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Pricing and Hedging Synthetic CDO Tranche Spread Risks

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Abstract

The recent credit crisis has focussed attention on the models used for pricing and assessing risk of structured credit transactions including bespoke CDO's. There are many models that have been proposed for pricing bespoke CDO's including the base correlation mapping methods with the market standard Gaussian copula model as well as the implied copula models. Methods commonly used in the market for hedging and pricing bespoke CDO's make explicit assumptions for the relationship between default probability and default correlation and calibrate the model to current CDO prices only. The ability of a model to hedge CDO tranche spread risks using a credit index is closely related to its ability to price CDOs on bespoke portfolios. This paper examines the measurement and hedging of synthetic CDO tranche spread risks based on market spread data following the sub-prime crisis. A range of methods proposed for pricing bespoke CDOs are examined to assess their ability to hedge the credit spread risk. The methods assessed are calibrated to the traded CDO index spread and then compared based on the mean absolute pricing errors over a time period including the sub-prime crisis. Standard pricing methods and variations used to price bespoke CDOs generally perform poorly in hedging credit spread risk. Past data can be used to improve the performance of the methods. The results of this analysis also raise concerns with the accuracy of "mark-to-model" valuations of bespoke CDOs using standard market methods.

Keywords: credit risk, CDO, Gaussian copula, base correlation, implied copula
JEL classification: G01, G13

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1 Introduction

The market for credit derivatives has been one of the fastest growing financial markets over recent years. These products include credit default swaps (CDS) providing credit protection on a single company, whose value reflects the default risk of that company, as well as portfolio based products such as basket default swaps, cash and synthetic CDOs, which require modelling of the dependence structure of the underlying companies. The most actively traded products are standardized contracts based on the European DJ iTraxx and American CDX IG portfolios. Both of these are constructed with 125 equally weighted investment grade companies across a range of industries. As well as the index there are a number of standard CDO tranches for these two indices that have been traded in the market including the equity, junior mezzanine, senior mezzanine, junior senior and super senior tranches. There are also bespoke CDO's offering protection, including tranches, on non-standard portfolios of credit risks. The International Swaps and Derivatives Association (ISDA) Market Survey showed USD\$54.6 trillion of credit default swaps (CDS) outstanding for in the first half of 2008 and a Bloomberg news article on 22 October 2008 [2] stated the Collateralized Debt Obligation (CDO) market was a \$1.2 trillion market.

More recently, the subprime credit crunch has caused many investment firms, banks as well as insurers specializing in credit protection, to write down losses on portfolios of CDOs. Many of these investors had exposure to super senior tranches. However the exposure to the risk of credit spread widening on these CDOs has resulted in major losses following the increase in default probabilities as the default risks were reassessed following the subprime crisis. Clearly any hedging strategies used by these firms have been ineffective. The standard market model for pricing has been the base correlation model derived from the Homogenous Large Portfolio One-Factor Gaussian Copula Model (OFGC), originally applied to credit derivatives by Li (2000) [10]. For standard CDO's, the model uses the market index spread to determine the default probability and determines a default correlation from the CDO tranche spreads given the default probability. Modifications have been made to the model to price bespoke CDO's, to provide better fits to market tranche spreads and to improve the hedging performance of the model. Finger (2008) [5] assesses the ability of the OFGC model to hedge credit spreads by calibrating the model to market spreads and, assuming perfect foresight for future index spreads and that the calibrated correlations remain constant, predicts the CDO tranche prices for the next 5 days. The predicted tranche spreads are compared with the actual spreads. The results shows high prediction errors and that the tranche prices using the model do not capture credit spread risks as they should if the model is to be useful for hedging.

This paper considers models developed for the purpose of pricing bespoke portfolios and how they can be assessed for the purpose of measuring and hedging credit spread risks. These methods have been designed to address shortcomings of the standard method of pricing based on the OFGC and constant correlation and should measure credit spread risks more accurately. The approach used by Finger (2008) [5] to assess the standard model is extended to consider methods used for pricing bespoke CDO's and their hedging ability based on market data following the subprime crisis. The methods considered include the correlation mapping methods as discussed in Baheti and Morgan (2007) [1]. These methods are used in the market for pricing bespoke portfolios. A recent method is the implied copula model introduced in Hull and White (2006) [8]. Hull and White (2008) [9] assess the pricing errors for a range of calibration methods including the implied copula method. By examining the hedging performance assuming perfect foresight of future credit spreads for the index, it is possible to assess the ability of the methods to hedge credit risk spreads for the tranches as well as for bespoke portfolios. If these market pricing methods do not capture the credit spread risk then this highlights the need for better models and methods, not only for pricing, but also for determining hedging strategies. In Baheti and Morgan (2007) [1] a number of methods are tested by mapping one standard index to another, where they calibrate the base correlation curve to the iTraxx tranche spread and then predict the tranche spread of CDX given the CDX index spread on 31/01/2007. They compare the actual and predicted tranche spread of CDX. Because the ratio of default probability of iTraxx and CDX are close to constant, this does not provide an assessment for portfolios with much higher default probabilities.

The results of our study are interesting and to some extent alarming. The performance of all

the models is generally disappointing with large hedging errors for the period used for the study. This was a period with substantial market turmoil and so the performance of these methods could be expected to be downgraded. However the hedging errors indicate significant concerns for most standard market methods. Using past data to determine a best fit relationship between default probability and default correlation improved the hedging performance and this is something that should be included in these methods to improve pricing and hedging performance. Finally, the study also indicates the need for caution with "mark-to-model" valuations of bespoke CDO portfolios because of the relationship between hedging and valuation of bespoke CDOs.

This paper begins with a brief review of CDOs and the OFGC model. A discussion of hedging follows along with the standard pricing methods including correlation mapping and the implied copula. The results of the hedging study are then presented and discussed. Finally a summary of conclusions from the study are presented.

2 CDOs and the Standard Market Model

Standard market CDOs provide credit protection based on an index portfolio as well as tranches on the portfolio. A synthetic CDO tranche contract is equivalent to an insurance contract with a deductible, or attachment point, a and a policy limit, or detachment point, d that provides different levels of protection against losses resulting from default on an index of a portfolio of firms. The index of the portfolio is based on specified underlying companies and a CDO tranche on the index covers a certain portion of the portfolio loss depending on the attachment a and detachment d points. The index itself can be considered as an index tranche with attachment point 0 and detachment point 1.

iTraxx Europe is a portfolio consisting of 125 equally weighted investment grade European companies. The traded CDOs on this portfolio have standard attachment and detachment points (0-3%), (3-6%), (6-9%), (9-12%), (12-22%). Table 1 shows the mid quotes for iTraxx tranches on 31 January 2007 and 31 July 2008. The quotes for the 0-3% equity or first loss tranche show the up-front payment (as a percent of principal) in addition to 500 basis points running spread per year. The quotes for the other tranches are the annual payment rates in basis points per year.

Dates	0-3%	3-6%	6-9%	9-12%	12-22%	Index	Default probab
7/31/2008	31.48%	355.7	220	141	69.8	93	0.0154
1/31/2007	10.34%	41.59	11.95	5.6	2	23	0.0038

Table 1: iTraxx mid quotes. Source: Bloomberg

The portfolio loss for the commonly traded index iTraxx at time t is given by

$$L_t = \frac{N_t}{N} \cdot (1 - R)$$

where N is the number of companies in the index, which is 125 for iTraxx, N_t the number of defaults in the portfolio up to time t , and R is the recovery rate which is assumed to be fixed at 40% for iTraxx. Synthetic CDO tranches are derivatives on the portfolio loss process L_t and can be considered as a swap of two series of payments, a premium leg and a default leg. The value of both legs are based on the outstanding notional principal of the tranche $K \cdot O_t$ (called notional), where K is the face value of the contract, and O_t is the proportion of the notional outstanding, which can be expressed as a function of attachment point a , detachment point d , and the portfolio loss L_t as

$$O_t = 1 - \left((d - a) - (d - L_t)^+ \right)^+ / (d - a) \quad (1)$$

The credit protection seller receives a payment of premium each quarter. If the portfolio loss does not exceed the attachment point a of the tranche before maturity, the premium is paid on

the full notional agreed in the contract and $O_t = 1$. If sufficiently many firms default such that the total portfolio loss exceeds the attachment point a , then the outstanding notional O_t is reduced according to Equation (1) and the future premium payments will be based on the reduced notional. This is the cashflow for the premium leg.

At the same time, whenever O_t is reduced because of defaults, the protection seller is obliged to pay the protection buyer an amount equal to the loss ($K \cdot (O_t - O_{t-1})$). This is the cashflow for the default leg. The fair or market spread of the tranche is then determined as the spread that equates the expected present value of the two legs.

Modeling the number of defaults N_t requires assumptions for

- probability of default of individual companies
- the recovery upon default (or loss given default), and
- dependencies between individual defaults.

Bluhm and Overbeck (2007) [3] and Giesecke (2004) [6] provide a comprehensive coverage of CDO's including the popular credit risk models. There are two main types of models used in credit risk modelling. Structural models are where the firm's asset process is modelled and default happens when the firm's assets fall below a threshold level. Dependence is introduced by including dependence between the different firms' asset processes. Intensity, or reduced form, models directly model the default intensity for a particular firm and the dependence is introduced in the default intensity of different firms over time. In both of these approaches the default probability and default dependence are usually modelled using a Gaussian factor model, or more generally using a copula model for dependence between marginal default times.

2.1 One-Factor Gaussian Copula as a Market Standard (OFGC)

The market standard pricing model was introduced by Li (2000) [10] and Vasicek (1987) [13]. These models assume the firm defaults when it's asset value falls below a specified level. The asset return X_i for firm i is assumed to be given by

$$X_i = \rho_i Y + \sqrt{1 - \rho_i^2} Z_i$$

so that X_i is modelled with a single common factor Y and an idiosyncratic term Z_i . The term ρ_i measures how much of the variation in X_i can be explained by Y . The term Z_i (for $i = 1, \dots, M$) and Y are all *i.i.d.* standard Normal variables. Under this assumption the X_i are also standard Normals, and the correlation between X_i and X_j is given by $\rho_i \rho_j$. Given Y , the X_i 's are independent. Since the model is driven by a single common factor and the dependence structure is a Gaussian copula, the model is referred to as the One-Factor Gaussian Copula (OFGC) model.

The distribution of X_i is mapped to the distribution of default time τ_i on a percentile to percentile basis using:

$$\begin{aligned} F_i(t) &= P(\tau_i < t) = P(X_i < D_{i,t}) = \Phi(D_{i,t}) \\ \implies D_{i,t} &= \Phi^{-1}(F_i(t)). \end{aligned}$$

where Φ is the standard cumulative normal distribution.

Following Vasicek (1987) [13] conditional on the single common factor Y

$$\begin{aligned} P(\tau_i < t | Y) &= P(X_i < D_{i,t} | Y) \\ &= P\left(\rho_i Y + \sqrt{1 - \rho_i^2} Z_i < \Phi^{-1}(F_i(t))\right) \\ &= P\left(Z_i < \frac{\Phi^{-1}(F_i(t)) - \rho_i Y}{\sqrt{1 - \rho_i^2}}\right) \\ &= \Phi\left(\frac{\Phi^{-1}(F_i(t)) - \rho_i Y}{\sqrt{1 - \rho_i^2}}\right) \end{aligned} \tag{2}$$

and for the portfolio

$$\begin{aligned}
P(\tau_i < t_1, \dots, \tau_M < t_M) &= P(X_1 < D_{1,t}, \dots, X_M < D_{M,t}) \\
&= \int_{-\infty}^{\infty} \prod_{i=1}^M P(X_i < D_{i,t} | Y) \cdot f(Y) \, dY \\
&= \Phi^M(F_1(t), \dots, F_M(t), \Sigma)
\end{aligned}$$

The joint distribution can be shown to be multivariate normal, which corresponds to a Gaussian copula. The model is simple and efficient to implement, which explains its popularity as a market standard model. To simulate the portfolio default process requires only to simulate U_1, \dots, U_M , from a Gaussian copula with covariance matrix Σ and to then determine individual default times from $\tau_i = F_i^{-1}(U_i)$.

The standard market model used for pricing synthetic CDOs also assumes:

- that the underlying portfolio is homogeneous such that $F_i(t)$ are the same for all i , corresponding to the same probability of default,
- all companies in the portfolio have the same correlation, so the ρ_i are the same for all i , and
- the recovery rate is constant and same for all companies.

The distribution of the number of defaults in the portfolio by time t is determined by noting that, given Y , the defaults are independent and the default status of an individual company at time t is Bernoulli with parameter

$$P(\tau_i < t | Y) = \Phi \left(\frac{\Phi^{-1}(F_i(t)) - \rho_i Y}{\sqrt{1 - \rho_i^2}} \right).$$

Assuming an homogenous portfolio this probability is the same for all firms and the distribution of the total defaults for M companies is

$$N_t | Y \sim \text{Binomial}(M, P(\tau < t | Y)).$$

The unconditional distribution for the total defaults is therefore:

$$P(N_t = n) = \int_{-\infty}^{\infty} P(N_t = n | Y) \cdot f(Y) \cdot dY \quad (3)$$

When calibrated to market credit spreads, these probabilities are used to price CDO tranches. The main feature of the copula model is that it separates the default probability and dependence structure and provides the flexibility of using different combinations of copula and marginals. It makes it possible to calibrate the default probability $F_i(t)$ and the correlation parameter ρ_i separately. The probability of default $F_i(t)$ is calibrated to the index tranche spread, since this price is independent of the default correlations. This can be done by assuming defaults follow an homogeneous Poisson process, and the default time of the underlying companies are independent exponential distributions when calibrating $F_i(t)$. Given $F_i(t)$ the model is calibrated to the market tranche spreads by choosing ρ so that the pricing formula spread equals the the market spreads.

2.2 Base correlation: OFGC in practice

In practice the OFGC does not fit all the tranche spreads with a single ρ , therefore different values of ρ are fitted to different tranches. These are called "compound correlations". A plot of these against the detachment points of CDO tranches exhibits the "correlation smile" as illustrated in Figure 1.

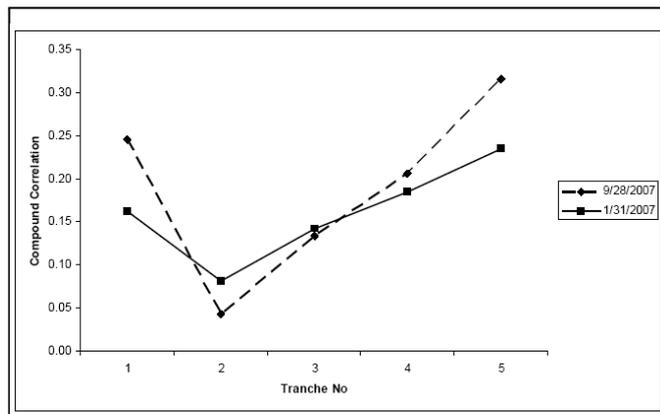


Figure 1: Market implied compound correlation smiles as at 1/31/2007 and 9/28/2007.

This is similar to the "volatility smile" observed in pricing equity options with the Black-Scholes formula. Although the OFGC has been regarded as the Black-Scholes model for credit derivatives, there are major differences between them. In option pricing when the volatility increases the price of an equity option increases, but the effect of correlation on the CDO tranche prices varies by the seniority of the tranche. When the correlation parameter increases the equity (first loss) tranche spread generally increases and the senior tranche spread generally decreases. The effect of the correlation parameter on mezzanine tranches varies so that there may even be 2 correlation parameters (one very high, one very low) that can fit mezzanine tranche spreads or there may be no correlations (between 0 and 1) that fit. Table 2 gives the fitted compound correlation for 13 selected dates. Note that the OFGC model can not fit market data when the default probability is high.

Dates	0-3%	3-6%	6-9%	9-12%	12-22%	Default probab
7/31/2008	0.47	0.87	NaN	0.14	0.25	0.0154
5/30/2008	0.38	0.88	NaN	0.19	0.29	0.0132
3/31/2008	0.47	0.85	NaN	NaN	0.22	0.0205
1/31/2008	0.39	0.84	0.12	0.23	0.35	0.0128
11/30/2007	0.33	NaN	0.11	0.21	0.33	0.009
9/28/2007	0.25	0.04	0.13	0.21	0.32	0.006
7/31/2007	0.27	NaN	0.12	0.18	0.28	0.009
6/29/2007	0.19	0.09	0.14	0.19	0.25	0.0042
5/31/2007	0.18	0.11	0.16	0.19	0.26	0.0033
4/30/2007	0.17	0.09	0.14	0.18	0.25	0.0038
3/30/2007	0.18	0.10	0.14	0.18	0.24	0.004
3/1/2007	0.17	0.09	0.15	0.19	0.24	0.0038
1/31/2007	0.16	0.08	0.14	0.18	0.24	0.0038

Table 2: Market implied base correlations (NaN means no correlation between 0 and 1 exists)

This drawback of the model when fitting to market data results in limitations in using compound correlations to value non-standard tranches since interpolation of the compound correlation curve is unreliable and sometimes impossible when there are 2 or no correlations that correspond to market tranches.

In order to avoid this problem the market has developed the method of "base correlation". The idea of base correlation was introduced in McGinty, Beinstein, Ahluwalia and Watts (2004) [11] and is uses the fact that any tranche can be represented as the difference between 2 equity tranches with different detachment points. For example, selling protection on a 4-8% tranche (with notional K) is equivalent to selling a 0-8% tranche (with notional $2K$) and buying a 0-4% tranche (with notional K) at the same time. The correlations fitted to these equity tranches are

called "base correlations" or "detachment correlations". A plot of these market implied "base correlations" against the standard detachment points shows the "correlation skew" in Figure 2:

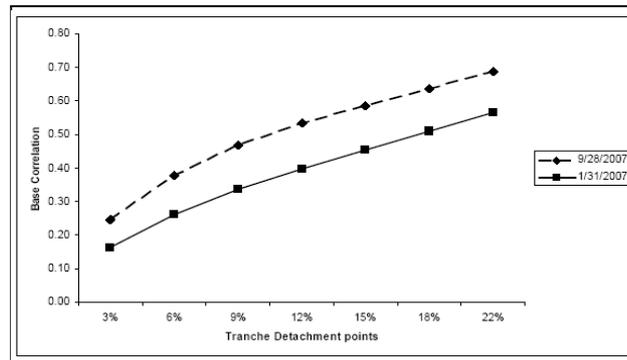


Figure 2: Market implied base correlations skew as at 1/31/2007 and 9/28/2007.

The base correlations are quoted with the tranche spreads in the CDO market and are used to price non-standard tranches using interpolation since they show a nearly linear relationship in Figure 2. The base correlation approach overcomes the problem of determining the correlation for mezzanine tranches, because the equity tranche spreads are always monotonic with correlation. It has also been found that the base correlation method fails to price the senior tranches at the time of high default probabilities. Table 3 gives the fitted base correlations corresponding to the dates in Table 2. No correlation parameter can fit recent prices of senior tranches after the credit crunch as default probabilities have increased.

Dates	0-3%	3-6%	6-9%	9-12%	12-22%	Default probab
7/31/2008	0.47	0.61	0.69	0.77	NaN	0.0154
5/30/2008	0.38	0.54	0.63	0.71	NaN	0.0132
3/31/2008	0.47	0.59	0.66	0.71	0.85	0.0205
1/31/2008	0.39	0.54	0.62	0.69	NaN	0.0128
11/30/2007	0.33	0.51	0.62	0.71	NaN	0.009
9/28/2007	0.25	0.38	0.47	0.53	0.69	0.006
7/31/2007	0.27	0.37	0.46	0.52	0.69	0.009
6/29/2007	0.19	0.29	0.37	0.43	0.61	0.0042
5/31/2007	0.18	0.26	0.33	0.39	0.54	0.0033
4/30/2007	0.17	0.26	0.33	0.40	0.56	0.0038
3/30/2007	0.18	0.26	0.33	0.39	0.56	0.004
3/1/2007	0.17	0.27	0.34	0.40	0.57	0.0038
1/31/2007	0.16	0.26	0.34	0.40	0.57	0.0038

Table 3: Market implied base correlations (NaN means no correlation between 0 and 1 can fit)

However, base correlations are only an advanced interpolation technique and do not rectify the limits of the assumptions underlying the OFGC model.

3 Credit Spread Risks

One of the lessons from the recent global financial crisis is that investors in CDOs did not properly take into account credit spread risks especially under scenarios involving increasing default probabilities. Credit spread risk is the risk that the credit spread of the traded CDO tranche will vary. Senior CDO tranches were believed to be low risk because losses were not expected to hit the attachment points. However an increase in the spread of a CDO contract results in a mark-to-market loss for the protection seller in it's trading book. In fact an increase in default probability has a larger effect on the credit spread of the senior tranches. Table 1 given previously shows the market price for iTraxx CDOs on two dates, one before and one after the subprime credit crisis

took effect. The quotes are in basis points except for the equity tranche (0-3% tranche). The senior tranche (12-22%) spread increased 35 times while the mezzanine tranche (3-6%) spread increased 9 times between these dates whereas the market implied default probability increased only 4 times. Credit spread risk on these tranches is highly sensitive to default probabilities. Neugebauer et.al (2006) [12] discusses how credit spread risk is measured using the delta or hedge ratio. This considers how CDO tranches can be hedged using the index. The hedge ratio (delta) is computed as the percentage of notional amount of the index tranche that needs to be bought/sold to hedge a long/short CDO tranche position. The deltas are quoted in the market along with the prices of CDO tranches. Currently the market computes the delta using the OFGC by varying the index spread, holding the correlation constant.

In order to assess this credit spread risk associated with writing these CDOs one way would be to stress test a portfolio using a set of scenarios for the economy that implied different default probabilities and to determine credit spreads for a particular CDO tranche under each scenario. This would indicate how much additional capital would be required in the event of adverse economic conditions and hence provide a measure of the credit spread risk. The problem is to decide which method to use to determine the credit spread of the CDO tranches in each scenario. For example: assuming the method was fitted to the market CDO prices at 1/31/2007, and the default probability scenario was changed from 0.0038 to 0.0154, then the method should produce tranche prices close to the market prices at 7/31/2008 as in Table 1. Deviations from these prices is an indication of credit spread risk for the portfolio.

Finger (2008) [5] assesses the ability of the OFGC model to hedge. One of the main reasons that the OFGC performs poorly in his study is because there is a relationship between default correlations and the default probabilities implied by the level of index spread in market data. Figure 3 plots the market implied equity tranche correlations against implied yearly default probabilities for 13 selected dates from 1/31/2007 to 7/31/2008.

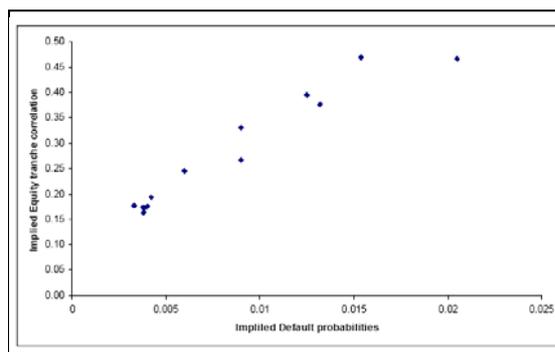


Figure 3: Default correlations and index default probabilities

A strong relationship is observed so that improved hedging models need to allow the fitted correlation to vary with default probabilities. This issue is closely related to the issue of pricing bespoke CDOs. A bespoke CDO on a portfolio which consists of low rated companies and has a default probability that's twice the standard iTraxx portfolio will need a correlation parameter to be determined for pricing the bespoke CDO tranches using the standard market model. Many models have been proposed for pricing bespoke CDOs. According to Finger (2004) [4], JP Morgan prices bespoke CDOs using the method of ATM (at-the-money) mapping, which implies a higher correlation for higher default probabilities, consistent with the observed relationship in Figure 3.

Understanding the relationship between CDO tranche prices and index default probabilities is fundamental to measuring hedging risk for credit risky portfolios. Credit spread risks involves determining the change in CDO tranche prices given a change in index default probability, hedging a CDO tranche position with the index tranche requires the delta which is the change in the CDO tranche prices given a change in the price (default probability) of the index tranche and pricing bespoke CDOs involves computing the difference between the price of CDOs on a standard portfolio and the bespoke portfolio, given the difference in default probability of the two portfolios. All of

these are closely related since default probabilities are calibrated from the index tranches, they all assess the change in CDO tranche prices given a change in the default probability. They differ only in the size of the change in default probability. For hedging, the change is usually small because the time interval is short. For pricing bespoke CDOs, the change is usually large. For example the spread of the iTraxx Europe Crossover index is usually more than 5 times the spread of the standard iTraxx Europe index. The market prices bespoke CDOs using the method of base correlation mapping. To measure credit spread risks, the change is usually small but it can be large because of extreme events and these are the events we are interested in from a risk management perspective. Table 1 shows how the default probability increased 4 times from before to after the subprime crisis impacted. To assess a method's ability to measure credit spread risks we need to test if it can correctly price CDO tranches given a large change in default probability as well as small changes.

3.1 Assessing Credit Spread Risks

The approach used to assess the effectiveness of the different methods for quantifying credit spread risk is similar to that used by Finger (2008) [5] using longer time periods, since we want to test the ability of the methods to correctly price CDO tranches when the default probability changes significantly. The dataset consists of 101 observations of the mid quote of iTraxx Europe tranche spreads from 22/09/07 to 12/09/08. Each observation consists of the price for an index tranche and 5 CDO tranches, including the history of iTraxx Europe series 9 and series 8. The maturity of the CDOs is 5 years. The source of the data is from Bloomberg (source provider: CMAN New York).

For each of the methods tested, they are first fitted to the market prices at the date 1/01/08. Assuming a relationship between default correlation and default probabilities and assuming the future index spread is known, the CDO tranche spreads are then predicted for the next 71 dates up to 12/09/08 and compared with the actual spreads. This is a period including the sub-prime crisis effects and will provide a good assessment of performance under stress conditions. To assess how using past data can improve the methods, a calibration to the past 30 dates from 22/09/07 up to 1/01/08 is used, and the calibrated model is then used to predict the future CDO tranche spreads assuming the future index spread is known (perfect foresight). The performance of different methods are then compared using the mean absolute error which can be interpreted as the percentage that the estimated spread of a particular tranche differs from the actual tranche spread on average.

3.1.1 Basic regression model

This method is based on the assumption that there is a simple relationship between CDO tranche spreads and the index spread. It is assumed that

$$T_i = a_i \cdot I$$

where T_i is the spread of i th tranche, so there is assumed to be a proportional relationship with the index spread I . The parameters a_i are calibrated from the market spreads on 1/01/08, which are simply the ratios of the tranche spreads and index spread at that date. Tranche spreads after 1/01/08 up to 12/09/08, are determined based on the index spreads on those dates and assuming the parameters a_i remain constant. Calculated spreads are compared with the actual spreads to determine an overall mean absolute error.

3.1.2 Regression on past price data

The simple regression calibrates the parameters only using the current date and does not take into account past information. A natural extension is to assume that $T_i = f(I)$, and to estimate the parameters by regressing the tranche spreads on the index spreads using the historical 30 day

data from 22/09/07 up to 1/01/08. The calibrated parameters are then used to predict the future tranche spreads up to 12/09/08.

For the equity tranche the following relationship was found to fit:

$$T_1 = 1 - a_1 \cdot I^{\beta_1}$$

and for the other tranches a linear relationship such that

$$T_i = a_i + I \cdot \beta_i$$

was found to fit well.

The equity tranche is modelled differently to other tranches mainly because it is quoted as a percentage of up-front payment which is always less than 1. By incorporating past information the performance of the regression model should be greatly improved. Although it may be obvious to include past data using a regression model, most methods currently used in the market don't include past data for either hedging or pricing bespoke portfolios.

3.1.3 Base Correlation Mapping

When hedging CDO tranches with the index, market practice is to compute the delta using the OFGC by varying the index spread, holding the correlation constant (Neugebauer et.al (2006) [12]). This assumption is not consistent with empirical observations since default correlations vary with default probabilities in market data. This problem is similar to the problem of pricing bespoke CDOs, where in practice it is not assumed that the default correlation for a bespoke portfolio that has higher probability of default is equal to the default correlation of the standard iTraxx or CDX portfolios. As noted in Baheti and Morgan (2007) [1], once the base correlation curve is calibrated to market prices of liquid market tranches, mapping methods are required to apply these calibrated correlation parameters to derive a base correlation curve for the bespoke portfolio in order to price CDOs on these portfolios. There are 3 approaches that will be considered.

No Mapping This method assumes correlations are not related to the default probabilities hence the correlation parameters calibrated to the standard portfolio are directly used to price bespoke CDOs. This is similar to the market method of hedging assuming the correlation curve is constant. Finger (2004) [4] also notes that the effect of hedging differs significantly between holding the compound correlation constant or holding the base correlation constant. To test this method, a base correlation curve is fitted to the market data at 1/01/08 and then the same correlations are used to predict the CDO tranche spreads for the next 71 dates up to 12/09/08.

ATM (At-The-Money) Mapping According to Finger (2004) [4], this is the method adopted by JP Morgan to price bespoke CDOs. The method assumes that if the ratio of default probability of the bespoke portfolio and the standard portfolio is a , then the 0 to $X\%$ tranche of the bespoke portfolio should be valued with the same correlation as the 0 to $aX\%$ tranche of the standard portfolio. To test the model, the base correlation curve is fitted to the data at 1/01/08 as a benchmark. If the future index spread implies a different default probability, the actual base correlation used to price the tranches is the 0 to $aX\%$ tranche correlation of the benchmark base correlation curve using linear interpolations, where a is the ratio of the implied default probability from the future index spread to the default probability implied on 1/01/08.

TLP (Tranche Loss Proportion) Mapping Beheti and Morgan (2007) [1] show that ATM mapping miss-prices the senior tranches of bespoke portfolios and suggested that the method of TLP mapping outperforms the other currently used mapping rules. This method assumes that:

$$\frac{ETL_S(K_S, \rho(K_S, T))}{EPL_S} = \frac{ETL_B(K_B, \rho(K_S, T))}{EPL_B} \quad (4)$$

where ETL is the expected tranche loss. Equation (4) implies that an equity tranche of bespoke portfolio with detachment point K_B should be valued with the same correlation as an equity tranche of the standard portfolio with detachment point K_S , if the expected tranche loss of these 2 equity tranches as a proportion of the respective expected portfolio loss are the same. A root search procedure is used to find the K_S corresponding to the bespoke strike K_B by first discretising the strikes from 1% to 100% and determining which equity tranche of the standard portfolio has the same expected tranche loss proportion as the bespoke equity tranche with strike K_B . More details of this procedure are given in Appendix A.

3.1.4 OFGC with parameterized base correlation

The above mapping rules assume a relationship between the default correlation and default probabilities, and calibrate the model only to the CDO prices at 1/1/08. Past data can be used to estimate the relationship between default probability and correlation. To incorporate past data with the OFGC the following method is proposed. Calibrate the OFGC to the 30 dates from 22/09/07 up to 1/01/08, giving 30 calibrated base correlation curves. Explicitly parameterize the base correlation as function of the default probabilities. Figure 4 shows the fitted base correlations for the 5 standard iTraxx tranches from 22/09/07 up to 1/01/08 using a linear function:

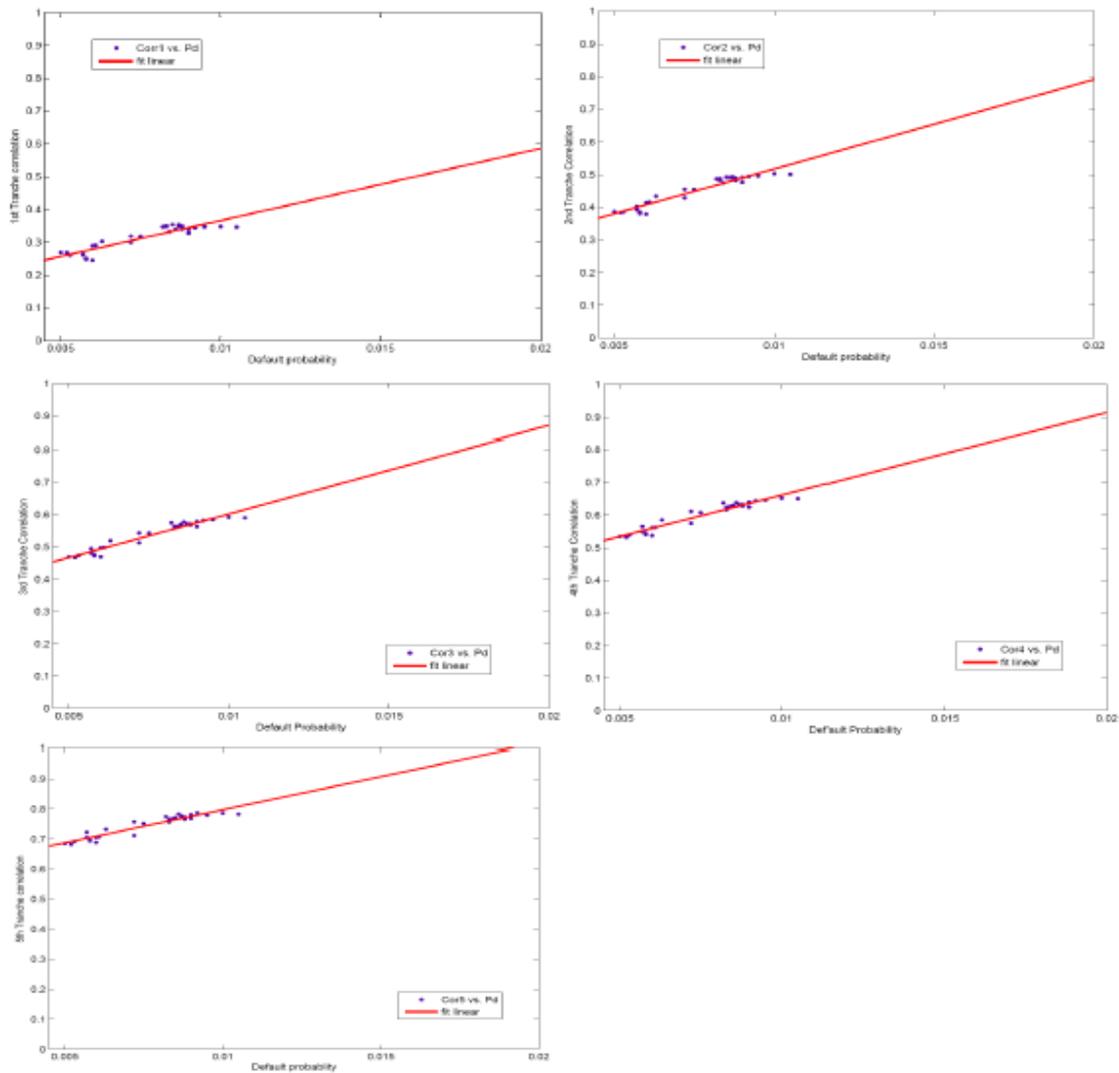


Figure 4: Linear fit for base correlations

The base correlations are highly correlated with default probabilities, Table 4 gives the adjusted R^2 for the fits of the linear function $corr = a + b * P$, where P is the default probability, and for the power function $corr = a * P^b$. The trend appears to not be linear with the increments decreasing. Also a linear relationship between default probability and default correlation may not be adequate because it can lead to correlations greater than 1 when attempting to price with a very high probability of default.

	Corresponding Tranches				
	0-3%	0-6%	0-9%	0-12%	0-22%
	Adjusted R ² for fitted model				
Linear Function:	0.82	0.91	0.93	0.93	0.89
Power Function:	0.84	0.92	0.94	0.94	0.90

Table 4: Fit of linear function for base correlation

Using the fitted relationship between default probability and base correlations, a base correlation curve for each of the 71 dates from 5/01/08 up to 12/09/08 is predicted based on the default probability implied from the index tranche on that date, and then used to price the CDO tranches on those 71 dates and compared with the actual tranche spreads.

3.1.5 The Implied Copula Approach:

Many extensions on the Gaussian copula have been studied by various researchers to overcome its limitations mainly in an attempt to better fit the market prices. One method is the implied copula approach introduced by Hull and White (2006) [8]. For the OFGC, given the common factor Y , the defaults are independent and the default state of an individual company at time t is Bernoulli. Assuming an homogenous portfolio this probability is the same for all firms, therefore the sum of the default of M companies is binomial. The unconditional distribution is:

$$P(N_t = n) = \int_{-\infty}^{\infty} P(N_t = n|Y) \cdot f(Y) \cdot dY$$

The implied copula approach directly models the distribution of the unconditional default probabilities. The simplest implementation of this approach assumes individual defaults follow an homogenous Poisson process therefore the default probability at time t given the hazard rate is:

$$P(\tau_i < t|\lambda) = 1 - \exp(-\lambda t).$$

The conditional distribution of the number of portfolio defaults is therefore

$$N_t|\lambda \sim Binomial(M, P(\tau < t|\lambda))$$

and the unconditional distribution is therefore:

$$P(N_t = n) = \int_0^{\infty} P(N_t = n|Y) \cdot f(\lambda) \cdot d\lambda \quad (5)$$

The implied copula approach determines an implied distribution of λ that will fit the market CDO tranche spreads. The distribution of λ is assumed to be a discrete L -point distribution (multinomial) with L possible values $(\lambda_1 \dots \lambda_L)$ and $Pr(\lambda = \lambda_i) = P_i$, with $\sum_{i=1}^L P_i = 1$. The P_i are chosen to fit the market prices. Hull and White (2006) [8], used a 50-point distribution with a constraint to smooth the distribution and showed that the distribution fits perfectly to the market tranche spreads. The distribution of λ is directly related to the default correlation implied by the model. According to Hull and White (2006) [8], the default correlation depends on the dispersion

of λ , when the variance of λ increases the default correlation increases. This can be shown by considering the OFGC setting, where

$$P(\tau_i < t|Y) = \Phi \left(\frac{\Phi^{-1}(F_i(t)) - \rho_i Y}{\sqrt{1 - \rho_i^2}} \right).$$

The variance of the unconditional default probability depends on $\frac{\rho_i}{\sqrt{1 - \rho_i^2}}$, and when ρ_i increases the variance increases, and vice-versa.

Valuing bespoke portfolio with implied copula: Hull and White (2006) [8] proposed the following way of pricing bespoke CDOs. An additional parameter β is introduced, and the hazard rate of the bespoke portfolio is related to the standard portfolio by:

$$\lambda^* = \beta\lambda \tag{6}$$

Details of fitting the implied copula model are given in Appendix B.

They assume $\beta = 1$ and calibrate the distribution of λ to the tranche prices of standard portfolios. Then they vary β so the index spreads of the bespoke portfolio is matched. The model should reasonably price the CDO tranches on the bespoke portfolio. To test the method, it is fitted to the market price at 1/01/08. The parameter β is chosen for each date from 5/01/08 to 12/09/08, such that the index tranche spread calculated from the model matches the market index spread. The bespoke tranche spread produced by the model is the predicted value which is compared with the actual market spreads.

3.1.6 An Improved Implied Copula Model:

Hull and White (2008) [9] propose a parameterized version of the implied copula model, in which it is assumed that the variable:

$$\frac{\ln \lambda - \mu}{\sigma} = t_v \tag{7}$$

has a Student t distribution with v degrees of freedom. Thus 3 parameters μ, σ and v are used to describe the probability distribution of λ . Hull and White (2008) [9] show that this 3 parameter model fits well to market tranche spreads, and the parameter μ increases when the index spread increases but the parameters σ and v are remarkably similar on any given day. It is also found that the two parameters that describe the dispersion of the distribution, σ and v , tend to move together, and it was proposed to use $v = 2.5$. To value a bespoke portfolio with the improved implied copula, Hull and White (2008) [9] propose that since σ and v don't vary much when the default probability changes, it can be assumed that the parameters σ and v fitted to the standard portfolio apply to the bespoke portfolio. The parameter μ is then chosen to match the average spread for the companies underlying the bespoke, and the model should reasonably price the bespoke tranches. The improved implied copula method for pricing bespoke portfolio is tested by fitting the 3 parameters μ, σ, v to the market prices at 1/01/08, assuming the parameters σ and v remain constant and by varying μ the index tranche spread calculated from the model is matched to the market index spread.

3.1.7 Using past price data

A method using past price data when calibrating the model is to explicitly assume a relationship between default correlation and default probabilities, and then calibrate directly to past CDO prices. This method can be applied to the OFGC and the mapping methods. Consider the ATM mapping method as an example. Recall that the ATM mapping method assumes that if the ratio of default probability of the bespoke portfolio and the standard portfolio is a , then the 0 to $X\%$ tranche of the bespoke portfolio should be valued with the same correlation as the 0 to $aX\%$ tranche of the standard portfolio. The model can be calibrated to past data by selecting a standard

default probability, then assuming there exists a standard base correlation curve such that if the ratio of the market implied default probability on a date and the standard default probability is a , then the 0 to $X\%$ CDO tranche on that date is valued with the same correlation as the 0 to $aX\%$ tranche of the standard base correlation curve. An optimal standard base correlation curve can be fitted to past data by minimizing the sum of squared pricing errors.

The same method can be applied to the implied copula model by calibrating an optimal distribution of λ that best fits all the past CDO prices, assuming that the distribution of λ is constant across time but β changes. An advantage of this method is that if after optimizing, there are still high pricing errors, then the assumptions of the method do not adequately explain the market prices. The disadvantage of this method is that it takes a long time to calibrate to past data. This time can be reduced in a copula model, because default probabilities can be calibrated separately, while in the implied copula method the parameter β has to be calibrated for each past date, which increases the time to calibrate the model dramatically. Because of this, this method has not been assessed in this paper.

4 Results and discussions

4.1 Hedging risk

Market tranche spreads are not solely determined by the default probability. Therefore when the market CDO tranche spreads differ whenever the index spread is the same, any method that assumes default correlations have a constant relationship to the default probability will produce errors. In this case the CDO tranches cannot be perfectly hedged only by trading the index contract. The unhedged component measures the best a model can achieve hedging CDO tranches with the index. This is estimated as follows. The CDO tranche prices with the same index spreads are grouped and it's assumed that the best estimate a method can achieve is the mean of these CDO tranche prices within the same group. The mean absolute error found for the difference between this mean and the actual tranche spread is shown in Table 5:

Tranche	0-3%	3-6%	6-9%	9-12%	12-22%
Error	0.0456	0.0377	0.0431	0.0519	0.0584

Table 5

This is interpreted as follows. For the 0-3% tranche, the minimal error for the tranche spread that a hedging model can achieve is 4.56%. Thus the best hedging model would, on average, miss-price the equity tranche by 4.56% based on the change of the index spread. Similar comment applies for the other tranches.

4.2 Model Results and Comparisons

The results of the comparison of the different methods is given in Table 6. This table shows the mean absolute difference between actual and estimated tranche spreads as a proportion of the actual tranche spreads. For example the "Reg1" model has on average, miss-priced the 0-3% tranche by 21.08% and the 3-6% tranche by 22.05% and so on. For the methods

- "Reg1" refers to the proportional regression model based on current data
- "Reg2" refers to the linear regression method
- "Ccc" refers to the constant compound correlation method
- "Cbc" refers to the constant base correlation method
- "ATM" and "TLP" refers to the method of ATM mapping and TLP mapping

- "Pbc" is where a power function was fitted to the base correlation as a function of default probability.
- "IC" refers to the implied copula model
- "IIC" refers to the improved implied copula model.

Model	0-3%	3-6%	6-9%	9-12%	12-22%	Average
Reg1	21.08%	22.05%	28.11%	30.29%	20.90%	24.48%
Reg2	6.67%	17.82%	22.45%	10.51%	10.65%	13.62%
Ccc	19.62%	24.53%	54.01%	34.18%	36.43%	33.93%
Cbc	19.62%	7.31%	9.14%	10.02%	13.28%	11.87%
ATM	9.03%	19.29%	31.89%	64.17%	195.28%	63.93%
TLP	7.71%	15.35%	9.14%	15.53%	11.92%	11.93%
Pbc	7.89%	34.60%	24.52%	10.16%	13.98%	18.23%
IC	45.30%	17.36%	15.70%	13.89%	19.95%	22.44%
IIC	53.74%	62.56%	12.19%	13.56%	9.41%	30.29%

Table 6: Mean absolute errors for standard market methods

From comparing the result of "Ccc" and "Cbc" it can be seen that hedging holding the base correlation constant is much better than holding the compound correlation constant. Both methods give the same prediction error for the equity tranche which is expected since it's priced the same way. However, hedging with constant base correlation gives the minimal error on average especially for the mezzanine tranches, despite the empirical observation that correlation increases with default probabilities. An explanation of this is that in the base correlation approach every tranche, except the equity tranche, is priced with 2 correlations, and the price is mostly affected by the difference between the 2 correlations instead of their level.

"Cbc" gives a high error when predicting the equity tranche, whereas other methods (ATM, TLP, Pbc) that assume the correlation increases with default probabilities provide much better results for equity tranches. For the same reason, when we parameterize the base correlation using past data, it improves the prediction for the equity tranche but provides worse results for the other tranches, since it considers the level of the base correlations instead of the difference between them. This suggests that the base correlation method may not be suitable as a basis to find the best fit relationship between default correlation and default probabilities.

TLP mapping method gives nearly as good a result overall and is significantly better in predicting the equity tranche spreads. ATM mapping gives the worst result overall and greatly miss-prices the senior tranches. These results are consistent with the results found in Baheti and Morgan (2007) [1]. This is not surprising since the assumption in the TLP mapping method is more reasonable than ATM mapping. This suggests that the base correlation mapping method can be used to hedge CDO tranche spread risks with the index, and TLP mapping can be considered a good alternative to the constant base correlation method in hedging.

The implied copula models gives poor results overall for hedging large moves in default probability. This may be because the method tends to overfit current market price data and can not capture movements away from the calibrated price data well.

Finally the regression models don't provide very good results as expected but surprisingly do not perform too badly. When past data is used to fit the regression model it significantly improves it's performance, being less than 2% worse than the "Cbc" and "TLP" models on average and it's the best method to predict the equity tranche spreads. Incorporating past information can improve a method's ability to hedge.

5 Conclusions and future research

The ability of a model or method to hedge CDO tranche spread risks with the index and it's ability to price CDOs on bespoke portfolios are closely related. Models that the market uses to price bespoke portfolios can be used for the purpose of hedging. In this paper a number of models are tested using a similar method for testing hedging as based on the approach in Finger (2008) [5]. The results of the study show that

- Hedging by holding the base correlation constant is significantly better than holding the compound correlation constant
- the ATM mapping method improves the prediction for equity tranches but fails badly for predicting the senior tranches
- the TLP mapping method is significantly better than ATM mapping particularly for equity tranches but provides worse result when predicting mezzanine tranches compared to the method of using constant base correlation (no mapping).

These results are consistent with the results reported in Baheti and Morgan (2007) [1] where the ability of the models to price bespoke portfolios is tested using a different approach. They fit the model to iTraxx and predict the CDX. This supports the proposition that the test for hedging can be applied to test a model's ability to also price bespoke portfolios. This is worthy of further research, since the method adopted in this paper could be a better approach to test a model's ability for pricing bespoke CDOs. The difference between the implied default probability for iTraxx and CDX are relatively constant through time and do not provide a good basis to extrapolate to portfolios with much higher default probabilities. The TLP mapping method can be applied to hedging as an alternative to the method of holding the base correlation constant, since it gives as good results in general and is significantly better for hedging the equity tranche.

In order to include past information a method was proposed by first calibrating the model to past data and then finding a best fit relationship between default correlation and default probabilities. From the observation that the market implied correlations are positively correlated with default probabilities, default correlation was fitted as a deterministic function of default probability from the past data. This significantly improves the hedging for equity tranches, however it does not give satisfactory results for the mezzanine tranches. The reason for this is that in the base correlation method every tranche other than the equity tranche is priced with 2 correlations, and the tranche's price mainly depends on the difference between correlations instead of the level of correlations. The base correlation method may not be suitable for this purpose. The base correlation has also been proven to be unable to price senior tranches with high default probability..

Implied copula models were considered as an alternative to the base correlation approach, and its ability to hedge and price bespoke CDOs are assessed. The results show that they perform worse than the current market models suggesting that it should be used with some caution. The implied copula is considered a good model for pricing because it is able to construct an empirical distribution to fit the market prices. However this will tend to overfit the data and caution should be exercised in using the model to hedge large changes in default probability. A further

disadvantage of the implied copula model is that it cannot calibrate the default probability from the index tranche separately. The ability to separate the default probability and the dependence structure is one of the biggest advantages of the copula models. This enables the model to be calibrated much quicker, specially when it comes to hedging.

The current methods used in the market for hedging or pricing bespoke CDOs have explicit assumptions of the relationship between default probability and default correlation and generally ignore past data. Incorporating past data can improve the performance of these methods.

Finally, although this study has considered hedging tranche spreads, because of the similarity between pricing bespoke CDOs and the methods considered here, the results of this study also indicate the need for caution in using "mark-to-model" valuations of bespoke CDOs based on standard market methods.

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APPENDIX A: TLP Mapping

First the base correlations are fitted to the 5 standard tranches of iTraxx on the date 1/1/08. From these 5 values, a base correlation curve consisting of 100 values for equity tranches with detachment points 1% to 100% is constructed with linear interpolation.

Denote the market implied default probability at 1/1/08 as P_S , the expected present value of the default leg of CDO tranches with attachment point 0 and detachment point d are calculated for $d = 1\%$ to 100% using the constructed base correlation curve and the corresponding correlations (for e.g. 0-1% tranche are valued with 0-1% correlation). These values are stored in a vector ETL_S .

Define a (100×100) Matrix ETL_B . Given market implied default probability at another date P_B . the expected present value of the default leg of CDO tranches with detachment point $d = 1\%$ to 100% are calculated using each of the 100 correlation values of the base correlation curve and the results are stored in the matrix ETL_B . (for e.g. 0-1% tranche are valued with 0-1% correlation and stored in $ETL_B(1, 1)$, then valued with 0-2% correlation and stored in $ETL_B(1, 2)$... so on).

EPL_S is the last element in the vector ETL_S and EPL_B is the last element in the vector ETL_B . (Expected portfolio loss is the expected loss of the 0-100% tranche, which is not affected by the value of correlations used).

The K_S corresponding to the bespoke strike K_B is found such that

$$\frac{ETL_B(K_B, K_S)}{EPL_B} = \frac{ETL_S(K_S, K_S)}{EPL_S}.$$

Note since a discrete approximation is used, the equation cannot be matched exactly, K_S is found as the value that best fits.

APPENDIX B: Fitting the Implied Copula Model

Hull and White (2006) [8] stated that the implied copula model cannot be fitted with constant recovery rate, and the following recovery rate specification reported in Hamilton et al (2005) [7] was recommended:

$$R(Q) = \max[0.52 - 6.9 \times Q(1), 0] \quad (8)$$

where $Q(1)$ is the yearly default probability which implies a negative correlation between default rates and recovery rates and is consistent with empirical observations.

Figure 5 shows the fitted distribution of λ for iTraxx data at 1/1/08 assuming a 17-point distribution. The value of λ_i were chosen such that the expected number of defaults in 5 year intervals is (0.5, 0.75, 1, 2, 3, 4, 6, 8, 12, 16, 24, 32, 48, 64, 96, 124) in the iTraxx portfolio of 125 names.

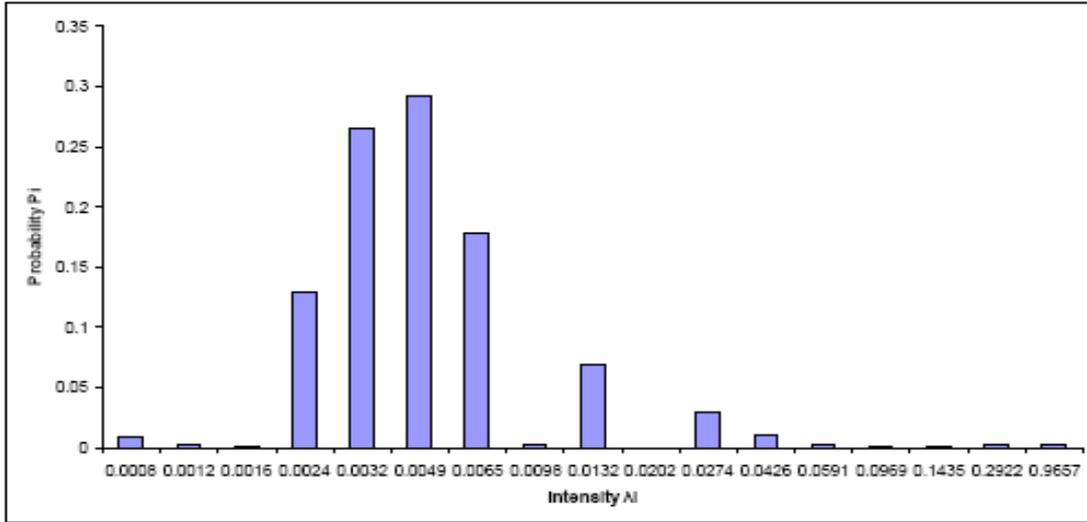


Figure 5: Implied hazard rate distribution for iTraxx on 1/1/08

The distribution was smoothed by minimizing the following constraint suggested in Hull and White (2006) [8] when fitting the model:

$$\sum_{i=2}^{L-1} \frac{(P_{i-1} + P_{i+1} - 2P_i)^2}{(\lambda_{i+1} - \lambda_{i-1})}$$

It can be observed that the market implies a small probability to extreme value of hazard rates, for example 0.4% probability for a hazard rate of 0.9657, which implies an expected default of 99.2% of the portfolio over 5 years. This result is common to many intensity models, without assigning a positive probability to a very high value of default intensities, the senior tranches cannot be correctly priced.