Optimal construction of a fund of funds

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1 Introduction

We study the problem of diversifying a given initial capital over a finite number of investment funds that follow different trading strategies. The investment funds operate in a market where a finite number of underlying assets may be traded over a finite discrete time. Our goal is to find a diversification that is optimal in terms of a given convex risk measure; see e.g. (Föllmer and Schied 2004, Chapter 4). We formulate an optimization problem where a portfolio manager is faced with uncertain asset returns as well as liabilities.

The main contribution of this paper is a description of a computational procedure for finding an optimal diversification between funds. The procedure combines simulations with large scale convex optimization and it can be efficiently implemented with modern solvers for linear programming.

We illustrate the optimization process on a problem coming from the Finnish pension insurance industry. The liabilities are taken as the claim process associated with current claims portfolio of the private sector occupational pension system and the investment horizon is 82 years. The results reveal a significant improvement over a set of standard investment styles that are often recommended for long term investors.

The rest of this paper is organized as follows. We begin by reviewing some well-known parametric investment strategies in Section 2. Section 3 states the optimization problem and Section 4 outlines the numerical procedure for its solution. The application to pension fund management is reported in Section 5. The market model used in the case study is described in the Appendix.

2 Basic investment strategies

Consider a financial market where a finite set J of securities can be traded over a finite discrete time $t = 0, \dots, T$. The return on asset $j \in J$ over holding period $[t-1, t]$ will be denoted by $R_{t,j}$. The interpretation is that if $h_{t-1,j}$ units of cash is invested in asset $j \in J$ at time $t-1$, the investment will be worth $R_{t,j}h_{t-1,j}$ at time t.

We study dynamic trading strategies from the perspective of an investor who has given initial capital w_0 and given liabilities $c = (c_t)_{t=1}^T$. Here c_t denotes a claim the investor has to pay at time t. The claim process c is allowed to take both positive and negative values so it can be used to model liabilities as well as income. The return processes $R_j = (R_{t,j})_{t=1}^T$ are assumed positive but otherwise their joint distribution with the claim process c is arbitrary.

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Several rules have been proposed for updating an investment portfolio in an uncertain dynamic environment. Below, we recall four well-known examples modified to accommodate for claim payments.

The simplest strategies are the *buy and hold* (BH) strategies where an initial investment portfolio is held over time without updates. When the claim process c is nonzero, BH strategies may be infeasible. A natural modification is to liquidate each asset in the proportion of the initial investments to cover the claims. The resulting strategy consists of investing

$$h_{t,j} = \begin{cases} \pi_j w_0 & t = 0, \\ R_{t,j} h_{t-1,j} - \pi_j c_t & t = 1, \dots, T, \end{cases}$$

units of cash in asset $j \in J$ at the beginning of the holding period starting at time t. Here π_j is the proportion invested in asset $j \in J$ at time $t = 0$. Such strategies will be "self-financing" in the sense that they allow for paying out the claims without need for extra capital after time $t = 0$. If the claim process c is null, the BH strategy requires no transactions after time $t = 0$.

Another well-known strategy is the *fixed proportions* (FP) strategy where at each time and state the allocation is rebalanced into proportions given by a vector $\pi \in \mathbb{R}^J$ whose components sum up to one. In other words,

$$h_t = \pi w_t,$$

where for $t = 1, \dots, T$,

$$w_t = \sum_{j \in J} h_{t-1,j} R_{t,j} - c_t.$$

A *target date fund* (TDF) is a popular strategy in the pension industry (Bodie and Treussard (2007)). In a TDF, the proportion invested in risky assets is decreased as retirement date approaches. In our multi-asset setting we implement TDFs as investment strategies that adjust the allocation between two complementary subsets J^r and J^s of the set of all assets J. Here J^s consists of "safe" assets and J^r consists of the rest. In a TDF, the *proportional exposure*, i.e. the proportion of wealth invested in J^r at time t is given by

$$e_t = a - bt.$$

The parameter a gives the initial proportional exposure in the risky assets and b specifies how fast the proportional exposure is decreased with time. Nonnegative proportional exposure in the risky assets can be guaranteed by choosing a and b so that

$$a \geq 0 \quad \text{and} \quad a - bT \geq 0.$$

A TDF is defined by

$$h_t = \pi_t w_t$$

where the vector π_t is dynamically adjusted to give the specified proportional exposure:

$$\sum_{j \in J^r} \pi_{t,j} = e_t.$$

To complete the definition, one has to determine how the wealth is allocated within J^r and J^s. We do this according to FP rules.

One of the best known strategies is the *constant proportion portfolio insurance* (CPPI) strategy; see e.g. Black and Jones (1987), Black and Perold (1992) and Perold and Sharpe

(1995). In a CPPI, the proportional exposure in the risky assets follows a rule of the form

$$\begin{aligned} e_t &= \frac{m}{w_t} \max\{w_t - F_t, 0\} \\ &= m \max\{1 - \frac{F_t}{w_t}, 0\}, \end{aligned}$$

where the "floor" F_t represents the time t value of a claim that should be paid in the future and the parameter $m \geq 0$ gives the fraction invested in risky assets of the excess of wealth over the floor. In our setting F_t would represent the value of the part of c remaining at time t. If one wishes to limit the maximum proportional exposure to a given upper bound l the strategy becomes

$$e_t = \min\{m \max\{1 - \frac{F_t}{w_t}, 0\}, l\}.$$

3 The optimization problem

Given an initial capital w_0 and a sequence $(c_t)_{t=1}^T$ of claims representing the liabilities of the investor, it is a natural idea to diversify among different strategies in order to better suit the risk preferences of the owner. The overall strategy obtained with diversification will also cover the claims $(c_t)_{t=1}^T$ so one is free to search for an optimal diversification. Diversifying among parametric classes of investment strategies, such as those listed above, may produce new strategies which do not belong to the original parametric classes; see Section 5.3.

The problem of diversifying among a finite set $\{h^i \mid i \in I\}$ of strategies can be written as

$$\underset{\alpha \in X}{\text{minimize}} \quad \rho(\sum_{i \in I} \alpha^i w_T^i),$$

where w_T^i is the terminal value of a wealth process w^i obtained by following strategy $i \in I$,

$$X = \{\alpha \in \mathbb{R}_+^I \mid \sum_{i \in I} \alpha^i = 1\}$$

and ρ is a *convex risk measure* that quantifies the preferences of the decision maker over random terminal wealth distributions; see e.g. Föllmer and Schied (2004) or Rockafellar (2007).

Several choices of ρ may be considered. We will concentrate on the Conditional Value at Risk ($CV@R$) which is particularly convenient in the optimization context. According to Rockafellar and Uryasev (2000), $CV@R_\delta$ at confidence level δ of a random variable w can be expressed as

$$CV@R_\delta(w) = \inf_\gamma E\left[\frac{1}{1-\delta} \max\{\gamma - w, 0\} - \gamma\right].$$

Moreover, the minimum over γ is achieved by Value at Risk at confidence level δ. The problem of optimal diversification with respect to $CV@R_\delta$ can be written as

$$\underset{\alpha \in X, \gamma}{\text{minimize}} \quad E\left[\frac{1}{1-\delta} \max\{\gamma - \sum_{i \in I} \alpha^i w_T^i, 0\} - \gamma\right]. \tag{1}$$

The problem thus becomes that of minimizing a convex expectation function over a finite number of variables. Mathematically, it is close to the classical problem of maximizing the expected utility in a one period setting and, consequently, similar techniques can be applied for its solution; see e.g. Sharpe (2007).

4 Numerical procedure

In order to solve (1), we will first make a quadrature approximation of the objective; see Pennanen and Koivu (2005), Koivu and Pennanen (to appear). That is, we generate a finite number N of return and claim scenarios (R^k, c^k), $k = 1, \dots, N$ over the planning horizon $t = 0, \dots, T$ and approximate the expectation by

$$\frac{1}{N} \sum_{k=1}^{N} \left[\frac{1}{1-\delta} \max\{\gamma - \sum_{i \in I} \alpha^i w_T^{i,k}, 0\} - \gamma \right],$$

where $w_T^{i,k}$ is the terminal wealth along scenario k obtained with strategy h^i. The computation of $w_T^{i,k}$ is straightforward: given realizations of R^k and c^k the corresponding wealth process $w^{i,k}$ is given recursively by

$$w_t^{i,k} = \begin{cases} w_0 & \text{for } t = 0, \\ \sum_{j \in J} R_{t,j}^k h_{t-1,j}^{i,k} - c_t^k & \text{for } t > 0, \end{cases}$$

where $h_{t-1}^{i,k} = \pi_{t-1}^i w_{t-1}^{i,k}$ and π_{t-1}^i is one of the weight vectors specified in the previous section.

Algorithmically, the solution procedure can be summarized as follows.

1. Generate N scenarios of asset returns R_t and claims c_t over $t = 1, \dots, T$.
2. Evaluate each basic strategy $i \in I$ along each of the scenarios $k = 1, \dots, N$ and record the corresponding terminal wealth $w_T^{i,k}$.
3. Solve the optimization problem

$$\underset{\alpha \in X, \gamma}{\text{minimize}} \quad \frac{1}{N} \sum_{k=1}^{N} \left[\frac{1}{1-\delta} \max\{\gamma - \sum_{i \in I} \alpha^i w_T^{i,k}, 0\} - \gamma \right] \tag{2}$$

for the optimal diversification weights α^i.

There are several possibilities for solving (2). We follow Rockafellar and Uryasev (2000) and reformulate (2) as the linear programming problem

$$\begin{aligned} \underset{\alpha \in \mathbb{R}^I, \gamma \in \mathbb{R}, s \in \mathbb{R}^N}{\text{minimize}} \quad & \frac{1}{N} \sum_{k=1}^{N} \left(\frac{1}{1-\delta} s^k - \gamma \right) \\ \text{subject to} \quad & s^k \geq \gamma - \sum_{i \in I} \alpha^i w_T^{i,k} \quad k = 1, \dots, N, \\ & \sum_{i \in I} \alpha^i = 1, \\ & \alpha^i, s^k \geq 0. \end{aligned}$$

This LP has $|I| + N + 1$ variables, where $|I|$ is the number of funds and N is the number of scenarios in the quadrature approximation of the expectation. Modern commercial solvers are able to solve LP problems with millions of variables and constraints.

5 Case study: pension fund management

Consider a closed pension fund whose aim is to cover its accrued pension liabilities with given initial capital. The pension claims are of the defined benefit type and they depend on the wage and consumer price indices. According to the current Finnish mortality tables, all the liabilities will be amortized in 82 years. The following section describes the stochastic return and claim processes $R = (R_t)_{t=1}^T$ and $c = (c_t)_{t=1}^T$ and Section 5.2 lists the basic strategies that will be used in the numerical study in Section 5.3.

5.1 Assets and liabilities

The set J of primitive assets consists of

1. Euro area money market,
2. Euro area government bonds,
3. Euro area equity,
4. US equity,
5. Euro area real estate.

These are the assets in which the individual funds described in Section 2 invest. On the other hand, the above asset classes may be viewed as investment funds themselves.

For the money market fund, the return over a holding period of Δt is determined by the short rate Y_1,

$$R_{t,1} = e^{\Delta t Y_{t-1,1}},$$

The short rate will be modeled as a strictly positive stochastic process which will imply that $R_1 > 0$. The return of the government bond fund will be approximated by the formula

$$R_{t,2} = \Delta t Y_{t-1,2} + \left(\frac{1 + Y_{t,2}}{1 + Y_{t-1,2}}\right)^{-D},$$

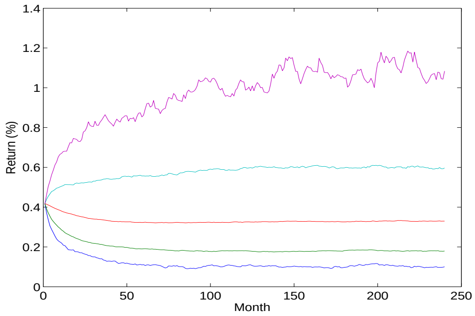
where $Y_{t,2}$ is the average yield to maturity of the bond fund at time t and D is the modified duration of the fund. The total returns of the equity and real estate funds are given simply in terms of the total return indices S_j,

$$R_{t,j} = \frac{S_{t,j}}{S_{t-1,j}}, \quad j = 3, 4, 5.$$

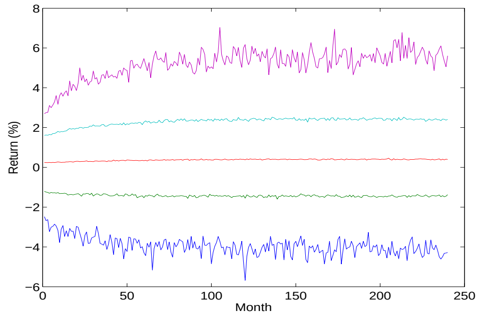
The pension fund's liabilities consist of the accrued benefits of the plan members. The population of the pension plan is distributed into different cohorts based on members' age and gender. The fraction of retirees in each cohort increases with age and reaches 100% by the age of 68. The youngest cohort is 18 years of age and all the members are assumed to die by the age of 100. The defined benefit pensions depend on stochastic wage and consumer price indices.

We will model the evolution of the short rate, the yield of the bond portfolio, the total return indices as well as the wage and consumer price indices with a Vector Equilibrium Correction-model (Engle and Granger (1987)) augmented with GARCH innovations. A detailed description of the model together with the estimated model parameters is given in the Appendix.

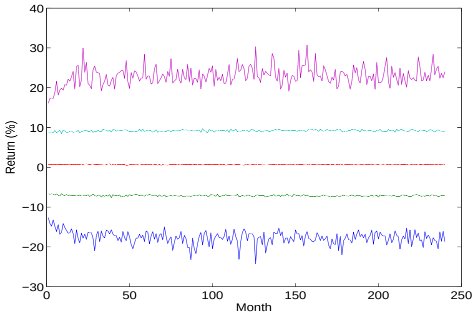
Figure 1 displays the 0.1%, 5%, 50% (median), 95% and the 99.9% percentiles of the simulated asset return distributions over the first twenty years of the 82 year investment horizon. Figure 2 displays the development of the median and the 95% confidence interval of the yearly pension claims over the 82 year horizon.



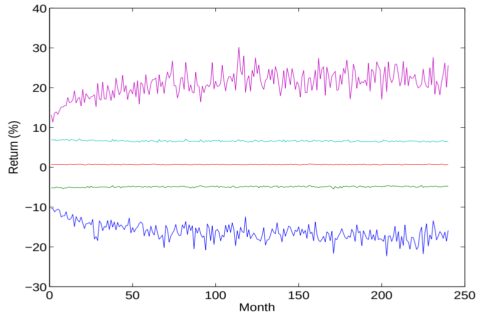
(a) Money market fund



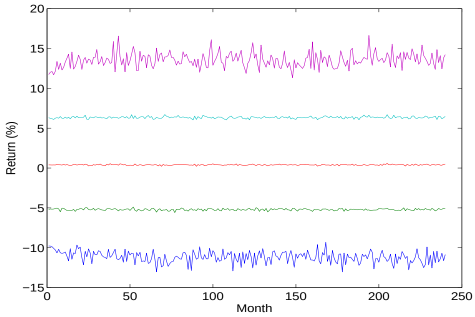
(b) Bond fund



(c) Euro area equity fund



(d) US equity fund



(e) Euro area real estate fund

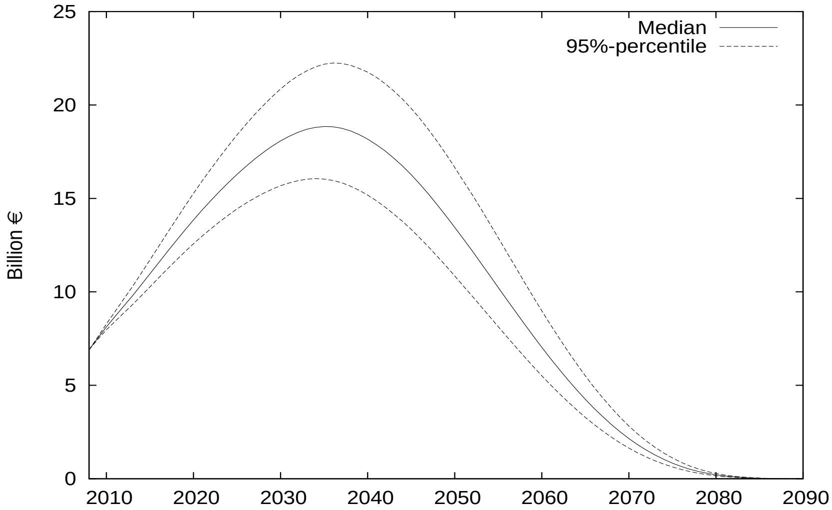
Figure 1: Evolution of the 0.1%, 5%, 50%, 95% and 99.9% percentiles of monthly asset return distributions over twenty years.

5.2 The investment funds

We will diversify a given initial capital among different investment funds as described in Section 3. The considered funds follow the trading rules listed in Section 2 with varying parameters. The set J^s of "safe assets" consists of the money market and bond investments.

We take five buy and hold strategies each of which invest all in a single asset. More general BH strategies can be generated by diversifying among such simple BH strategies. We use 11 FP strategies with varying parameters π. In TDF and CPPI strategies, we always use fixed proportion allocations within the safe assets J^s and the risky assets J^r. We use 20 TDF strategies with varying values for α and β. In the case of CPPI strategies, we define

Figure 2: Median and 95% confidence interval of the projected pension expenditure c over the 82 year horizon.

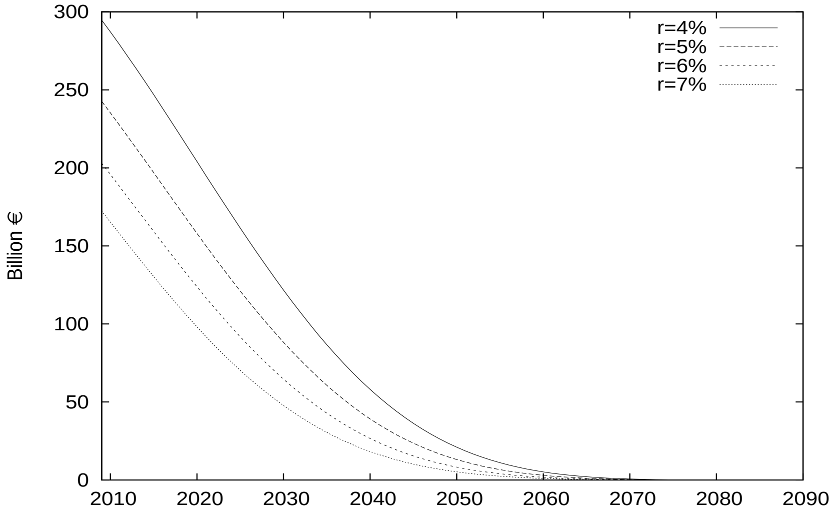


the floor through

$$
\begin{aligned}
F_T &= 0, \\
F_t &= (1+r)F_{t-1} - \bar{c}_t \quad t = 0, \ldots, T,
\end{aligned}
$$

where r is a deterministic discount factor and $\bar{c}_t$ is the median of claim amount at time t; see Figure 3. This corresponds to the traditional actuarial definition of "technical reserves" for an insurance portfolio. We generate 40 CPPI strategies with varying values for the multiplier m and the discount factor r in the definition of the floor.

Figure 3: Development of the floor F with different discount factors r over the 82 year horizon.



5.3 Results

We computed an optimal diversification over the above funds assuming an initial capital of 225 billion euros. We constructed the corresponding linear programming problem with 20000 scenarios as described in Section 4. The resulting LP consisted of 20072 variables and 20001 constraints. The LP was solved with MOSEK interior point solver and AMD 3GHz processor in approximately 30 seconds.

The optimal solution is given in Table 1 with the characteristics of the funds in the optimal diversification. The optimal allocation in terms of the primitive assets at time $t = 0$ is given in Figure 4. The $CV@R_{97.5\%}$ of the optimally constructed fund of funds is 251. The last column of Table 1 gives the $CV@R$ numbers obtained with the individual funds in the optimal fund of funds. The constructed fund of funds clearly improves upon them. The best $CV@R_{97.5\%}$ value among all individual funds is 1020, which means that the best individual fund is roughly 300% riskier than the optimal diversification. Surprisingly, this fund is not included in the optimal fund of funds. All the $CV@R$-values were computed on an independent set of 100000 scenarios.

Table 1: Optimally constructed fund of funds.

Weight (%)	Type	Parameters	$CV@R_{2.5\%}$ (billion €)
66.5	BH	Bonds	1569
2.9	BH	Euro Equity	6567
10.4	BH	US Equity	5041
2.2	FP	$m = 0.8$	3324
3.9	CPPI	$m = 1, r = 4\%, l = 100\%$	1420
9.9	CPPI	$m = 2, r = 4\%, l = 100\%$	1907
4.2	CPPI	$m = 2, r = 5\%, l = 100\%$	2417

Notes: The first column gives the optimal weight of each of the investment strategies. The second column indicates the type of the investment strategy; see section (2). The third column gives the parameters of the investment strategies, with m denoting the weight of the risky assets, r the deterministic discount factor and l the upper bound of the risky assets. The last column gives the $CV@R_{2.5\%}$ for each strategy in billions of euros.

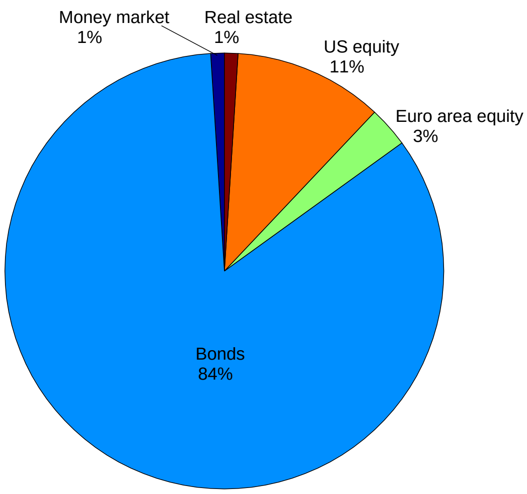
6 Conclusions

This paper applied the computational technique developed in Koivu and Pennanen (to appear) to a long term asset liability management problem with dynamic portfolio updates. The technique reduces the original problem to that of diversifying a given initial capital over a finite number of investment funds that follow dynamic trading strategies with varying investment styles. The simplified problem was solved with numerical integration and optimization techniques. When evaluated on an independent set of 100000 scenarios the optimized fund of funds outperformed the best individual investment strategy by a wide margin. This opens ample possibilities for future research. An interesting possibility would be to apply the approach to risk measure based pricing of insurance liabilities in incomplete markets.

A The time series model

As described above, the returns of the investment funds and pension cash flows can be expressed in terms of seven economic factors; short term (money market) interest rate (Y_1),

Figure 4: Optimal initial allocation in the primitive assets.



yield of a euro area government bond fund (Y_2), euro area total return equity index (S_3), US total return equity index S_4, euro area total return real estate investment index (S_5), Finnish wage index (W) and euro area consumer price index (C). We will model the evolution of the stochastic factors with a Vector Equilibrium Correction-model (Engle and Granger (1987)) augmented with GARCH innovations. To guarantee the positivity of the processes Y_1, Y_2, S_3, S_4, S_5, W and C we will model their natural logarithms as real-valued processes. More precisely, we will assume that the vector process

$$\xi_t = \begin{bmatrix} \ln Y_{t,1} \\ \ln Y_{t,2} \\ \ln S_{t,3} \\ \ln S_{t,4} \\ \ln S_{t,5} \\ \ln W_t \\ \ln C_t \end{bmatrix}$$

follows a VEqC-GARCH process

$$\Delta\xi_t - \delta = \mu_t + \sigma_t \varepsilon_t, \tag{3}$$

where

$$\mu_t = A(\Delta\xi_{t-1} - \delta) + \alpha(\beta^T \xi_{t-1} - \gamma) \tag{4}$$

and

$$\sigma_t^2 = C\sigma_{t-1}\varepsilon_{t-1}(C\sigma_{t-1}\varepsilon_{t-1})^T + D\sigma_{t-l}^2 D^T + \Omega. \tag{5}$$

In (4) the matrix A captures the autoregressive behavior of the time series, the second term takes into account the long-term behavior of ξ_t around statistical equilibria described by the linear equations $\beta'\xi = \gamma$ and δ is a vector of drift rates. The time varying volatilities, and hence covariances, of the time series are modelled through a multivariate GARCH specification (5), where matrices C, D and Ω are parameters of the model.

In its most general form the above model specification has a very high number of free parameters that need to be estimated. To simplify the estimation procedure and to maintain the model parsimonious, while still capturing the most essential features observed in the historical time series, we will assume that the matrices A, C and D are diagonal and fix the matrix β as

$$\beta = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T.$$

The specification of the matrix β implies that the government bond yield and the spread between the bond yield and the short rate are mean reverting processes.

We take the parameter vectors δ and γ as *user specified parameters* and set their values to

$$\delta = 10^{-3} \begin{bmatrix} 0 & 0 & 7.5 & 7.5 & 5.0 & 2.0 & 3.0 \end{bmatrix}^T,$$

$$\gamma = \begin{bmatrix} \ln(5) \\ \ln(5/4) \end{bmatrix}.$$

The vector δ allows the user to specify the expected *median* values of the equity and real estate returns as well as the growth rates of consumer prices and wages. Correspondingly, through the specification of the vector γ the user can control the long term median values of the government bond yield, the spread between the bond yield and short rate, and hence, the expected median level of the short rate. The set equilibrium values imply that the median values of the short rate $Y_{t,1}$ and the yield of the bond portfolio $Y_{t,2}$ will equal 4 and 5, respectively.

We estimated the remaining model parameters using monthly data between January 1991 and July 2008 by applying an estimation procedure where all insignificant parameters were deleted one by one until all remaining parameters were significant at a 5% confidence level. The time series used in the estimation are summarized in table 2 and the estimated parameter matrices are given below.

Table 2: Data series used in the estimation

Stochastic factor	Historical time series
Y_1	Three month EURIBOR (FIBOR prior to EURIBOR)
Y_2	Yield of a German government bond portfolio with an average modified duration of five years
S_3	MSCI Euro area total return equity index
S_4	MSCI US total return equity index
S_5	EPRA/NAREIT Eurozone total return real estate index
W	Seasonally adjusted Finnish wage index (Statistics Finland)
C	Seasonally adjusted Eurozone consumer price index (Eurostat)

$$A = 10^{-2} \begin{bmatrix} 41.995 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 14.807 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 96.233 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 93.422 \end{bmatrix}$$

$$\alpha = 10^{-2} \begin{bmatrix} 0 & -2.119 & 0 & 0 & 0 & 0 & 0 \\ 1.514 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T,$$

$$C = 10^{-2} \begin{bmatrix} 25.788 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 29.816 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 41.952 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 38.588 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 28.071 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 31.8125 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$D = 10^{-2} \begin{bmatrix} 88.301 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 91.236 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 86.412 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 91.373 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 94.117 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 81.056 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\Omega = 10^{-6} \begin{bmatrix} 202.241 & 71.004 & -0.460 & 0.723 & -1.622 & -0.015 & -0.105 \\ 71.004 & 170.507 & 30.889 & 9.200 & -3.682 & 0.134 & -0.277 \\ -0.460 & 30.889 & 202.430 & 53.547 & 54.036 & 0.021 & 0.199 \\ 0.723 & 9.200 & 53.547 & 25.330 & 14.050 & 0.003 & 0.021 \\ -1.622 & -3.682 & 54.036 & 14.050 & 44.769 & -0.094 & 0.179 \\ -0.015 & 0.134 & 0.021 & 0.003 & -0.094 & 0.010 & 0.019 \\ -0.105 & -0.277 & 0.199 & 0.021 & 0.179 & 0.019 & 0.198 \end{bmatrix}.$$

References

F. Black and R. Jones. Simplifying portfolio insurance. *Journal of Portfolio Management*, 14(1):48–51, 1987.

F. Black and A. F. Perold. Theory of constant proportion portfolio insurance. *Journal of Economic Dynamics and Control*, 16:403–426, 1992.

Z. Bodie and J. Treussard. Making investment choices as simple as possible, but not simpler. *Financial Analysis Journal*, 63(3):42–47, 2007.

R. F. Engle and C. W. J. Granger. Co-integration and error correction: representation, estimation, and testing. *Econometrica*, 55(2):251–276, 1987.

H. Föllmer and A. Schied. *Stochastic finance*, volume 27 of *de Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, extended edition, 2004. An introduction in discrete time.

M. Koivu and T. Pennanen. Galerkin methods in dynamic stochastic programming. *Optimization*, to appear.

T. Pennanen and M. Koivu. Epi-convergent discretizations of stochastic programs via integration quadratures. *Numer. Math.*, 100(1):141–163, 2005.

A. F. Perold and W. F. Sharpe. Dynamic strategies for asset allocation. *Financial Analysis Journal*, 51(1):149–160, 1995.

R. T. Rockafellar. Coherent approaches to risk in optimization under uncertainty. *Tutorials in Operations Research INFORMS 2007*, pages 38–61, 2007.

R. T. Rockafellar and S.P. Uryasev. Optimization of Conditional Value-at-Risk. *Journal of Risk*, 2:21–42, 2000.

W. F. Sharpe. Expected utility asset allocation. *Financial Analysis Journal*, 63(5):18–30, 2007.