A longitudinal study on Portfolio Optimization: Is the "Success" Time Dependent?

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ABSTRACT

In this paper we study the performance of several portfolio optimization strategies and their characteristics in different holding periods over the time horizon of almost three decades. The selected strategies are two of the classic mean–variance, two of the mean–CVaR and the log–optimal portfolio optimization strategies. For benchmark purposes the naïve portfolio is used. Four of the above approaches work in the risk-return plane, therefore we also intend to investigate how portfolio performance is affected by certain characteristics and known limitations of the associated risk measure. Equity index returns of 18 developed markets are used for an ex-ante analysis and the profitability of each optimization strategy is measured and compared to that of the US stock price index. We attempt to find out whether the most profitable strategy will be changing over the timeline. Furthermore, we intend to reveal whether the current financial crisis has any significant effect on the rank order of the strategies.

Keywords: Portfolio optimization, mean–variance portfolio, mean–CVaR portfolio, log-optimal portfolio

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1. Introduction

Out-of-sample back testing is a well accepted and widely used approach in the literature to get a picture about the “success” of portfolio allocation under realistic circumstances. With this method we can find out how different allocation strategies would perform in the presence of parameter uncertainty caused by estimation risk.

It has crucial importance to know the characteristics of a strategy one is going to follow in bringing an allocation decision into effect. If there is a downturn in the economy, investors are more sensitive to be exposed to risk. In contrast, in a booming period they are less reluctant to take higher risk. The choice of risk measure might have a strong influence on the construction of a given portfolio and on the performance of a given investment, especially in those situations when the stock markets are extremely volatile and the return/loss distributions are asymmetric and leptokurtic.[1]

Our objective is to study empirically various portfolio allocation strategies over a time horizon of almost three decades, to compare and contrast their characteristics along different “dimensions” such as the average realized return, volatility of realized return and risk adjusted performance. As a volatility measure we are going to apply the classical standard deviation as well as conditional value-at-risk (CVaR) which is a coherent measure of downside risk.[2]

The perspective of an American investor will be taken in simulating portfolio allocation in 18 developed stock market indexes. The selected strategies are two of the classic mean–variance, two of the mean–CVaR and the log–optimal portfolio optimization strategies. For benchmark purposes the naïve portfolio and the US stock price index (as a proxy of the domestic portfolio) was used.

The remainder of the paper is organised as follows. Section 2 gives an insight into the methodology. The results of the empirical analysis are discussed in Section 3. First the specification of data and the research design are given. The second part of Section 3 presents and interprets the findings. Section 4 provides concluding remarks.

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[1] See Szegő (2002).
[2] See Artzner et al. (1999).

2. Methodology

2.1 Mean-Variance Optimization

The concept of mean-variance efficiency was developed by Markowitz (1952). The strategy to set up the minimum-variance portfolio can be formulated as follows.[3]

$$\min V(\underline{x}) = \sum_{i=1}^{n}\sum_{k=1}^{n} x_i x_j \rho_{i,j}\sigma_i\sigma_j$$

subject to (1)

$$\sum_{i=1}^{n} x_i = 1, \quad x_i \geq 0, \quad i = 1, 2, ..., n$$

where x_i denotes the proportion of money allocated in the *i*th asset, σ_i stands for the standard deviation of return on the *i*th asset, $\rho_{i,j}$ is the correlation term between the returns on the *i*th and *j*th asset and *n* denotes the number of assets in the portfolio. We refer to the solution of model (1) as the minimum-variance portfolio (MVP).[4]

The MVP strategy aims to determine the portfolio (MVP) with the lowest estimated risk (measured by the standard deviation), not using any information on the asset specific expected returns.

In addition to the above-mentioned MVP, among the mean-variance efficient portfolios we tried to identify that one which results in at least as high expected return as that of the naïve portfolio[5]. In our study this portfolio is labelled as N-MVP. In order to create this portfolio the following constraint should be included in model (1).

$$\sum_{i=1}^{n} x_i E(R_i) \geq \frac{\sum_{i=1}^{n} E(R_i)}{n} \qquad (2)$$

where $E(R_i)$ denotes the expected return of the *i*th asset.

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[3] About the foundations of the theory see also Markowitz (1991).
[4] For a detailed description of this strategy and an application in international diversification see Bugár and Maurer (1999).
[5] In case of the naïve portfolio equal weights are invested in each asset.

2.2 Optimization by conditional value-at-risk (CVaR)

CVaR was introduced into portfolio optimization quite recently by Rockafellar and Uryasev (2002a, 2002b) as an alternative to VaR.

Let $\underline{R}_1, \underline{R}_2, ..., \underline{R}_q$ [6] be a sample set of return vectors. For a particular realisation of portfolio returns, i.e. for a specific return vector the loss on a portfolio can be determined as:

$$L_{p_k} = -\underline{x}^T \underline{R}_k = -\sum_{i=1}^{n} x_i R_{k,i} \tag{3}$$

where $\underline{x}^T$ is the transpose of vector $\underline{x}$.

In order to identify the portfolio with the minimum CVaR, as it is shown by Rockafellar and Uryasev (2002a), the following linear programming problem has to be solved:

$$\begin{aligned} &\min_{\underline{x},\zeta} CVaR(\underline{x},\zeta) = \zeta + \frac{1}{q(1-\alpha)} \sum_{k=1}^{q} u_k \\ &\text{subject to} \\ &\underline{x}^T \underline{R}_k + \zeta + u_k \geq 0, \\ &u_k \geq 0, \ k = 1,2,...,q \\ &\sum_{i=1}^{n} x_i = 1, \ x_i \geq 0, \quad i = 1, 2, ..., n \end{aligned} \tag{4}$$

By solving (4) we find the optimal portfolio weights ($\underline{x}^*$) as well as the corresponding VaR (ζ^*). The resulting portfolio will be referred to as MCVaR.

The MCVaR strategy intends to identify the portfolio (MCVaR) with the lowest estimated risk (measured by CVaR), not using any information on the asset specific expected returns.

In addition to MCVaR defined above, among the mean-CVaR efficient portfolios we tried to determine that one which results in at least as high expected return as that of the naïve portfolio. This portfolio is denoted by N-MCVaR. In order to set up this portfolio the following constraint (2) should be involved in model (4).

[6] The number of elements in the sample set equals the number of the return observations in the time series of returns, while the dimension of the vectors is equal to the number of assets in the portfolio.

2.3 Log-optimal portfolio strategy

Györfi, Ottucsák and A. Urbán (2007) have studied a strategy which strives to maximize the natural logarithm of portfolio return. The empirical version of this strategy utilizes the empirical distributions of the different asset returns. The optimization problem can be formulated as follows:

$$\max E[\ln R_p(\underline{x})] = \frac{1}{q} \cdot \sum_{k=1}^{q} \ln(\sum_{i=1}^{n} x_i R_{ik})$$

subject to (5)

$$\sum_{i=1}^{n} x_i = 1, \;\; x_i \geq 0, \quad i = 1, 2, ..., n$$

where R_{ik} stands for the *k*th return observation of the *i*th portfolio asset (*q* is the number of return observations in each return series).

3. Empirical analysis

3.1. Description of data and research design

The empirical analysis was performed on equity price index returns of 18 developed stock markets. We utilised MSCI equity indexes taken from Datastream[7]. The stock markets include Australia (AU), Austria (AT), Belgium (BE), Canada (CA), Denmark (DK), France (FR), Germany (DE), Hong Kong (HK), Italy (IT), Japan (JP), the Netherlands (NL), Norway (NO), Singapore (SG), Spain (ES), Sweden (SE), Switzerland (CH), the United Kingdom (GB) and the United States of America (US). The calculations were based on US-Dollar returns, i.e. the perspective of an American investor has been taken.

The time period comprised 29 years, starting from the 10th of July 1980 and ending on the same date 2009.

We studied the real profitability and risk adjusted performance of the six different portfolio optimization strategies described in Section 2 by applying an out-of-sample back-testing[8] procedure. In order to perform a longitudinal study, five-year long investment

[7] The authors are grateful to Barnabás Ács and Mónika Horváth (Thomson Reuters) for their assistance in data collection.
[8] In other words, we made an ex-ante analysis.

(holding) periods have been created on a roll-over basis. In forming the relevant holding periods the following procedure has been followed. First of all, the return data of the first two years, in particular the period from the 10th of July 1980 until the 9th of July 1982 has been kept for estimation purpose. Therefore, the first holding period comprised that one starting on the 10th of July 1982 and ending on the 10th of July 1987. The second holding period has been constructed by sliding the starting time point by one year, so we considered the period from the 10th of July 1983 until the 10th of 1988. This process bas been repeated until we reached the final time point. With this procedure we got altogether 23 holding periods which seem to be appropriate to simulate 23 different investment decisions for each strategy.

As it is well-known, the ex-ante estimation technique relies on an estimation period as well as on a forecasting period. The latter one is always considered to be equivalent with the holding period. As we already mentioned, we worked with a two-year estimation period. In particular, for each holding period we used the daily return data of the antecedent two-year period, namely the daily equity price index return series[9] of all the 18 developed markets to estimate the relevant portfolio weights for the subsequent week. The optimization has been performed in case of each strategy, i.e. the portfolio allocation decision resulted by the corresponding strategy has been simulated. Based on the index returns of the particular markets experienced on the subsequent day and the resulted portfolio weights into the different markets, the realised (daily) portfolio return has been calculated for each strategy.

Then the portfolio allocation decision has been revised in a week, i.e. rebalancing of the portfolio weights has been performed in every seventh day. By doing so, the estimation period has been moved one week forward, and the new portfolio weights along with the realised return have been computed again for the next adjacent week for each strategy. The realised returns for the days of the previous week have been calculated based on the old portfolio weights. Therefore, in accordance with the weekly rebalancing of the relevant portfolio, we applied a sliding window of one week to get the subsequent two-year estimation period in simulating the corresponding new portfolio allocation decision.

As a result of the above described process, we could rely on altogether 1250 realised daily returns[10] for each holding period and optimization strategy.

In this work, we consider return in percentage terms. Furthermore, loss is meant as the corresponding negative return. Therefore, it is also given in percent. As a consequence,

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[9] In general, calculating with 250 trading days in a year, we can rely on 500 data points in the estimation period.
[10] Under the condition that the number of trading days is 250 in each year of the five-year holding period.

CVaR is also determined as a percentage value. In CVaR computations 95 percent confidence level has been used.

In addition to the five portfolio optimization strategies described in Section 2, we also considered the naïve portfolio and the US stock index. As a result, altogether we compare and contrast seven different portfolio optimization strategies.

3.2. Presentation and interpretation of the results

In this section the different allocation strategies are compared and contrasted in terms of their *profitability*, *volatility* of the realized returns produced in the various 5-year investment holding periods as well as in terms of their *risk adjusted performance*. As volatility measure not only the standard deviation (SD) but also CVaR has been used. In order to be consistent, risk adjusted performance was evaluated both by the ratio of average daily realised return to SD and that of the average return to CVaR, respectively.

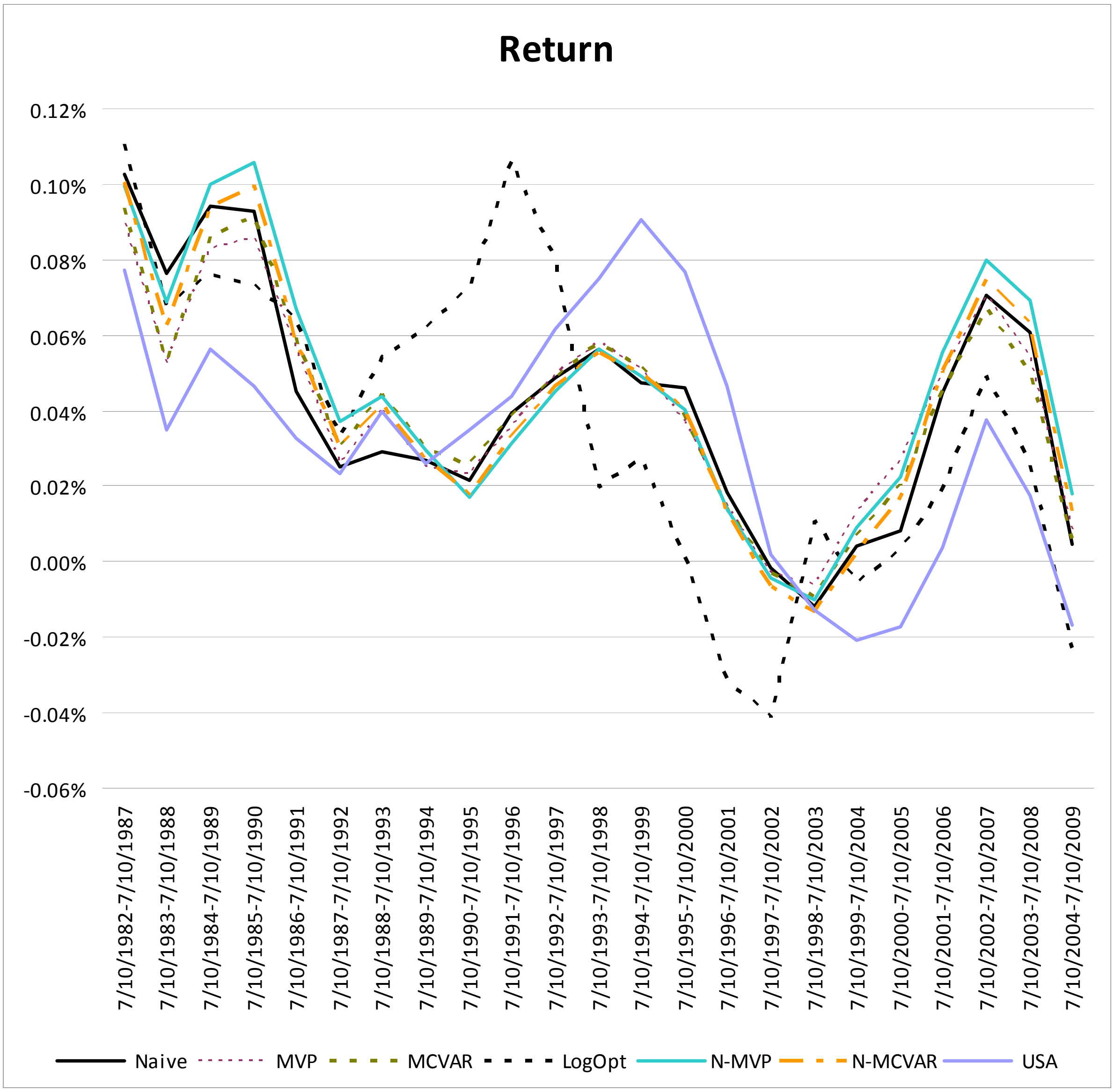
The final aim was to make a comparison between the different strategies regarding their long-term "success-pattern", i.e. to evaluate them in the whole ex-ante time horizon containing 23 investment holding periods.

Table 6 in the Appendix shows the results of the ex-analysis. In particular, the daily average realized return, the value of the above mentioned volatility as well as the risk adjusted performance measures is listed for each investment holding period and each strategy.

In order to make it more illustrative, here we present the graphs showing the long-term pattern of the various measures for all the seven strategies considered including the US stock price index as a proxy of the US domestic portfolio. All the other strategies represent an internationally diversified portfolio.

The daily average (realised) return provided by the different allocation strategies is shown on Figure 1. All the return curves except that of the log-optimal portfolio follow a similar pattern characterised by up and down movements of the stock exchanges dictated by the state of the world economy. The first declining part is due to the international debt crisis which broke out in August 1982. It has been followed by a relatively short upward tendency and then by a downturn again. One can also observe the negative effects of the Asian currency crises which began in mid-1997 as well as the Russian financial crisis on its track in 1998. After this there was a period of economic expansion. The very last downward slope depicted on Figure 1 is due to the recent financial crisis.

Figure 1 Realized return (daily average) given by seven different allocation strategies in different five-year holding periods over a timeline of 27 years



Note: All the proceeds were supposed to be reinvested during the relevant investment holding periods. Therefore, the average return is meant to be a geometric average.

The log-optimal strategy significantly outperformed all the others in terms of average profitability in five consecutive investment holding periods between 1988 and 1992 (these are starting time points of the relevant overlapping periods). Just the opposite has happened in the next six periods. In two of them it has even resulted in a negative daily average return. Following this, when all the other strategies had a descending average return, it has produced an ascending one. Based on the above mentioned facts, in most of the cases the pattern of average return provided by the log-optimal strategy is seemingly in “disparity” with those ones produced by the other six allocation strategies.

Table 1 summarizes the number of those occasions a particular strategy has produced a higher average return than another one for each pair of strategies. The number stands in a particular cell shows the number of investment holding periods (out of 23) when the strategy in the appropriate column had outperformed the strategy indicated in the corresponding row in terms of average profitability. For instance, the highest number (18) in Table 1 means that N-MVP resulted in a higher average return than N-MCVaR in 18 cases. So, there were only 5 cases when N-MCVaR has outperformed N-MVP. The last row shows the average rank for each strategy which has been computed based on the rank numbers ranging from 1 to 7. The rank number has been formed to indicate the place of a given strategy in the descending order of average realized returns in the different holding periods.

Table 1 Comparison of long-term profitability of the different portfolio allocation strategies

	Naïve	MVP	MCVAR	LogOpt	N-MVP	N-MCVAR	USA
Naïve	-	12	13	9	15	12	9
MVP	11	-	12	10	13	13	10
MCVAR	10	11	-	10	14	9	8
LogOpt	14	13	13	-	16	13	6
N-MVP	8	10	9	7	-	5	8
N-MCVAR	11	10	14	10	18	-	9
USA	14	13	15	17	15	14	-
Average rank	**4.04**	**4.00**	**3.70**	**4.26**	**3.04**	**4.13**	**4.83**

Note: The average rank for a particular strategy has been calculated by averaging the rank numbers given in the different investment holding periods. A rank number indicates the place of the particular strategy in the rank order of all allocation strategies in a given period. Therefore, the rank number can be any integer from 1 to 7 where 1 indicates the highest average return and 7 the lowest one in a particular holding period.

In terms of average realized return, with the average rank of 3.04, the N-MVP has proved to be the best among all strategies considered. The MCVaR was the second best (3.7), followed by the MVP (4.0) and the Naïve (4.04). The Log-optimal portfolio was the not so successful on the long run as one could expect it based on the fact that the profitability criterion has priority in this case. It is probably due to the high estimation error in forecasting the expected portfolio return from past data. The US domestic portfolio has proved to be the least profitable. It suggests that international diversification pays in terms of higher realized return.

Figure 2 and Figure 3 present the volatility patterns of the daily realized return distributions. In the first case the standard deviation was used as a volatility measure while in the latter case the CVaR was applied. Looking at the above mentioned figures simultaneously one can recognize the patterns depicted by the different curves are very similar. It indicates that we should not expect the striking difference between the results given by the two measures. Based on the graphs of the average returns, it is even intuitively obvious that the profitability of log-optimal strategy as well as that of the US domestic portfolio show the highest volatility. It is also remarkable that the log-optimal strategy had the highest volatility over the whole time horizon studied. All the other five strategies resulted in a significantly lower volatility than those two mentioned above. Furthermore, the volatility curves also seem to be more stable in time in the earlier cases than in the latter ones. However, it is remarkable that due to the recent financial crisis there is a sharp increase in both volatility measures from the last but one to the last holding period. For most of the strategies the values of the measures have doubled.

Table 2 Comparison of long-term volatility (measured by standard deviation) of the realized return for the different portfolio allocation strategies

	Naïve	MVP	MCVAR	LogOpt	N-MVP	N-MCVAR	USA
Naïve	-	23	23	0	22	22	3
MVP	0	-	6	0	0	0	0
MCVAR	0	17	-	0	8	0	0
LogOpt	23	23	23	-	23	23	23
N-MVP	1	23	15	0	-	10	0
N-MCVAR	1	23	23	0	13	-	0
USA	20	23	23	0	23	23	-
Average rank	**5.04**	**1.26**	**2.09**	**7.00**	**3.13**	**3.61**	**5.87**

Note: The average rank for a particular strategy has been calculated by averaging the rank numbers given in the different investment holding periods. A rank number indicates the place of the particular strategy in the rank order of all allocation strategies in a given period. Therefore, the rank number can be any integer from 1 to 7 where 1 indicates the lowest volatility of the realized return and 7 the highest one in a particular holding period.

Table 2 and Table 3 present a pair-wise comparison between the volatility measures of the different portfolio allocation strategies. The number stands in a particular cell shows the number of investment holding periods (out of 23) when the strategy in the appropriate column had outperformed the strategy indicated in the corresponding row in terms of possessing lower volatility of the realized return.

Figure 2 Volatility (measured by the standard deviation) of the daily realized returns given by seven different allocation strategies in different five-year holding periods over a timeline of 27 years

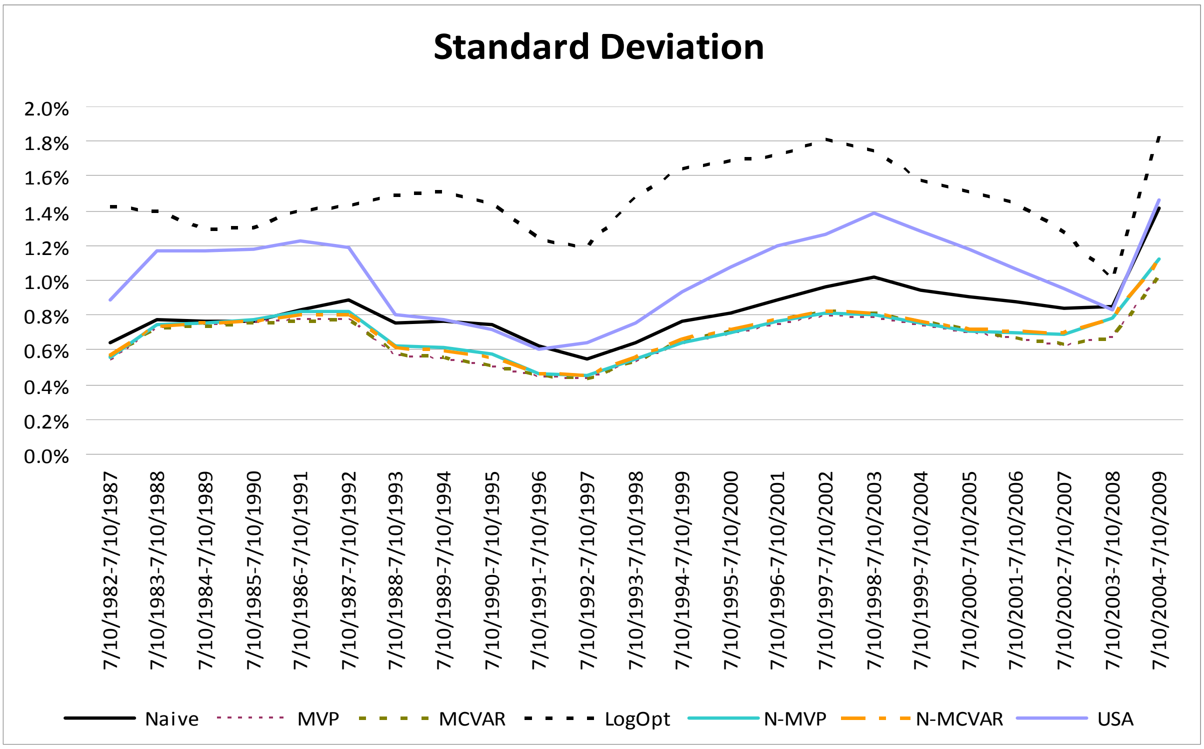


Figure 3 Volatility (measured by CVaR) of the daily realized returns given by seven different allocation strategies in different five-year holding periods over a timeline of 27 years

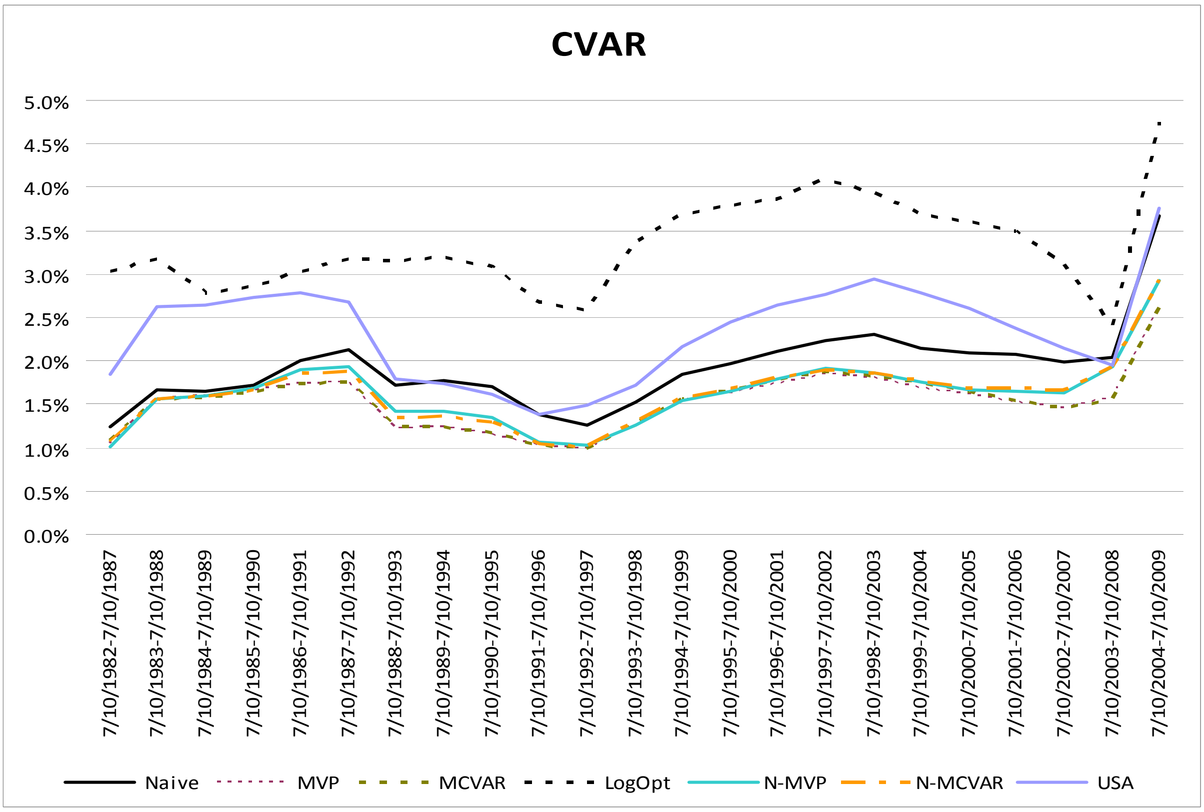


Table 3 Comparison of long-term volatility (measured by CVaR) of the realized return for the different portfolio allocation strategies

	Naïve	MVP	MCVAR	LogOpt	N-MVP	N-MCVAR	USA
Naïve	-	23	23	0	23	23	3
MVP	0	-	7	0	5	2	0
MCVAR	0	16	-	0	5	1	0
LogOpt	23	23	23	-	23	23	23
N-MVP	0	18	18	0	-	11	0
N-MCVAR	0	21	22	0	12	-	0
USA	20	23	23	0	23	23	-
Average rank	**5.13**	**1.61**	**1.96**	**7.00**	**3.04**	**3.39**	**5.87**

In accordance with our observation based on Figure 2 and Figure 3 there was not a single period when the volatility of the log-optimal strategy would have been lower than that of any of the other strategies. That explains that the average rank for this strategy takes the highest possible value (7).

Either looking at Table 2 or Table 3 we can set up the same rank order of the strategies in terms of volatility. The MVP has proved to be the best with respect to both measures. Considering the standard deviation (CVaR) its average rank is 1.26 (1.61) indicating it had the lowest volatility in most of the cases. The second best is the MCVaR with the average rank of 2.09 (1.96).[11] The results above are not surprising because of the fact that both strategies are striving for minimizing the risk. The rank order continues with the N-MVP (it average rank is 3.13 and 3.04, respectively). It is followed by the N-MCVaR (the respective values are 3.61 and 3.39). These latter two strategies have an aim to reach at least as a high expected return as that of the naïve diversification strategy with the lowest possible risk. As a result, unlike the MVP and MCVaR, in those cases the risk produced by in-sample estimation is not at the global minimum level. In the volatility order of the strategies the Naïve portfolio takes the fifth place, followed by the US price index, and finally the log-optimal portfolio comes last.

Finally, we analyze the results regarding the performance indicators of the different strategies. The development of average return per standard deviation is depicted on Figure 4, while Figure 5 shows the graph of average return per CVaR. Despite the fact that the numeric

[11] As it can be seen there is a difference in the average ranks given by the standard deviation and CVaR as a volatility measure but it does not modify rank order of the strategies.

values are different, the curves on both figures follow almost identical patterns. It is a consequence of the high level of “coincidence” in the volatility measures.

The overall performance of the risk optimization strategies is quite favorable compared to those of the log-optimal and US domestic portfolio. In case of the latter one, however, we can identify a time period when it outperformed all the others. Indeed, it had a superior performance in four consecutive investment holding periods between 1994 and 1997 (these are starting time points of the relevant overlapping periods). However, right after this “success” a “disaster” has followed mostly due to the terror attack against the United States in 2001.

Table 4 and Table 5 show a pair-wise comparison between the performance measures of the different portfolio allocation strategies. The number stands in a particular cell shows the number of investment holding periods (out of 23) when the strategy in the appropriate column had outperformed the strategy indicated in the corresponding row in terms of having higher performance. For instance, the MCVaR has performed better in 20 holding periods than the log-optimal strategy (see either Table 4 or Table 5).

Based on the average rank in the last rows of Table 4 and Table 5 we can set up the performance order of the strategies studied. It is in the descending order of performance as follows: N-MVP, MVP, MCVaR, N-MCVAR, Naïve, the US domestic portfolio and the log-optimal portfolio. Despite the fact that the values are different the performance order does not change if we switch from one performance measure to the other one.

Figure 4 Risk adjusted performance given by seven different portfolio allocation strategies in different five-year holding periods over a timeline of 27 years

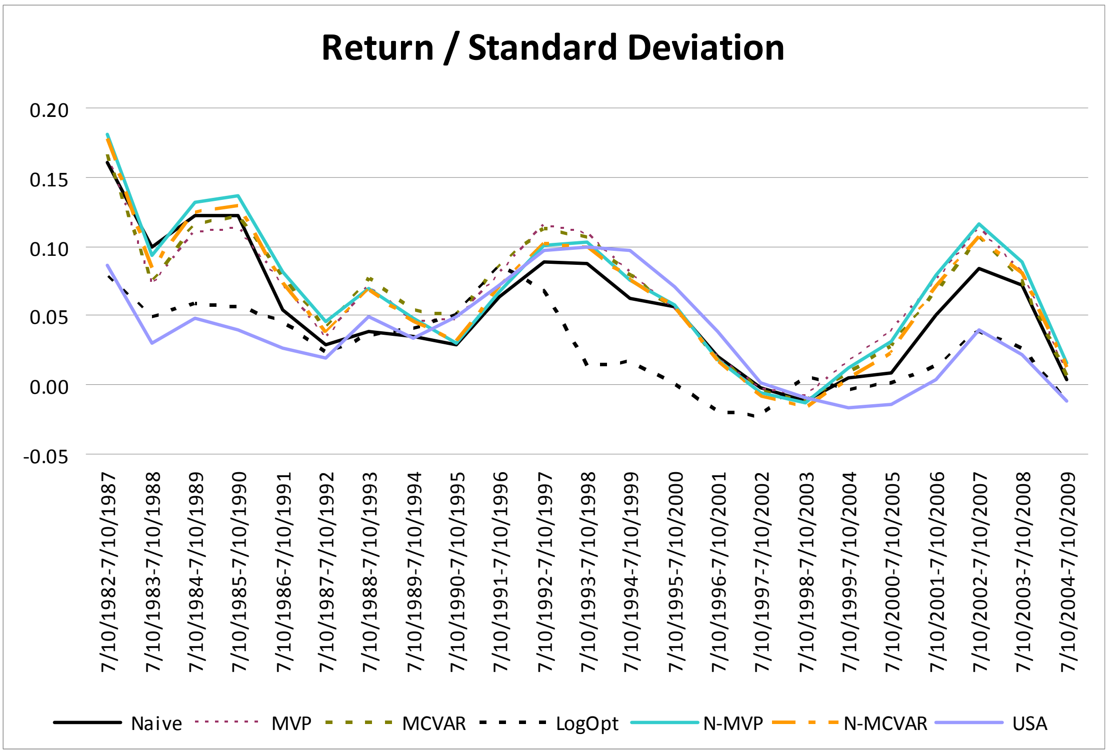


Figure 5 Risk adjusted performance given by seven different portfolio allocation strategies in different five-year holding periods over a timeline of 27 years

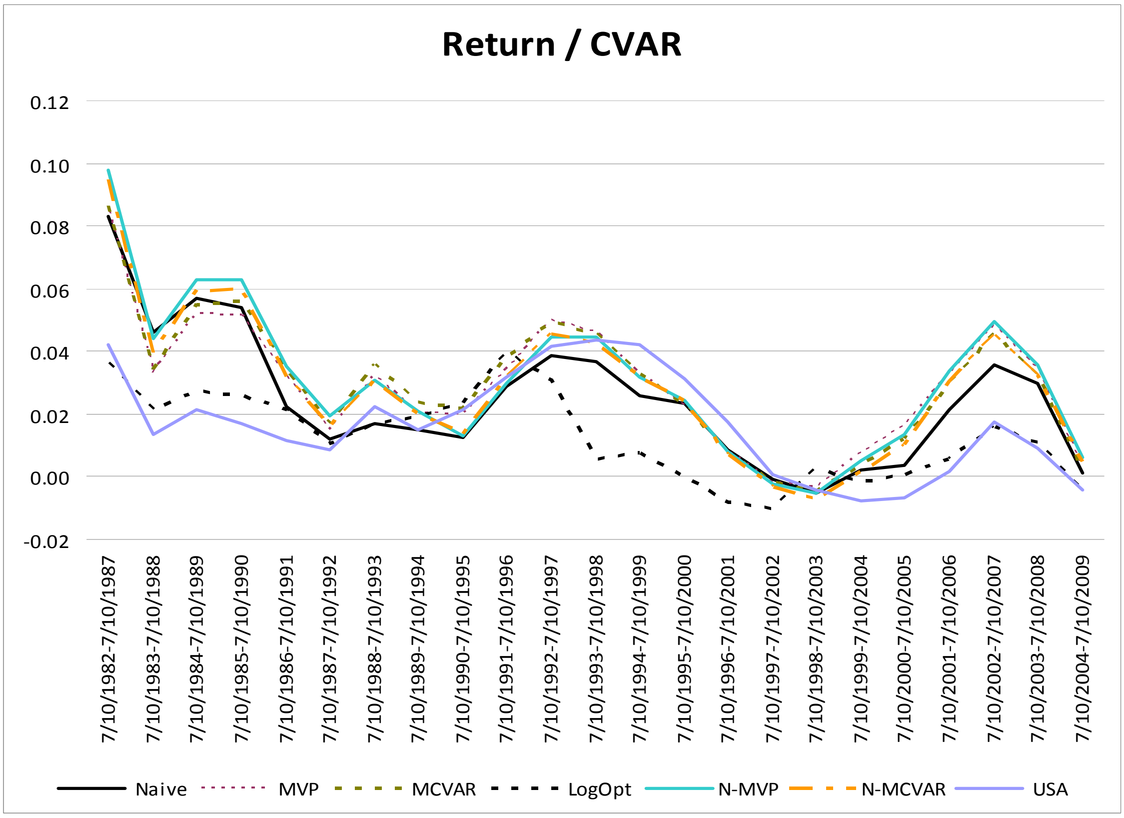


Table 4 Comparison of risk adjusted performance (measured by average holding period return per standard deviation) of the different portfolio allocation strategies

	Naïve	MVP	MCVAR	LogOpt	N-MVP	N-MCVAR	USA
Naïve	-	17	18	4	19	17	10
MVP	6	-	12	3	12	9	5
MCVAR	5	11	-	3	14	9	5
LogOpt	19	20	20	-	20	20	10
N-MVP	4	11	9	3	-	3	7
N-MCVAR	6	14	14	3	20	-	8
USA	13	18	18	13	16	15	-
Average rank	**4.70**	**3.04**	**3.04**	**5.74**	**2.61**	**3.83**	**5.04**

Table 5 Comparison of risk adjusted performance (measured by average holding period return per CVaR) of the different portfolio allocation strategies

	Naïve	MVP	MCVAR	LogOpt	N-MVP	N-MCVAR	USA
Naïve	-	17	18	5	19	18	10
MVP	6	-	11	3	12	7	5
MCVAR	5	12	-	3	14	8	5
LogOpt	18	20	20	-	20	20	10
N-MVP	4	11	9	3	-	3	7
N-MCVAR	5	16	15	3	20	-	8
USA	13	18	18	13	16	15	-
Average rank	**4.78**	**2.91**	**3.04**	**5.70**	**2.61**	**3.91**	**5.04**

Note: The average rank for a particular strategy has been calculated by averaging the rank numbers given in the different investment holding periods. A rank number indicates the place of the particular strategy in the rank order of all allocation strategies in a given period. Therefore, the rank number can be any integer from 1 to 7 where 1 indicates the highest risk adjusted performance and 7 the lowest one in a particular holding period.

4. Conclusions

In this paper we studied the performance of several portfolio optimization strategies and their characteristics in different 5-year holding periods over the time horizon of almost three decades. The selected strategies were two of the classic mean–variance (MVP and N-MVP), two of the mean–CVaR (MCVaR and N-MCVaR) and the log–optimal portfolio optimization strategies. For benchmark purposes the naïve portfolio and the US stock price index was used. Four of the above approaches work in the risk-return plane. Therefore, we also intended to investigate how portfolio performance is affected by the choice of the volatility measure. Equity index returns of 18 developed markets are used for an ex-ante analysis and the profitability of each optimization strategy is measured and compared to that of the US stock price index.

The main conclusions of the present study can be summarized as follows.

In terms of average realized return the N-MVP has proved to be the best among all strategies considered. The MCVaR was the second best, followed by the MVP, the Naïve portfolio and the N-MCVaR, respectively. The log-optimal portfolio was not so successful on the long run as one could expect it based on the fact that the profitability criterion has priority in this case. It is probably due to the high estimation error in forecasting the expected portfolio return from the past empirical return distribution. The US domestic portfolio has proved to be the least profitable. It suggests that international diversification pays in terms of higher realized return.

The MVP has proved to be the best strategy with respect to both volatility measures. The second best is the MCVaR. The results above are not surprising because of the fact that both strategies are striving for minimizing the risk. The rank order continues with the N-MVP. It is followed by the N-MCVaR. These latter two strategies have an aim to reach at least as a high expected return as that of the naïve diversification strategy with the lowest possible risk. As a result, unlike the MVP and MCVaR, in those cases the risk produced by in-sample estimation is not at the global minimum level. In the volatility order of the strategies the Naïve portfolio takes the fifth place, followed by the US price index, and the log-optimal portfolio comes very last.

It is also remarkable that the realized returns of the log-optimal strategy had the highest volatility over the whole time horizon studied. It is worth mentioning that due to the recent financial crisis there is a sharp increase in both volatility measures from the last but one to the last holding period. For most of the strategies the values of the measures have doubled.

The overall performance of the risk optimization strategies was quite favorable compared to those of the log-optimal and the US domestic portfolio. Based on the average rank we can set up the performance order of the strategies studied. It is in the descending order of performance as follows: N-MVP, MVP, MCVaR, N-MCVAR, Naïve, the US domestic portfolio and the log-optimal portfolio.

An imperfection of the present study is that we have not calculated with transaction costs which occur in reality. Despite the fact that there was a significant drop in them during the last decade, they might modify our findings. It is a task for future research to take them into consideration.

We have defined multiple dimensions to measure the success of portfolios in our study. These dimensions are return, risk and the risk adjusted performance of the portfolios. We can conclude that the ranking of strategies based on riskiness is stable during the investigated timeline, suggesting that the use of the risk minimization strategies MVP and MCVaR are just as capable of achieving their objective on a turbulent market as in normal market circumstances.

The ranking of the strategies using the return dimension, however, is much more volatile. There are very significant differences in the results, and we are also unable to observe the kind of correlation between the returns of the strategies that we could see with their risk. Looking at risk measures we could observe different trends during the investigated time period that have moved the riskiness of all strategies in the same direction.

The performance dimension inherits the volatility of the return dimension, but the volatility is somewhat decreased by the stable risk values.

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Appendix

Table 6 Profitability, volatility and risk adjusted performance given by seven different allocation strategies in different five-year holding periods over a timeline of 27 years

Holding Period #1	7/10/1982-7/10/1987						
Strategies	Naive	MVP	MCVaR	LogOpt	N-MVP	N-MCVaR	USA
Average Return ($\overline{R}$)	0.103%	0.089%	0.094%	0.111%	0.100%	0.100%	0.077%
CVaR	1.24%	1.05%	1.08%	3.04%	1.02%	1.06%	1.84%
SD	0.64%	0.54%	0.56%	1.43%	0.55%	0.57%	0.89%
$\overline{R}$ /SD	0.160	0.165	0.167	0.077	0.180	0.177	0.087
$\overline{R}$ /CVaR	0.083	0.085	0.087	0.036	0.098	0.095	0.042

Holding Period #2	7/10/1983-7/10/1988						
Strategies	Naive	MVP	MCVaR	LogOpt	N-MVP	N-MCVaR	USA
Average Return ($\overline{R}$)	0.076%	0.052%	0.053%	0.068%	0.069%	0.062%	0.035%
CVaR	1.66%	1.57%	1.56%	3.17%	1.56%	1.56%	2.62%
SD	0.77%	0.73%	0.72%	1.40%	0.74%	0.74%	1.17%
$\overline{R}$ /SD	0.099	0.071	0.073	0.049	0.093	0.084	0.030
$\overline{R}$ / CVaR	0.046	0.033	0.034	0.021	0.044	0.040	0.013

Holding Period #3	7/10/1984-7/10/1989						
Strategies	Naive	MVP	M CVaR	LogOpt	N-MVP	N-MCVaR	USA
Average Return ($\overline{R}$)	0.094%	0.083%	0.086%	0.076%	0.100%	0.094%	0.056%
CVaR	1.66%	1.59%	1.58%	2.78%	1.59%	1.59%	2.63%
SD	0.77%	0.75%	0.74%	1.30%	0.76%	0.75%	1.17%
$\overline{R}$ /SD	0.122	0.111	0.117	0.059	0.132	0.125	0.048
$\overline{R}$ / CVaR	0.057	0.052	0.055	0.027	0.063	0.059	0.021

Holding Period #4	7/10/1985-7/10/1990						
Strategies	Naive	MVP	M CVaR	LogOpt	N-MVP	N-MCVaR	USA
Average Return ($\overline{R}$)	0.093%	0.085%	0.092%	0.073%	0.106%	0.100%	0.046%
CVaR	1.72%	1.66%	1.64%	2.85%	1.69%	1.66%	2.74%
SD	0.76%	0.76%	0.75%	1.30%	0.77%	0.77%	1.18%
$\overline{R}$ /SD	0.122	0.113	0.122	0.056	0.137	0.130	0.039
$\overline{R}$ / CVaR	0.054	0.051	0.056	0.026	0.063	0.060	0.017

Holding Period #5	7/10/1986-7/10/1991						
Strategies	Naive	MVP	M CVaR	LogOpt	N-MVP	N-MCVaR	USA
Average Return ($\overline{R}$)	0.045%	0.056%	0.059%	0.064%	0.067%	0.059%	0.033%
CVaR	2.01%	1.73%	1.73%	3.03%	1.90%	1.86%	2.79%
SD	0.83%	0.77%	0.77%	1.40%	0.82%	0.80%	1.22%
$\overline{R}$ /SD	0.054	0.073	0.078	0.046	0.082	0.073	0.027
$\overline{R}$ / CVaR	0.022	0.032	0.034	0.021	0.035	0.032	0.012

Holding Period #6	7/10/1987-7/10/1992						
Strategies	Naive	MVP	M CVaR	LogOpt	N-MVP	N-MCVaR	USA
Average Return ($\overline{R}$)	0.025%	0.026%	0.030%	0.033%	0.037%	0.030%	0.023%
CVaR	2.13%	1.75%	1.76%	3.17%	1.94%	1.88%	2.68%
SD	0.88%	0.78%	0.77%	1.42%	0.82%	0.80%	1.19%
$\overline{R}$ /SD	0.029	0.033	0.039	0.023	0.045	0.037	0.020
$\overline{R}$ / CVaR	0.012	0.015	0.017	0.010	0.019	0.016	0.009

Holding Period #7	7/10/1988-7/10/1993						
Strategies	Naive	MVP	M CVaR	LogOpt	N-MVP	N-MCVaR	USA
Average Return ($\overline{R}$)	0.029%	0.040%	0.044%	0.054%	0.044%	0.042%	0.040%
CVaR	1.72%	1.23%	1.23%	3.16%	1.41%	1.35%	1.78%
SD	0.75%	0.57%	0.57%	1.49%	0.63%	0.61%	0.80%
$\overline{R}$ /SD	0.039	0.070	0.077	0.036	0.070	0.068	0.050
$\overline{R}$ / CVaR	0.017	0.032	0.036	0.017	0.031	0.031	0.022

Holding Period #8	7/10/1989-7/10/1994						
Strategies	Naive	MVP	M CVaR	LogOpt	N-MVP	N-MCVaR	USA
Average Return ($\overline{R}$)	0.027%	0.025%	0.030%	0.061%	0.029%	0.027%	0.026%
CVaR	1.77%	1.23%	1.24%	3.19%	1.42%	1.36%	1.75%
SD	0.77%	0.55%	0.56%	1.51%	0.61%	0.60%	0.77%
$\overline{R}$ /SD	0.035	0.046	0.054	0.041	0.048	0.045	0.034
$\overline{R}$ / CVaR	0.015	0.020	0.024	0.019	0.021	0.020	0.015

Holding Period #9	7/10/1990-7/10/1995						
Strategies	Naive	MVP	M CVaR	LogOpt	N-MVP	N-MCVaR	USA
Average Return ($\overline{R}$)	0.021%	0.023%	0.026%	0.072%	0.017%	0.018%	0.035%
CVaR	1.71%	1.16%	1.17%	3.08%	1.34%	1.30%	1.62%
SD	0.74%	0.50%	0.51%	1.44%	0.57%	0.56%	0.71%
$\overline{R}$ /SD	0.029	0.046	0.050	0.050	0.030	0.032	0.049
$\overline{R}$ / CVaR	0.013	0.020	0.022	0.024	0.013	0.014	0.022

Holding Period #10	7/10/1991-7/10/1996						
Strategies	Naive	MVP	M CVaR	LogOpt	N-MVP	N-MCVaR	USA
Average Return ($\overline{R}$)	0.039%	0.036%	0.038%	0.106%	0.031%	0.033%	0.044%
CVaR	1.38%	1.03%	1.02%	2.68%	1.05%	1.05%	1.38%
SD	0.62%	0.44%	0.45%	1.24%	0.46%	0.47%	0.61%
$\overline{R}$ /SD	0.064	0.080	0.085	0.086	0.068	0.071	0.072
$\overline{R}$ / CVaR	0.029	0.034	0.038	0.040	0.030	0.032	0.032

Holding Period #11	7/10/1992-7/10/1997						
Strategies	Naive	MVP	M CVaR	LogOpt	N-MVP	N-MCVaR	USA
Average Return ($\overline{R}$)	0.049%	0.050%	0.049%	0.080%	0.045%	0.046%	0.062%
CVaR	1.25%	1.00%	0.98%	2.58%	1.02%	1.02%	1.48%
SD	0.55%	0.43%	0.43%	1.19%	0.45%	0.45%	0.64%
$\overline{R}$ /SD	0.089	0.115	0.112	0.067	0.100	0.102	0.097
$\overline{R}$ / CVaR	0.039	0.050	0.050	0.031	0.044	0.046	0.042

Holding Period #12	7/10/1993-7/10/1998						
Strategies	Naive	MVP	M CVaR	LogOpt	N-MVP	N-MCVaR	USA
Average Return ($\overline{R}$)	0.056%	0.058%	0.058%	0.020%	0.056%	0.056%	0.075%
CVaR	1.52%	1.26%	1.28%	3.38%	1.26%	1.30%	1.72%
SD	0.64%	0.53%	0.54%	1.47%	0.55%	0.56%	0.75%
$\overline{R}$ /SD	0.087	0.110	0.107	0.013	0.103	0.100	0.100
$\overline{R}$ / CVaR	0.037	0.046	0.045	0.006	0.045	0.043	0.044

Holding Period #13	7/10/1994-7/10/1999						
Strategies	Naive	MVP	M CVaR	LogOpt	N-MVP	N-MCVaR	USA
Average Return ($\overline{R}$)	0.048%	0.051%	0.051%	0.027%	0.049%	0.050%	0.091%
CVaR	1.85%	1.54%	1.57%	3.69%	1.54%	1.57%	2.16%
SD	0.76%	0.63%	0.65%	1.64%	0.64%	0.66%	0.94%
$\overline{R}$ /SD	0.062	0.080	0.079	0.017	0.076	0.075	0.097
$\overline{R}$ / CVaR	0.026	0.033	0.033	0.007	0.032	0.032	0.042

Holding Period #14	7/10/1995-7/10/2000						
Strategies	Naive	MVP	M CVaR	LogOpt	N-MVP	N-MCVaR	USA
Average Return ($\overline{R}$)	0.046%	0.037%	0.039%	0.001%	0.040%	0.040%	0.077%
CVaR	1.96%	1.63%	1.67%	3.79%	1.65%	1.68%	2.45%
SD	0.81%	0.69%	0.71%	1.69%	0.70%	0.71%	1.08%
$\overline{R}$ /SD	0.056	0.054	0.054	0.001	0.058	0.056	0.071
$\overline{R}$ / CVaR	0.023	0.023	0.023	0.000	0.024	0.024	0.031

Holding Period #15	7/10/1996-7/10/2001						
Strategies	Naive	MVP	M CVaR	LogOpt	N-MVP	N-MCVaR	USA
Average Return ($\overline{R}$)	0.018%	0.014%	0.014%	-0.032%	0.014%	0.013%	0.046%
CVaR	2.11%	1.73%	1.78%	3.87%	1.80%	1.81%	2.65%
SD	0.89%	0.75%	0.78%	1.72%	0.77%	0.78%	1.20%
$\overline{R}$ /SD	0.021	0.019	0.018	-0.018	0.018	0.017	0.039
$\overline{R}$ / CVaR	0.009	0.008	0.008	-0.008	0.008	0.007	0.018

Holding Period #16	7/10/1997-7/10/2002						
Strategies	Naive	MVP	M CVaR	LogOpt	N-MVP	N-MCVaR	USA
Average Return ($\overline{R}$)	-0.002%	-0.003%	-0.003%	-0.041%	-0.004%	-0.007%	0.002%
CVaR	2.23%	1.85%	1.88%	4.10%	1.91%	1.90%	2.77%
SD	0.96%	0.79%	0.82%	1.81%	0.81%	0.82%	1.27%
$\overline{R}$ /SD	-0.002	-0.004	-0.004	-0.023	-0.005	-0.008	0.002
$\overline{R}$ / CVaR	-0.001	-0.002	-0.002	-0.010	-0.002	-0.003	0.001

Holding Period #17	7/10/1998-7/10/2003						
Strategies	Naive	MVP	M CVaR	LogOpt	N-MVP	N-MCVaR	USA
Average Return ($\overline{R}$)	-0.012%	-0.006%	-0.009%	0.011%	-0.010%	-0.013%	-0.013%
CVaR	2.31%	1.80%	1.83%	3.94%	1.86%	1.86%	2.95%
SD	1.01%	0.78%	0.81%	1.75%	0.80%	0.81%	1.38%
$\overline{R}$ /SD	-0.012	-0.008	-0.011	0.006	-0.013	-0.016	-0.009
$\overline{R}$ / CVaR	-0.005	-0.003	-0.005	0.003	-0.005	-0.007	-0.004

Holding Period #18	7/10/1999-7/10/2004						
Strategies	Naive	MVP	M CVaR	LogOpt	N-MVP	N-MCVaR	USA
Average Return ($\overline{R}$)	0.004%	0.013%	0.007%	-0.006%	0.009%	0.003%	-0.021%
CVaR	2.15%	1.69%	1.75%	3.70%	1.76%	1.78%	2.78%
SD	0.94%	0.74%	0.76%	1.57%	0.76%	0.76%	1.29%
$\overline{R}$ /SD	0.004	0.017	0.009	-0.004	0.012	0.003	-0.016
$\overline{R}$ / CVaR	0.002	0.008	0.004	-0.002	0.005	0.001	-0.007

Holding Period #19	7/10/2000-7/10/2005						
Strategies	Naive	MVP	M CVaR	LogOpt	N-MVP	N-MCVaR	USA
Average Return ($\overline{R}$)	0.008%	0.027%	0.020%	0.003%	0.022%	0.017%	-0.017%
CVaR	2.09%	1.61%	1.65%	3.60%	1.67%	1.69%	2.61%
SD	0.90%	0.70%	0.71%	1.51%	0.71%	0.72%	1.18%
$\overline{R}$ /SD	0.009	0.039	0.028	0.002	0.032	0.023	-0.015
$\overline{R}$ / CVaR	0.004	0.017	0.012	0.001	0.013	0.010	-0.007

Holding Period #20	7/10/2001-7/10/2006						
Strategies	Naive	MVP	M CVaR	LogOpt	N-MVP	N-MCVaR	USA
Average Return ($\overline{R}$)	0.045%	0.050%	0.045%	0.019%	0.056%	0.051%	0.004%
CVaR	2.08%	1.53%	1.54%	3.48%	1.65%	1.68%	2.38%
SD	0.88%	0.66%	0.67%	1.44%	0.70%	0.71%	1.06%
$\overline{R}$ /SD	0.051	0.076	0.067	0.013	0.080	0.071	0.003
$\overline{R}$ / CVaR	0.022	0.033	0.029	0.006	0.034	0.030	0.001

Holding Period #21	7/10/2002-7/10/2007						
Strategies	Naive	MVP	M CVaR	LogOpt	N-MVP	N-MCVaR	USA
Average Return ($\overline{R}$)	0.071%	0.070%	0.067%	0.049%	0.080%	0.075%	0.037%
CVaR	1.99%	1.45%	1.46%	3.11%	1.62%	1.66%	2.14%
SD	0.84%	0.63%	0.63%	1.27%	0.69%	0.70%	0.95%
$\overline{R}$ /SD	0.084	0.113	0.105	0.038	0.116	0.107	0.039
$\overline{R}$ / CVaR	0.036	0.049	0.046	0.016	0.049	0.045	0.017

Holding Period #22	7/10/2003-7/10/2008						
Strategies	Naive	MVP	MCVaR	LogOpt	N-MVP	N-MCVaR	USA
Average Return ($\overline{R}$)	0.061%	0.054%	0.050%	0.026%	0.069%	0.063%	0.018%
CVaR	2.03%	1.58%	1.57%	2.42%	1.93%	1.94%	1.94%
SD	0.85%	0.67%	0.67%	1.00%	0.78%	0.78%	0.83%
$\overline{R}$ /SD	0.072	0.081	0.075	0.026	0.088	0.081	0.021
$\overline{R}$ / CVaR	0.030	0.034	0.032	0.011	0.036	0.032	0.009

Holding Period #23	7/10/2004-7/10/2009						
Strategies	Naive	MVP	M CVaR	LogOpt	N-MVP	N-MCVaR	USA
Average Return ($\overline{R}$)	0.005%	0.008%	0.006%	-0.023%	0.018%	0.013%	-0.017%
CVaR	3.66%	2.62%	2.62%	4.76%	2.92%	2.92%	3.76%
SD	1.41%	1.02%	1.02%	1.83%	1.13%	1.11%	1.46%
$\overline{R}$ /SD	0.003	0.008	0.006	-0.013	0.016	0.012	-0.011
$\overline{R}$ / CVaR	0.001	0.003	0.002	-0.005	0.006	0.005	-0.004

Note: Realized (ex-ante) returns were computed on a daily basis. Average return refers to the (geometric) average of realized daily returns in the relevant holding period. CVaR and SD (standard deviation) were used to measure the volatility of the realized return distribution. In accordance with the chosen measures of volatility, the risk adjusted performance also was gauged in two different ways, namely with the ratio of average return to CVaR and average return to SD, respectively.