adapted to the insurance liabilities. For example, if one of the asset classes $j \in \bar{J}$ corresponds e.g. to reinsurance, its returns process $\bar{R}_j$ may be completely determined by the pension claims c. Such an instrument might be a good ingredient in a hedging strategy.

The search for the minimum capital requirement leads to an optimization problem

$$
\begin{array}{lll}
\underset{h\in\mathcal{N},\bar{h}\in\mathbb{R}^{\bar{J}}}{\text{minimize}} & \displaystyle\sum_{j\in J} h_{0,j} + \sum_{j\in\bar{J}} \bar{h}_j & \\
\text{subject to} & \displaystyle\sum_{j\in J} h_{t,j} + c_t \leq \sum_{j\in J} R_{t,j} h_{t-1,j} + \sum_{j\in\bar{J}} \bar{R}_{t,j}\bar{h}_j & t = 1,\dots,T, \\
& h_{t,j}, \bar{h}_l \geq 0 \quad j \in J\setminus\{0\}, l \in \bar{J}, & \\
& \rho(\displaystyle\sum_{j\in J} h_{T,j}) \leq 0, &
\end{array}
\tag{3}
$$

where $\mathcal{N}$ denotes the $\mathbb{R}^J$-valued portfolio processes adapted to the filtration $(\mathcal{F}_t)_{t=0}^T$. The above problem cannot be solved analytically, except in some simple special cases. In practice, one has to rely on expert knowledge of the problem or numerical approximation schemes or both.

One way to approach the solutions of (3) is to apply the optimization procedure developed in Koivu and Pennanen [to appear] to the problem

$$
\begin{array}{lll}
\underset{h\in\mathcal{N},\bar{h}\in\mathbb{R}^{\bar{J}}}{\text{minimize}} & \rho(\displaystyle\sum_{j\in J} h_{T,j}) & \\
\text{subject to} & \displaystyle\sum_{j\in J} h_{0,j} + \sum_{j\in\bar{J}} \bar{h}_j \leq w & \\
& \displaystyle\sum_{j\in J} h_{t,j} + c_t \leq \sum_{j\in J} R_{t,j} h_{t-1,j} + \sum_{j\in\bar{J}} \bar{R}_{t,j}\bar{h}_j & t = 1,\dots,T, \\
& h_{t,j}, \bar{h}_l \geq 0 \quad j \in J\setminus\{0\}, l \in \bar{J}, &
\end{array}
\tag{4}
$$

for varying values of initial wealth $w \in \mathbb{R}$. If w is such that the optimum value of (4) is zero we can conclude that w is the optimum value of (3). The optimal investment strategies of (4) are then optimal also in (3). If ρ is the Conditional Value at risk with some confidence level, the procedure presented in Hilli et al. [2009] is directly applicable to (4).

Case study: Finnish private sector pension liabilities

On the strategic level, the assets of a typical Finnish pension fund are usually allocated in interest rate, equity and real estate funds. In our numerical study, the liquid assets are modeled accordingly. A more detailed breakdown of the available

	5%	50%	95%
Money market	2.9	3.6	4.4
Bonds	-0.6	4.4	10.8
Nordic equities	-26.8	7.8	58.2
European equities	-17.9	6.7	38.6
US equities	-19.7	6.7	41.7
Asian equities	-22.9	7.7	50.6
Real estate	-17.4	6.2	36.5

Table 3: Quantiles of the annualized returns of the liquid asset classes (%)

asset classes is given in Table 3. A description of the stochastic model for asset returns used in this study can be found in Hilli et al. [2007]. Table 3 gives the median and the 90% confidence intervals of the annualized returns of the liquid asset classes based on the model. The illiquid instruments consist of fixed coupon bonds with a 4% annual coupon rate and maturities of 10, 20 and 30 years. The prices of the illiquid bonds were computed by discounting the cash flows with swap rates. The cash flows of the pension claims are computed as in Section 3.

In the numerical study we evaluated 529 parametric dynamic investment strategies with varying investment styles. The strategies are based on well-known and widely applied *buy and hold*, *fixed proportion* and *constant proportion portfolio insurance*. In buy and hold strategies the initial asset allocation is held over time without rebalancing. In fixed-proportion strategies the asset allocation is rebalanced, in each period, to fixed portfolio weights. In constant proportion portfolio insurance-strategies the portfolio weights are adjusted according to the "cushion" which is defined as the difference of the total assets and a rough estimate of the capital requirement calculated using a deterministic model much as in Section 2. The larger the cushion, the higher the weight of risky assets; see Black and Perold [1992].

As in in Hilli et al. [2009] all strategies were modified to accommodate for claim payments and for the possibility that the insurer runs out of funds before the settlement of the claims so that the budget constraints in (4) are satisfied. In case the earned investment returns are insufficient to cover the liabilities it is assumed that the insurance provider borrows the required funds for the claim payments from the money market.

For each basis strategy we solved for the minimum initial capital V_0 such that the corresponding final wealth is acceptable in the sense of the specified risk measure ρ. We then used the optimization procedure of Hilli et al. [2009] to find the capital requirement of a dynamic strategy where the given capital is optimally diversified among the individual basis strategies. All the computations were based on the Conditional Value at Risk with varying confidence levels.

Our computations were based on approximating the probability distribution of the risk factors $(R, \bar{R}, c)$ by a sample of 200000 scenarios. In order to avoid bias in the case of optimal diversification over the basis strategies, the optimization was based on an independent set of 100000 scenarios.

The capital requirements based on the numerical computations are given in Table 4 and the corresponding solvency ratios in Table 5. For each confidence level, the *best basis* gives the minimum capital requirement attainable by using the best individual strategy among the evaluated 529 basis strategies. As the capital requirements obtained with the best individual investment strategies are significantly lower than the ones computed in Section 3, it is safe to conclude that the choice of an investment strategy plays an important role in the determination of capital requirements of a pension insurance portfolio. The capital requirement can be even further reduced by optimally diversifying the initial capital among the individual basis strategies as in Koivu and Pennanen [to appear] and Hilli et al. [2009]. The constructed optimal strategies significantly reduce the required capital compared to the best individual strategy, as evidenced by the results in Table 4. The optimized hedging strategies as well as the best basis strategies may vary considerably depending on the chosen confidence level.

	Confidence level				
	95%	90%	85%	80%	66%
Section 3	305	288	276	268	252
Best basis	296	284	273	261	239
Optimization	288	271	254	236	202

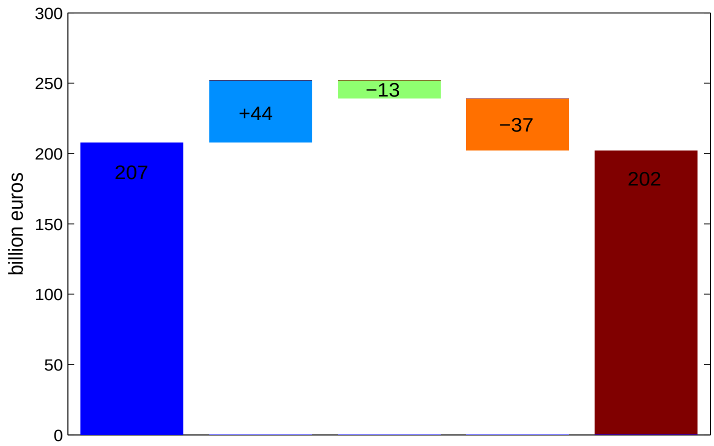
Table 4: V_0 (billion €) with varying investment strategies and confidence levels.

	Confidence level				
	95%	90%	85%	80%	66%
Section 3	23.6	25.1	26.1	26.7	28.7
Best basis	24.3	25.4	26.4	27.6	30.1
Optimization	25.0	26.6	28.3	30.5	35.6

Table 5: Solvency ratios (%) with varying investment strategies and confidence levels.

Figure 6 illustrates the valuation effects in our case study stemming from uncertain asset returns and claim payments, diversification benefits, dynamic portfolio rebalancing and the choice of the optimal investment strategy. The first bar in Figure 6 gives the deterministic value of V_0 (Section 2), the second gives the contribution of stochastic returns and liabilities (Section 3), the third reflects the diversification benefits among asset classes (best basis strategy), the fourth reflects the effects of optimal hedging strategy and the last bar gives the optimized value of V_0 with a 66% confidence level and $CV@R$ risk measure. By chance, with the chosen confidence level, the valuation benefits stemming from diversification between asset classes and the optimal investment strategy roughly equal the costs of uncertainty in asset returns and claim payments. This however, is a pure coincidence and does not apply in general or even with other confidence levels in our case study; see Table 4.

Figure 6: Contributions to V_0 with $CV@R_{66\%}$ risk measure. Notes: The first bar gives the deterministic value of V_0, the second the contribution of stochastic returns and liabilities, the third reflects the diversification effects among asset classes, the fourth reflects the effects of optimal investment strategy and the last bar gives the optimized V_0.



5 Conclusions

This paper developed a computational framework for market consistent valuation of insurance claims in incomplete markets. The framework was applied to valuation of a pension insurance portfolio. The results revealed the fundamental role of uncertainty in asset returns and claims, risk preferences and the dynamic hedging strategy in the valuation of non-replicable insurance claims. The developed approach leaves ample possibilities for future research. An interesting avenue for future research is to study how uncertainties in mortality forecasts affect the value of pension liabilities. The presented framework can readily accommodate stochastic mortality models.

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References

Ph. Artzner, F. Delbaen, and P. Koch-Medona. Risk measures and efficient use of capital. *Astin Bulletin*, 39(1):101–116, 2009.

K. Biström, Elo, P., T. Klaavo, I. Risku, and H. Sihvonen. Lakisääteiset eläkkeet, pitkänaikavälin laskelmat 2075. Eläketurvakeskuksen raportteja, Eläketurvakeskus, 2007.

F. Black and A. F. Perold. Theory of constant proportion portfolio insurance. *Journal of Economic Dynamics and Control*, 16:403–426, 1992.

F. Black and M. Scholes. The pricing of options and corporate liabilities. *Journal of Political Economy*, 81(3):637–654, 1973.

N. L. Bowers, H. U. Gerber, J. C. Hickman, D. A. Jones, and C. J. Nesbitt. *Actuarial Mathematics*. Society of Actuaries, Shaumburg, IL, 2 edition, 1997. ISBN 0-9389594-6-8.

H. Bühlmann. *Mathematical methods in risk theory*. Die Grundlehren der mathematischen Wissenschaften, Band 172. Springer-Verlag, New York, 1970.

H. Föllmer and A. Schied. *Stochastic finance*, volume 27 of *de Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, extended edition, 2004. ISBN 3-11-018346-3. An introduction in discrete time.

P. Hilli, M. Koivu, and T. Pennanen. A stochastic model for assets and liabilities of a pension institution. 2nd PBSS Section Colloquium, 2007.

P. Hilli, M. Koivu, and T. Pennanen. *Financial Risk Management in the Statutory Occupational Pension System*. 2008:19. Reports of the Ministry of Social Affairs and Health, 2008.

P. Hilli, M. Koivu, and T. Pennanen. Optimal construction of a fund of funds. In A.B. Berkelaar, J. Coche, and K. Nyholm, editors, *Interest Rate Models, Asset Allocation and Quantitative Techniques for Central Banks and Sovereign Wealth Funds*. Palgrave Macmillan, 2009.

International Association of Insurance Supervisors. The IAIS common structure for the assessment of insurer solvency. Structure paper, 2007.

P. Jorion. *Value at Risk, the New Benchmark for Controlling Market Risk*. McGraw-Hill, 1997.

M. Koivu and T. Pennanen. Galerkin methods in dynamic stochastic programming. *Optimization*, to appear.

R. T. Rockafellar. Coherent approaches to risk in optimization under uncertainty. *INFORMS tutorial for November 2007*, 2007.

R. T. Rockafellar and S.P. Uryasev. Optimization of Conditional Value-at-Risk. *Journal of Risk*, 2:21–42, 2000.

V. Savela. Työeläkerahastojen sijoitusrakenne 31.12.2009. TELA, 2008.

M.V. Wüthrich, H. Bühlmann, and H. Furrer. *Market-Consistent Actuarial Valuation*. Springer-Verlag, 2008.