An integrated Cost of Risk model and its application to company valuation.

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Abstract

This paper proposes a new and integrated approach of measuring risk and the associated cost. The model is developed from the simple practical example of a bond spread and then generalized. It is shown that is also encompasses the popular measures Value at Risk and Tail Value at Risk. A suficient condition is established under which the measure is coherent and in an application section market data is used to parametrize the measure and evaluate the capital cost of an example company.

1 Introduction

It is not only since Solvency II that a very active discussion in the actuarial community takes place regarding appropriate risk measures and their practical application. A very often used approach is to take a risk capital of some definition, set a rate of return and multiply both to derive the cost. If the actual return of an undertaking exceeds this cost, it is deemed to be profitable. The answers given are obviously dependent on both the capital definition used and the rate of return chosen.

Coincidentally the target rate of return in most companies seems to be 15% - no matter the risk measure. Furthermore Milne and Onorato have shown in [5] that use of return on capital with a constant hurdle rate can lead to substantial loss of shareholder value. Goldfarb in [1] comes to the similar conclusion that RAROC is not an unambiguous measure.

The motivation for the research underlying this paper has been that the popular capital measures Value at Risk (VaR) and Tail Value at Risk (TVaR) are not capable of sufficiently distinguishing critical characteristics of risk. In the practical experience of the author this has lead to counterintuitive steering indications and intense discussions with decision makers. One example is the commercial value of reinsurance purchased. For a VaR and TVaR with practically used confidence levels two excess of loss layers, one with an exhaustion probability of 10% the other with 2%, will look fairly similar. But it is generally accepted that the risk loads (i.e. the commercial value) of both should be significantly different. Another example are bonds with different ratings. A "C" rated bond will yield significantly higher spreads than a "B" bond. But again for practical VaR and TVaR measures both look the same.

Furthermore, for application in enterprise steering the risk capital is allocated down to risk drivers and/or business lines. Again there are various approaches available for such a task and hence another degree of uncertainty with regard to which numbers ultimately to believe. Gründl and Schmeiser in [6] even go so far as to question whether it is at all sensible to steer a financial service provider by capital allocation.

Bottom line the business steering approaches as described are based on a triangle of three more or less separate models which are tied together to produce some figures. The obvious ambiguity of the results of such a setup have been and still are a source for intensive discussions - not only in the actuarial community, but particularly with practitioners.

To mitigate at least part of the inherent ambiguity the present paper proposes an integrated approach to measure risk without fully giving up the flexibility of looking at different aspects of it at certain times. Depending on the stakeholder of the company he will have varying concerns. Whereas the policyholder or the regulator will put their emphasis on the extreme events, i.e. small percentiles, an investor will rather look to the more likely events, i.e. central percentiles.

In section 2 some general notational conventions have been put down for ease of reference. For a bottom up idea of how an integrated approach to measure risk could look like, section 3 looks at a simple model for bonds as in the respective markets there is already an implicit consensus measurement of risk via the return requirements. The conclusions are then generalized into the risk cost model in section 4. Sections 5 and 6 then are dedicated to establish that the risk cost model encompasses some of the most widespread risk measures and that the measure is under certain conditions coherent. The definition of the risk measure implicitly already contains a "natural" decomposition, which is taken a look at in section 7. The theory is then used in section 8 for the valuation of an example company. Finally in section 9 some issues that need particular attention in the application are summarized.

2 Definitions

The random variables X on which the results of this paper are intended to be applied are valuation distributions of insurance companies. In particular when talking about risk the concern with respect to X will be large negative deviations from the mean – as opposed to the case where X denotes a claims distribution. The time horizon T for the considerations may be arbitrary but shall be fixed further on[1]. It is intentionally not broken down any further to claims, premium, cost and investment result as the interest of any external stakeholder will not

[1] Typically it is one year as for instance in the Solvency II framework.

differentiate between where the risk comes from but rather focus on the mere existence of it.

For a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ the set of all real valued random variables shall be denoted by L and for each $X \in L$ its distribution function F_X shall be defined as usual:

$$F_X : \mathbb{R} \to [0,1],\ F_X(x) = \mathbf{P}(X \leq x).$$

The function of upper quantiles is defined as

$$Q_X^u:[0,1] \to \mathbb{R},\ Q_X^u(\alpha) = \sup\{x \in \mathbb{R}:\mathbf{P}(X \leq x) \leq \alpha\}.$$

In this definition it is always the case that $Q_X^u(1) = +\infty$. For regularity reasons we want to eliminate this behavior for random variables with bounded value set by letting

$$Q_X:[0,1] \to \mathbb{R},\ Q_X(\alpha) = \begin{cases} Q_X^u(\alpha), & \alpha < 1 \\ \lim\limits_{\omega \to 1} Q_X^u(\omega), & \alpha = 1 \end{cases}$$

Complementary to the function of the upper quantiles the lower quantiles are denoted by

$$Q_X^l:[0,1] \to \mathbb{R},\ Q_X^l(\alpha) = \sup\{x \in \mathbb{R}:\mathbf{P}(X \leq x) < \alpha\}.$$

Note: With this terminology the expected solvency capital requirement measure (SCR) under the Solvency II regime, which is defined as the value at risk with a probability of default of 200 years, would be expressed as $\text{SCR} = -Q_X(0,5\%)$.

The distribution function and the upper quantiles function are non decreasing, right continuous functions. The corresponding borel measures exist and shall be denoted by $\mathrm{d}Q_X$ and $\mathrm{d}F_X$. Moreover Q_X , Q_X^l and Q_X^u are right inverse of F_X in the sense that

$$F_X \circ Q_X(\alpha) = F_X \circ Q_X^l(\alpha) = F_X \circ Q_X^u(\alpha) = \alpha,\ \alpha \in F_X(\mathbb{R}).$$

In then special case where X has a density $f(x)$ the quantile functions coincide. If furthermore the density has continuous support, i.e. $\text{supp}(f) \stackrel{\text{def}}{=} \{x \in \mathbb{R}:f(x) \neq 0\} =]a;b[$ for $a, b \in [-\infty, +\infty]$, then the distribution function $F_X(x) = \displaystyle\int_{-\infty}^{x} f(t)\mathrm{d}t$ is bijective and the quantile function $Q_X = F_X^{-1}$ is its inverse.

In the formulae we will occasionally use indicator functions. They are defined on an arbitrary set Ω' , which typically is the set $\mathbb{R}$ of real numbers. For a subset $B \subset \Omega'$ the indicator function $\mathbf{1}_B$ is the mapping

$$\mathbf{1}_B : \Omega' \to \mathbb{R},\ \mathbf{1}_B(\omega) = \begin{cases} 1, & \omega \in B \\ 0, & \omega \notin B \end{cases}$$

3 Motivation: A simple bond investment model

The following is an intentionally simplified model for fixed income securities. It shall serve the purpose of motivating the overall approach and not as a full model for investments in bonds. For instance we shall only consider the possibilities of no repayment or full repayment. The case of partial repayment or effects of appraisal values are not taken into account.

Suppose a company issues a bond with a face value of n and a rating of “B”. There is a possibility of default, i.e. that the bond will not be repaid upon maturity. An investor will require a certain spread s over the return r of a secure investment to compensate him for the possibility that the company may default on repaying the bond. The required spread depends on the rating of the bond. On the other hand the investment rating gives information about how secure an investment in the bond is, i.e. how likely it is that the bond will not be repaid. If we denote the probability of default with α then the spread becomes a function $\alpha \to s(\alpha)$ thereof.

For the following the bond shall be a zero bond, i.e. the price for the bond is paid upfront by way of a discount on the face value N. As noted above we only consider the two possibilities no default and total default. Then the effective result for the investor at the time of remuneration of the bond is either that he has to write off the full value N or he exchanges two assets, namely the bond for cash. In the latter case the impact on the balance sheet is ± 0. The impact on the balance sheet from the investor's point of view is denoted by the random variable X. It's distribution can be written as

$$F_X(x) = \begin{cases} 0, & x < -n \\ \alpha, & -n \le x < 0 \\ 1, & 0 \le x \end{cases}.$$

The cost of this transaction for the company can be referred to as *Cost of Risk* (in short CoR) as emitting the bond is a means of raising capital for the operation. This CoR is easily calculated as

$$\mathrm{CoR} = n \cdot s(\alpha) = -Q^*_X(\alpha) \cdot s(\alpha) = -Q_X(0) \cdot s(\alpha).$$

The model can be generalized to the situation where multiple bonds with different seniority (and ratings) are being issued: Suppose the total capital need N of the company is being issues via $k \in \mathbb{N}$ tranches with nominal values $n_1, ..., n_k$ and $N = n_1 + ... + n_k$. Each tranche has its own rating with the associated spreads denoted by $s_1, ..., s_k$. The default probabilities of the tranches $\alpha_1, ..., \alpha_k$ are assumed to be pairwise different. The indices shall be such that the default probabilities are increasing $\alpha_i < \alpha_j,\ i < j$. For notational purposes we add $\alpha_0 = 0$ and $\alpha_{k+1} = 1$. Note that from a risk-return-logic then the spreads will also be increasing.

In this situation the amount lost to the investor with probability α_i is equal to $l_i = -(n_i + ... + n_k)$. With the convention that $l_{k+1} = 0$ the distribution of the result X can be written as

$$F_X(x) = \sum_{i=1}^{k} \alpha_i \cdot \mathbf{1}_{[l_i, l_{i+1}[}(x) + \mathbf{1}_{[0,\infty]}(x).$$

The cost of raising the capital this way is

$$\text{CoR} = \sum_{i=1}^{k} n_i \cdot s(\alpha_i) = \sum_{i=1}^{k} (Q_X(\alpha_i) - Q_X(\alpha_{i-1})) \cdot s(\alpha_i). \tag{1}$$

This model still can be applied if the bonds don't have varying seniority but the trigger for loosing the investment is defined sequentially for each trance. This for instance could be the case for a series of natural catastrophe bonds that are being issued.

4 The risk cost model

The fundamental idea of generalizing the above model beyond an investor's view is that the information on where a stakeholder is most concerned should be somehow coded in the required spread. As a central theorem we will show that the most common risk capital models are special cases thereof.

Definition 4.1. *Let $s{:}[0,1] \to \mathbb{R}_+$ be a non-decreasing, left continuous, bounded function. Let further*

$$\mathcal{D}_s = \left\{ X \in L{:}Q_X(1) < \infty \wedge \int_0^1 s(\alpha) dQ_X < \infty \right\}.$$

Then the cost of risk associated with this function s is defined as the functional

$$\text{CoR}_s{:}\mathcal{D}_s \to \mathbb{R},\ \text{CoR}_s(X) = \int_0^1 s(\alpha) dQ_X - s(1)Q_X(1). \tag{2}$$

Note 4.2. *The investor's model discussed above is a special case of the general definition if we set*

$$s_{inv} \stackrel{\text{def}}{=} \sum_{i=1}^{k} s_i \cdot \mathbf{1}_{]\alpha_{i-1}, \alpha_i]} + s_k \cdot \mathbf{1}_{]\alpha_k, 1]}.$$

Proof. As per the definition of the random variable X in the prior chapter we have $Q_X(\omega) = 0$ for all $\omega \in]\alpha_k, 1]$. Hence the second term in the definition of s_{inv} does not contribute to the CoR_s:

$$\begin{aligned}\mathrm{CoR}_s(X) &= \int_0^1 s_{inv}(\omega)\mathrm{d}Q_X = \sum_{i=1}^{k} s_i \int_{]\alpha_{i-1},\alpha_i]} \mathrm{d}Q_X + s_k \int_{]\alpha_k,1]} \mathrm{d}Q_X \\ &= \sum_{i=1}^{k} s_i \left(Q_X(\alpha_i) - Q_X(\alpha_{i-1})\right). \end{aligned} \tag{3}$$

Which is equal to (1). □

In practical applications s will typically have a more simple structure than Q_X let alone that it is fixed for one application, whereas the random variable X will be changing. It would be preferable to express CoR_s by way of a measure induced by s. To this end we consider the right continuous variation of s denoted by

$$s^*(\omega) = \lim_{\varepsilon\to 0} s(\omega + \varepsilon).$$

Then we have the following

Proposition 4.3. *If the number of discontinuities $\{s - s^* \neq 0\} = \{\omega_1, ...\omega_n\}$ is finite then for every $X \in \mathcal{D}_s$*

$$\mathrm{CoR}_s(X) = -\int_0^1 Q_X(\omega)\, ds^* + \sum_{i=1}^{n} (s - s^*)(\omega_i) \cdot (Q_X^u - Q_X^l)(\omega_i) - \lim_{\varepsilon\to 0} s^*(\varepsilon) Q_X(\varepsilon). \tag{4}$$

Proof. For notational purposes we add $\omega_0 = 0$ and $\omega_{n+1} = 1$. With this the integral over the interval $[0, 1]$ naturally falls apart into into $n + 1$ local integrals over $[\omega_i, \omega_{i+1}], i = 0, ...n$

$$\int_0^1 s(\omega)\mathrm{d}Q_X = \sum_{i=0}^{n} \int_{\omega_i}^{\omega_{i+1}} s(\omega)\mathrm{d}Q_X = \sum_{i=0}^{n} \int_{\omega_i}^{\omega_{i+1}} (s - s^*)(\omega) + s^*(\omega)\mathrm{d}Q_X \tag{5}$$

In the interior of each integral $s = s^*$. Then the integration of the first term for $i = 0, ..., n - 1$ yields

$$\begin{aligned}\int_{\omega_i}^{\omega_{i+1}} (s - s^*)(\omega)\mathrm{d}Q_X &= (s - s^*)(\omega_i) \cdot \lim_{\varepsilon\to 0} \left(Q_X(\omega_i) - Q_X(\omega_i - \varepsilon)\right) \\ &= (s - s^*)(\omega_i) \cdot \left(Q_X^u - Q_X^l\right)(\omega_i).\end{aligned}$$

The respective integral for $i = n$ is zero as $\mathrm{d}Q_X$ by definition does not have any weight at 1. On the second term of the equation (5 we can apply partial integration which yields

$$\int_{\omega_i}^{\omega_{i+1}} s^*(\omega)\mathrm{d}Q_X = s^*(\omega_{i+1})Q_X(\omega_{i+1}) - s^*(\omega_i)Q_X(\omega_i) - \int_{\omega_i}^{\omega_{i+1}} Q_X(\omega)\mathrm{d}s^*.$$

Note that for $i = 0$ in fact this needs to be expressed by the limit, which has been omitted here for purpose of readability. Adding all local integrals back up we get

$$\begin{aligned}\int_0^1 s(\omega)\mathrm{d}Q_X &= -\int_0^1 Q_X(\omega)\mathrm{d}s^* + \sum_{i=1}^n (s - s^*)(\omega_i) \cdot (Q_X^u - Q_X^l)(\omega_i) \\ &\quad + s^*(1)Q_X(1) - \lim_{\varepsilon \to 0} s^*(\varepsilon)Q_X(\varepsilon).\end{aligned}$$

Keeping in mind that per definition $s^*(1) = s(1)$ the proposition is proved. □

5 Comprehensiveness of the family

A classical approach to measure risk cost is to come up with a model for the risk capital and applying a required rate of return to it. However, as is noted in [1] the methods by which the capital and it's cost are derived are not necessarily consistent. Nonetheless risk capital is a popular metric in the insurance industry as it is for instance necessary from a solvency point of view. Two risk (capital) measures that have found their way into the Solvency II discussion are the Value at Risk and the Tail Value at risk. They have for instance been discussed in [2]:

Definition 5.1. *For $X \in L$ the following risk measures for a confidence level $\alpha \in [0, 1]$ are defined in case they exist finitely:*

1. *Value at Risk:* $\mathrm{VaR}(\alpha)(X) = -Q_X(\alpha) - \mathrm{E}[X]$
2. *Tail Value at Risk:* $\mathrm{TVaR}(\alpha)(X) =$
$-\frac{1}{\alpha}\left(\mathrm{E}[X \cdot \mathbf{1}_{\{X \le Q_X(\alpha)\}}] + Q_X(\alpha) \cdot (\alpha - \mathbf{P}[X \le Q_X(\alpha)])\right) - \mathrm{E}[X].$

Theorem 5.2. *For a fixed $\alpha \in [0, 1]$ set*

1. $s_1(\omega) = \omega$
2. $s_2(\omega) = \mathbf{1}_{]\alpha,1]}(\omega) + \omega$
3. $s_3(\omega) = \frac{\omega}{\alpha} \cdot \mathbf{1}_{[0,\alpha]} + \mathbf{1}_{]\alpha,1]} + \omega$

Then the following identities hold

1. $\mathrm{CoR}_{s_1}(X) = -\mathrm{E}[X]$ *for all* $X \in \mathcal{D}_{s_1}$ *with* $\lim_{\varepsilon \to 0} \varepsilon Q_X(\varepsilon) = 0$.
2. $\mathrm{CoR}_{s_2}(X) = -\mathrm{VaR}_\alpha(X)$ *for all* $X \in \mathcal{D}_{s_2} = L$.
3. $\mathrm{CoR}_{s_3}(X) = -\mathrm{TVaR}_\alpha(X)$ *for all* $X \in \mathcal{D}_{s_3}$ *with* $\lim_{\varepsilon \to 0} \varepsilon Q_X(\varepsilon) = 0$.

Proof. 1. As s is continuous the equation (4) simplifies to

$$\mathrm{CoR}_{s_1}(X) = -\int_0^1 Q_X(\omega)\mathrm{d}\omega.$$

If we first assume that F_X has a density f . Then by substituting $\omega = F_X(x)$ we get

$$\mathrm{CoR}_{s_1}(X) = -\int_{\mathbb{R}} Q_X \circ F_X(x) \cdot f(x)\mathrm{d}x.$$

For $x \in \mathrm{supp}(f)$ we have the identity $Q_X \circ F_X(x) = x$. On the other hand $f(x) = 0$ for $x \notin \mathrm{supp}(f)$. Hence the value of the integral is not changed if we replace the integrand with $x \cdot f(x)$, which shows the proposition.

Now we assume that F_X is discrete and concentrated on $x_1, x_2, \ldots$ with $\alpha_i = \mathbf{P}(X = x_i)$, then the upper quantile function can be written as

$$Q_X(\omega) = x_i, \quad \sum_{\{j : x_j < x_i\}} \alpha_j \le \omega < \sum_{\{j : x_j < x_i\}} \alpha_j + \alpha_i.$$

From this it is clear that

$$-\int_0^1 Q_X(\omega)\mathrm{d}\omega = -\sum_{i=1}^{\infty} x_i \cdot \alpha_i = -\mathrm{E}[X].$$

The proof for a mixed distribution then is clear.

2. This is a straight forward calculation of the definition integral.

3. Again we are using the equation (4) and consider that the measure $\mathrm{d}s_3$ is zero on the interval $]\alpha, 1]$

$$\mathrm{CoR}_{s_2}(X) = -\int_0^1 Q_X(\omega)\mathrm{d}s_3 = -\frac{1}{\alpha}\int_0^\alpha Q_X(\omega)\mathrm{d}\omega - \mathrm{E}[X].$$

With the same logic as under 1. the result follows. □

6 Coherence of CoR_s

Coherence is a property, or rather a set of properties, introduced by Artzner et al. in [3] that is in the literature deemed to be desirable or even required for application in risk management of an insurance company. For easy reference we shall here quote the definition from [2]:

Definition 6.1. *A risk measure ρ being a mapping $\rho : L \to \mathbb{R}$ is called coherent if it satisfies the following properties:*

1. *Translational invariance: $\rho(X+\alpha) = \rho(X) - \alpha$ for all $X \in L$ and $\alpha \in \mathbb{R}$.*
2. *Subadditivity: $\rho(X+Y) \leq \rho(X) + \rho(Y)$ for all $X, Y \in L$.*
3. *Positive homogeneity: $\rho(\lambda X) = \lambda\rho(X)$ for all $X \in L$ and all $\lambda \geq 0$.*
4. *Monotonicity: $\rho(X) \leq \rho(Y)$ for all $X, Y \in L$ with[2] $X \geq Y$ $\mathbf{P}$ -a.s.*

Further on we will restrict us to the set of centered random variables $L_0 = \{X \in L{:}E[X] = 0\}$, which renders the condition 1 of the coherence obsolete.

It is a well known fact that *TVaR* is coherent, but *VaR* is not. Hence from the theorem in the prior chapter it is clear that the family of CoR_s -risk-measures is not in general coherent. In this chapter a sufficient criteria in respect of s for the coherence of the induced CoR_s-risk-measures will be established.

Lemma 6.2. 1. *Let s_1, s_2 be spreads for which $\mathrm{CoR}_{s_1}, \mathrm{CoR}_{s_2}$ are coherent on $L_0 \cap \mathcal{D}_{s_1}$ and on $L_0 \cap \mathcal{D}_{s_2}$ respectively. Then $\mathrm{CoR}_{s_1+s_2}$ is coherent on $L_0 \cap \mathcal{D}_{s_1} \cap \mathcal{D}_{s_2}$.*

2. *Let $s_1, s_2, \ldots$ be a sequence of spreads for which all CoR_{s_i} are coherent on $L_0 \cap \mathcal{D}_{s_i}$ and with $s_i(x) \leq s_j(x)$ for all $x \in [0,1]$. If the series converges point wise to a spread s, the CoR_s is coherent on $L_0 \cap \bigcap_{i\in\mathbb{N}} \mathcal{D}_{s_i}$.*

Proof. From the integral definition it is clear that $\mathrm{CoR}_{s_1+s_2} = \mathrm{CoR}_{s_1} + \mathrm{CoR}_{s_2}$. The proof of 1. is a straight forward calculation. Similarly it is clear that $\mathrm{CoR}_s = \lim_{i\to\infty} \mathrm{CoR}_{s_i}$. Then again the proof of 2. becomes a straight forward calculation. □

Theorem 6.3. *Let s be a continuous spread and concave in the sense that for every $0 \leq \alpha_1 < \alpha_2 \leq 1$ it holds that $s(\omega) \geq s(\alpha_1) + (\omega - \alpha_1) \cdot \dfrac{s(\alpha_2) - s(\alpha_1)}{\alpha_2 - \alpha_1}$ for every $\alpha_1 \leq \omega \leq \alpha_2$. Then CoR_s is coherent.*

Proof. We want to approximate s from below with a sequence of spreads that are finite sums of TVaR-type spreads. To this end let $\alpha_{i,j} = j/2^i$ for $i \in \mathbb{N}$

[2]The applicable condition is $X \geq Y$ in the case of result distribution and $X \leq Y$ on loss distributions.

and $j = 0, ..., 2^i$. If we denominate the growth of s from $\alpha_{i,j-1}$ to $\alpha_{i,j}$ with $\delta_{i,j} = \frac{1}{2^i}\left(s(\alpha_{i,j}) - s(\alpha_{i,j-1})\right)$ it follows that the series

$$s_i(\omega) := \sum_{j=1}^{2^i} \left((\omega - \alpha_{i,j-1}) \cdot \delta_{i,j} + s(\alpha_{i,j-1})\right) \cdot \mathbf{1}_{[\alpha_{i,j-1},\alpha_{i,j}[}(\omega) + s(1) \cdot \mathbf{1}_{\{1\}}.$$

converges to $s(\omega)$ from below and it is sufficient to show that the induced CoR_{s_i} are coherent. If we can show that s_i can be decomposed into the recursively defined

$$s_{i,j}(\omega) := \omega \cdot (\delta_{i,j} - \delta_{i,j+1})\mathbf{1}_{[0,\alpha_{i,j}[}(\omega) + \alpha_{i,j} \cdot (\delta_{i,j} - \delta_{i,j+1})\mathbf{1}_{[\alpha_{i,j},1]}(\omega))$$

with the convention of $\delta_{i,2^i+1}$ the issue reduces to showing that the $\mathrm{CoR}_{s_{i,j}}$ are coherent. The $s_{i,j}$ again have the form of a TVaR-type spread (on L_0) multiplied by a factor. As this factor according to the concavity criteria is positive, they are indeed coherent. On the other hand

$$\frac{\mathrm{d}}{\mathrm{d}\omega} \sum_{j=1}^{2^i} s_{i,j}(\omega_0) = \delta_{i,j} = \frac{\mathrm{d}}{\mathrm{d}\omega} s_i(\omega_0), \ \forall \omega_0 \in \left]\alpha_{i,j-1}, \alpha_{i,j}\right[$$

and $\sum s_{i,j}(0) = 0$. As the $s_{i,j}$ as well as the s_i have been constructed continuous this yields the proposition. □

7 Cost allocation

In analogy to [4] the CoR_s can be seen as a cost per percentile layer. Likewise it can be allocated to individual risk drivers. Say the total result of a firm (denoted by the random variable X) is composed of the result of two divisions denoted by Y and Z, i.e. $X = Y + Z$.

For applications X can be assumed to be bounded and concentrated on a finite probability space: In practical situations a numerical approximation method like a Monte Carlo simulation will be used thus effectively reducing the problem to a discrete probability space. Likewise the condition of bounded random variables will not be an issue. The potential loss will certainly be lower than the world's GDP.

Hence we let $\Omega = \{\omega_1, \omega_2, ..., \omega_n\}$ with $x_i = X(\omega_i)$, $p_i = \mathbf{P}(\omega_i)$, $\sum_{i=1}^{n} p_i = 1$. Thereby the index shall be selected thus that the series of possible results is not descending, i.e. $x_i \leq x_j, \forall 1 \leq i \leq j \leq n$. With the partial sums $\alpha_i := \sum_{j \leq i} p_j$ it follows that $Q_X(\omega) = x_i,\ \forall \alpha_{i-1} \leq \omega < \alpha_i$ and hence

$$\mathrm{CoR}_s(X) = -\sum_{i=1}^{n} s(\alpha_i)(x_{i+1} - x_i) = -\sum_{i=1}^{n} \left[s(\alpha_{i-1}) - s(\alpha_i)\right] x_i. \tag{6}$$

Letting $y_i := Y(\omega_i)$ and $z_i := Z(\omega_i)$ this yields a natural decomposition of CoR_s into

$$\mathrm{CoR}_s(X) = -\sum_{i=1}^{n} \left[s(\alpha_{j-1}) - s(\alpha_j)\right] y_i - \sum_{i=1}^{n} \left[s(\alpha_{j-1}) - s(\alpha_j)\right] z_i. \tag{7}$$

It has to be noted that the series y_i and z_i are not necessarily any more non-decreasing. Of course the two sums on the right of 7 could be resorted accordingly. But the spread function will generally be different. In fact the dependency structure of Y and Z will have to be factored into the spreads applied to the two random variables.

8 A practical example

For this example we will consider a mono line company, only writing motor third party liability, for which we will estimate its cost for financing its capital on markets and allocate the cost between the risk drivers.

The annual premium volume is taken as 20mEUR and the best estimate reserve is set at 60mEUR. Volatility and distribution assumptions for premium and reserve risk are taken as per the QIS3 parametrization (see [9]) to be 10% and 12,5% respectively and a lognorml distribution. The mean expected insurance result in both cases has been set to 0. Investments are assumed to be entirely made in "AAAťť rated government bonds with matched term and currency structure.

The spread function is calibrated to market observations. Given the market turmoil during 2008 data used will be as at the beginning of 2008. For ease of reference bond returns have been taken from a German investment magazine that also includes Standard & Poor's ratings. Hence in an additional step these ratings need to be linked to default probabilities. The rating classes have been mapped to an index according to $S\&P_1 := AAA$, $S\&P_2 := AA+$, $S\&P_3 := AA$, etc.

As a first step the observed one year defaults as published in [7] have been used as a basis. A logarithmic model has been fit to the measured data by minimizing the squared relative error using the solver of Microsoft EXCEL:

$$\mathbf{P}(\text{Default in class } S\&P_i) = 2,6 \cdot 10^{-5} \cdot \mathrm{e}^{i \cdot 0,52}. \tag{8}$$

The graphical fit in logarithmic scaling is shown in figure 1.

The second step of mapping spreads to rating classes has been based on the bond lists published in [8]. This data only contains the annualized return denoted by

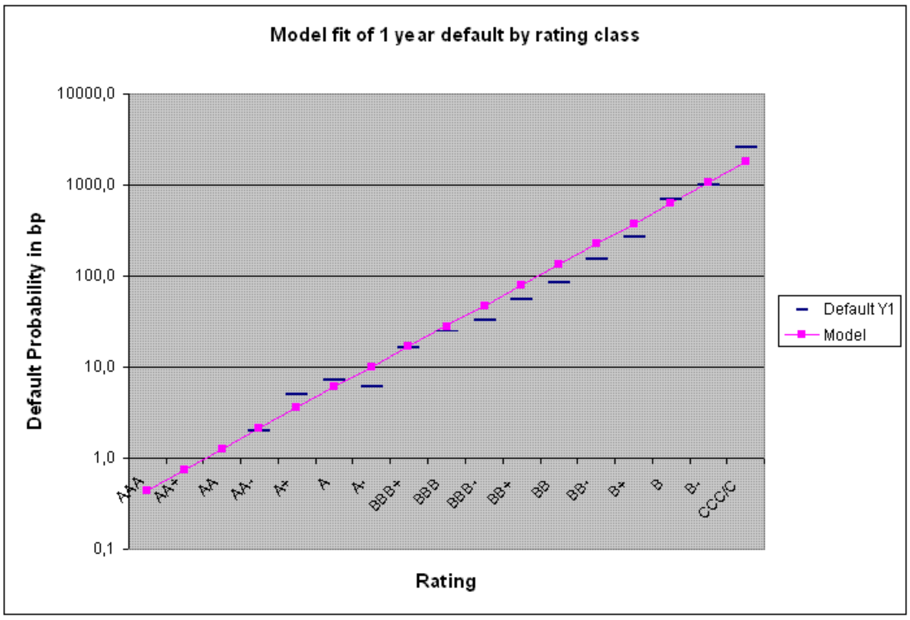

Figure 1: Fit of default to rating class

r and not the spread over the risk free rate. Again the raw data has been smoothed using a logarithmic model fit minimizing the mean squared error:

$$r(S\&P_i) = 3,84\% + 0,27\% \cdot \mathrm{e}^{i \cdot 0,2}. \tag{9}$$

In this model the 3,84% are the risk free return. The graphical fit is shown in figure 2.

It is easy to solve equation (8) for the index i and substitute it in equation (9), which yields the fairly simple expression

$$s(\omega) = 15,5\% \cdot x^{0,384}. \tag{10}$$

Through using the fitting approach the risk measure has been extended from a discrete measure on the rating classes to a continuous spread function. From theorem 6.3 it follows that it induces a coherent risk measure CoR_s.

For an estimate of the cost of the risk of the company described above, a simple Monte Carlo simulation approach with 1.000 iterations has been taken and formula 6 was used. To reflect the correlation as per [9] of 50% between the premium and reserve risk, for each iteration three independent uniform random numbers R_1, R_2, R_3 between 0 and 1 have been generated in Microsoft EXCEL. The partially dependent numbers $0,5 \cdot (R_1 + R_3)$ and $0,5 \cdot (R_2 + R_3)$ then have been transformed back with the built in lognormal percentile function, centered to reflect the zero expectation on the insurance result and summed in the iteration x_i of the corporate result.

The result of the calculation is a cost of risk of 1,2mEUR. The 200 year value at risk, i.e. the expected Solvency II level of confidence, in this simulation run

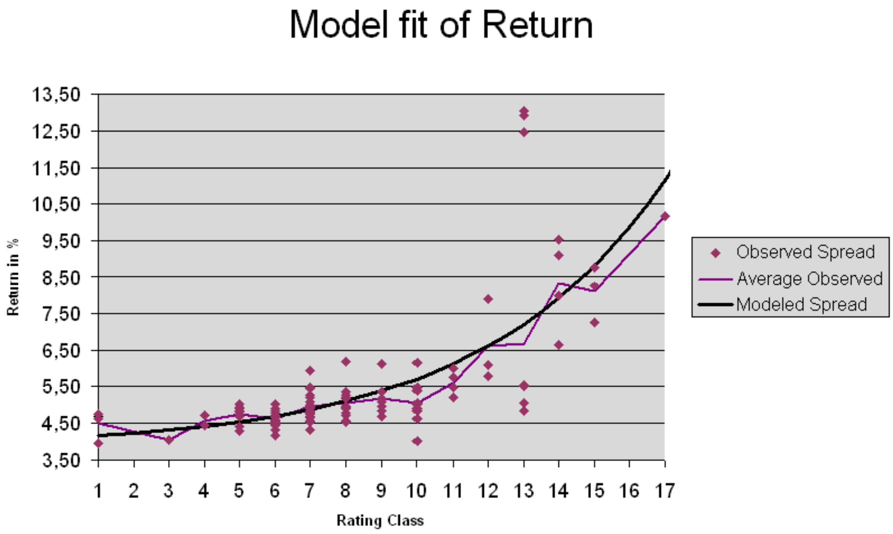

Figure 2: Fit of return to rating class

has been 26mEUR. So the cost of risk over solvency capital would yield 4,5%, which has to be added on top of the risk free rate of 3,8% to arrive at a total expected return of 8,4% on solvency capital. This seems to be i a reasonable range.

As a byproduct the total cost of risk has been decomposed according to section 7 to 12% premium risk and 88% reserve risk. At a first glance this might seem odd given the experience in German motor business. However, it needs to be kept in mind, that in the model we have taken the reserves to be set at best estimate without any prudence margin. The decomposition is fairly close to the allocation that would have been seen in a covariance allocation (14% and 86%).

9 Caveats and Calibration

The CoR_s theory described in this paper has the advantage of being an integrated approach to measure the cost of risk including a "natural" decomposition. There are no three potentially disjoint approaches for measuring the amount of capital, the cost of the capital and the allocation to lines of business. Nonetheless there is a certain degree of freedom in selecting the appropriate spread function s for an application.

When thinking about the return to shareholders perspective, the spread function can be estimated from the capital markets as shown in section 8. The flip side of this approach is the dependency of any valuation on psychological aspects during times of market turmoil as witnessed during 2008. Also the approach taken to use ratings of bonds to classify their risk is probably not the ultimate solution. And finally in the presented model there is no consideration of partial defaults or downgrades of a bond.

Another question that needs awareness when using the presented model in valuing a company or individual business segments is the asymmetry of information. If the valuation is done from inside the company there is a maximum knowledge about the most appropriate distribution of the results X. An external investor will typically not have the full picture and use different sources of information to form his opinion. This is particularly important when thinking about a comparison of the CoR_s model with a CAPM valuation. The CAPM reflects a market consensus and hence does not benefit from the internal view.

For an undertaking that has a CAPM valuation this could be used to regularly recalibrate the "raw", purely internal model CoR_s^{raw} with an adjustment factor

$$\lambda := \frac{CAPM}{\mathrm{CoR}_s^{raw}}.$$

Thus the model can serve as an approximate predictor of how changes in the risk profile of the undertaking can impact its value.

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