Superhedging in illiquid markets

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Abstract

We study superhedging of securities that give random payments possibly at multiple dates. Such securities are common in practice where, due to illiquidity, wealth cannot be transferred quite freely in time. We generalize some classical characterizations of superhedging to markets where trading costs may depend nonlinearly on traded amounts and portfolios may be subject to constraints. In addition to classical frictionless markets and markets with transaction costs or bid-ask spreads, our model covers markets with nonlinear illiquidity effects for large instantaneous trades. The characterizations are given in terms of stochastic term structures which generalize term structures of interest rates beyond fixed income markets as well as martingale densities beyond stochastic markets with a cash account. The characterizations are valid under a topological condition and a minimal consistency condition, both of which are implied by the no arbitrage condition in the case of classical perfectly liquid market models. We give alternative sufficient conditions that apply to market models with general convex cost functions and portfolio constraints.

Key words: Superhedging, illiquidity, claim process, premium process, stochastic term structure

1 Introduction

Much of trading in practice consists of exchanging random sequences of cash flows where payments occur at several dates. This is the case, for example, in swap contracts where a stochastic sequence is traded for a deterministic one. Other examples can be found in various insurance contracts where premiums are usually paid e.g. quarterly instead of a single payment at the beginning. Distinguishing between payments at different dates is important since, in real illiquid markets, wealth cannot be transfered quite freely in time. This is in contrast with most market models in the literature of superhedging. In real markets, there are also instantaneous illiquidity effects when transferring wealth between different assets. For example, in double auction markets, the cost of a market

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order is nonlinear in the traded amount. The nonlinearities bring up phenomena such as decreasing returns to scale that are not present in classical perfectly liquid market models nor in conical models with proportional transaction costs and conical constraints.

The present paper can be seen as an extension of Dermody and Rockafellar [11, 12] where superhedging of fixed income instruments was studied in a deterministic market model with nonlinear illiquidity effects. We extend [11, 12] by considering stochastic market models and dynamic trading strategies. Moreover, we study superhedging in terms of general *premium processes* which may give premium payments at several dates. This allows us to cover e.g. swaps and various insurance contracts where premiums are paid over time instead of a single payment at the beginning.

Superhedging of stochastic claim processes has been studied e.g. by Napp [31] and Jaschke and Küchler [20] but they considered conical market models which do not allow for nonlinear illiquidity effects. Staum [47] included nonlinearities in an abstract "market ask pricing function" but that suppresses the role of a premium process and its relationship with the market model. In [4], Bion-Nadal studied the dynamics of superhedging prices in an abstract convex market model with a cash account. Our study is closely related to the theory of convex risk measures for processes but there the emphasis is mostly on capital requirements as opposed to pricing in terms of general premium processes; see for example Pflug and Ruszczyński [37], Frittelli and Scandolo [18], Cheridito, Delbaen and Kupper [8] and Acciaio, Föllmer and Penner [1] and their references.

We examine superhedging in terms of general claim and premium processes in a market model with nonlinear illiquidity effects and portfolio constraints. We extend some classical dual characterizations to this more general setting. Although superhedging is often not quite a practical premise, it forms a basis for more realistic approaches based on risk preferences. The results of this paper contribute towards extending risk based pricing approaches for realistic illiquid markets with general premium and claim processes. Moreover, given the extensive literature on superhedging, our results are also of purely theoretical interest in showing how known phenomena in superhedging are affected when illiquidity is taken into account.

We will use a nonlinear discrete time model from Pennanen [34, 35] where trading costs are given by convex cost functions and portfolios may be subject to convex constraints. The existence of a cash account is not assumed a priori so that claim processes cannot be simply accumulated at the end using the cash account. The model generalizes many better-known models such as the classical linear model, the transaction cost model of Jouini and Kallal [21], the sublinear model of Kaval and Molchanov [26], the illiquidity model of Çetin and Rogers [6] as well as the linear models with portfolio constraints of Pham and Touzi [38], Napp [32], Evstigneev, Schürger and Taksar [14] and Rokhlin [44]. Our model covers nonlinear illiquidity effects associated with instantaneous trades (market orders) but we assume, like in the above references, that agents have no market power in the sense that trades do not affect the costs of subsequent trades. This is analogous to the models of Çetin, Jarrow and Protter [5], Çetin, Soner and Touzi [7] and Rogers and Singh [43], the last one of which gives economic motivation for the assumption. We avoid long term price impacts because they interfere with convexity which is essential in many aspects of pricing and hedging. Convexity becomes an important issue also in numerical calculations; see e.g. Edirisinghe, Naik and Uppal [13] or Koivu and Pennanen [28].

The notion of arbitrage is often given a central role when studying pricing and hedging of contingent claims in financial markets. In classical perfectly liquid market models, there are two good reasons for this. First, a violation of the no arbitrage condition leads to an unnatural situation where one can find self-financing trading strategies that generate infinite proceeds out of zero initial investment. Second, as discovered by Schachermayer [45], the no arbitrage condition implies the closedness of the set of claims that can be superhedged with zero cost. The closedness yields dual characterizations of superhedging conditions in terms of e.g. martingale measures and state price deflators.

In illiquid markets, however, things are different. A violation of the no arbitrage condition no longer means that one can generate infinite proceeds by simple scaling of arbitrage strategies. Indeed, illiquidity effects may come into play when trades get larger; see e.g. Dermody and Prisman [10], Çetin and Rogers [6] or [34, 35]. On the other hand, even in the case of linear models, the no arbitrage condition is not necessary for closedness of the set of claims hedgeable with zero cost. There may exist other economically meaningful conditions that yield the closedness and corresponding dual characterizations of superhedging conditions.

This paper gives sufficient closedness conditions that apply to claim processes under general nonlinear cost functions and portfolio constraints. The conditions are quite natural and they are satisfied e.g. in double auction markets when one is not allowed to go infinitely short in any of the traded assets. We also give a minimal condition under which pricing problems are well-defined and nontrivial in convex, possibly nonlinear market models. This is a simple algebraic condition generalizing the condition of "no arbitrage of the second kind" in Ingersoll [19] or the "weak no arbitrage" condition in [12]. It is also closely related to the "law of one price" in the case of classical perfectly liquid markets.

Under these two conditions, we obtain dual characterizations of superhedging in terms of *stochastic term structures* which generalize term structures of interest rates beyond fixed income markets as well as martingale densities beyond stochastic markets with a cash account. In the presence of nonlinear illiquidity effects, a nonlinear penalty term appears in pricing formulas much like in dual representations of convex risk measures which are not positively homogeneous. This extends Föllmer and Schied [16, Proposition 16] where analogous expressions were given in the presence of convex constraints in the classical linear model with a cash-account; see also Klöppel and Schweizer [27, Section 4].

The rest of this paper is organized as follows, The next section defines the market model. Sections 3 and 4 define the hedging and pricing problems for claim and portfolio processes and study their properties in algebraic terms. Section 5 derives dual characterizations of the superhedging conditions for integrable processes in terms of bounded stochastic term structures. This is done under the assumption that the set of claims hedgeable with zero cost is closed in probability. Section 6 derives sufficient conditions for the closedness. Section 7 makes some concluding remarks.

2 The market model

Consider a financial market where trading occurs over finite discrete time $t = 0, \dots, T$. Let $(\Omega, \mathcal{F}, P)$ be a probability space with a filtration $(\mathcal{F}_t)_{t=0}^T$ describing the information available to an investor at each $t = 0, \dots, T$. For simplicity, we assume that $\mathcal{F}_0$ is the trivial σ-algebra $\{\emptyset, \Omega\}$ and that each $\mathcal{F}_t$ is complete with respect to P. The Borel σ-algebra on $\mathbb{R}^J$ will be denoted by $\mathcal{B}(\mathbb{R}^J)$.

Definition 1 *A* convex cost process *is a sequence $S = (S_t)_{t=0}^T$ of extended real-valued functions on $\mathbb{R}^J \times \Omega$ such that for $t = 0, \dots, T$,*

1. *the function $S_t(\cdot, \omega)$ is convex, lower semicontinuous and vanishes at 0 for every $\omega \in \Omega$,*
2. *S_t is $\mathcal{B}(\mathbb{R}^J) \otimes \mathcal{F}_t$-measurable.*

A cost process S is said to be nondecreasing, nonlinear, polyhedral, positively homogeneous, linear, ... *if the functions $S_t(\cdot, \omega)$ have the corresponding property for every $\omega \in \Omega$.*

The interpretation is that buying a portfolio $x_t \in \mathbb{R}^J$ at time t and state ω costs $S_t(x_t, \omega)$ units of cash. The measurability property implies that if the portfolio x_t is $\mathcal{F}_t$-measurable then the cost $\omega \mapsto S_t(x_t(\omega), \omega)$ is also $\mathcal{F}_t$-measurable (see e.g. [42, Proposition 14.28]). This just means that the cost is known at the time of purchase. We pose no smoothness assumptions on the functions $S_t(\cdot, \omega)$. The measurability property together with lower semicontinuity in Definition 1 mean that S_t is an $\mathcal{F}_t$-measurable *normal integrand* in the sense of Rockafellar [39]; see also Rockafellar and Wets [42, Chapter 14].

Definition 1, originally given in [33], was motivated by the structure of double auction markets where the costs of market orders are polyhedral convex functions of the number of shares bought. The classical linear market model corresponds to $S_t(x, \omega) = s_t(\omega) \cdot x$, where $s = (s_t)_{t=0}^T$ is an $\mathbb{R}^J$-valued $(\mathcal{F}_t)_{t=0}^T$-adapted price process. Definition 1 covers also many other models from literature; see [35].

We allow for general convex portfolio constraints where at each $t = 0, \dots, T$ the portfolio x_t is restricted to lie in a convex set D_t which may depend on ω.

Definition 2 *A* convex portfolio constraint process *is a sequence $D = (D_t)_{t=0}^T$ of set-valued mappings from Ω to $\mathbb{R}^J$ such that for $t = 0, \dots, T$,*

1. *$D_t(\omega)$ is closed, convex and $0 \in D_t(\omega)$ for every $\omega \in \Omega$,*
2. *the set-valued mapping $\omega \mapsto D_t(\omega)$ is $\mathcal{F}_t$-measurable.*

A constraint process D is said to be polyhedral, conical, ... *if the sets $D_t(\omega)$ have the corresponding property for every $\omega \in \Omega$.*

The classical case without constraints corresponds to $D_t(\omega) = \mathbb{R}^J$ for every $\omega \in \Omega$ and $t = 0, \dots, T$. In addition to obvious "short selling" restrictions, portfolio constraints can be used to model situations where one encounters different interest rates for lending and borrowing. This can be done by introducing two separate "cash accounts" whose unit prices appreciate according to the two interest rates and restricting the investments in these assets to be nonnegative and nonpositive, respectively. A simple example that goes beyond conical and deterministic constraints is when there are nonzero bounds on market values of investments.

Remark 3 (Market values) *Large investors usually view investments in terms of their market values rather than in units of shares; see [23] and [29]. If $s = (s_t)_{t=0}^T$ is a componentwise strictly positive $\mathbb{R}^J$-valued process, we can write*

$$S_t(x, \omega) = \varphi_t(M_t(\omega)x, \omega)$$

where $\varphi_t(h) := S_t((h^j/s_t^j)_{j\in J})$ and $M_t(\omega)$ is the diagonal matrix with entries $s_t^j(\omega)$. Everything that is said below can be stated in terms of the variables $h_t^j(\omega) := s_t^j(\omega)x_t^j(\omega)$. If $s_t(\omega)$ is a vector of "market prices" of the assets J, then the vector $h_t(\omega)$ gives the "market values" of the assets held. Market prices are usually understood as the unit prices associated with infinitesimal trades. If the cost function $S_t(\cdot, \omega)$ is smooth at the origin, then $s_t(\omega) = \nabla S_t(0, \omega)$ is the natural definition. If $S_t(\omega)$ is nondifferentiable at the origin, then $s_t(\omega)$ could be any element of the subdifferential

$$\partial S_t(0, \omega) := \{s \in \mathbb{R}^J \mid S_t(x, \omega) \geq S_t(0, \omega) + s \cdot x \quad \forall x \in \mathbb{R}^J\}.$$

In double auction markets, $\partial S_t(0, \omega)$ is the product of the intervals between the bid and ask prices of the assets J; see [35].

3 Superhedging

When wealth cannot be transfered freely in time (due to e.g. different interest rates for lending and borrowing) it is important to distinguish between payments that occur at different dates. A *(contingent) claim process* is a real-valued stochastic process $c = (c_t)_{t=0}^T$ that is adapted to $(\mathcal{F}_t)_{t=0}^T$. The value of c_t is interpreted as the amount of cash the owner of the claim receives at time t. Such claim processes are quite common in practice. For example, most insurance contracts, fixed income products as well as dividend paying stocks have several payout dates. In the presence of a cash account, discrimination between payments at different dates would be unnecessary (see Example 4 below) but in real markets it is essential. The set of claim processes will be denoted by $\mathcal{M}$.

In problems of superhedging, one usually looks for the initial endowments (premiums) that allow, without subsequent investments, for delivering a claim with given maturity. Since in illiquid markets, cash at different dates are genuinely different things, it makes sense to study superhedging in terms of "premium processes". A *premium process* is a real-valued adapted stochastic process $p = (p_t)_{t=0}^T$ of cash flows that the seller receives in exchange for delivering a claim $c = (c_t)_{t=0}^T$. Allowing both premiums as well as claims to be sequences of cash flows is not only mathematically convenient (claims and premiums live in the same space) but also practical since much of trading consists of exchanging sequences of cash flows. This is the case e.g. in swap contracts where a stochastic sequence of payments is exchanged for a deterministic one. Also, in various insurance contracts premiums are paid annually instead of a single payment at the beginning.

We say that $p \in \mathcal{M}$ is a *superhedging premium* for $c \in \mathcal{M}$ if there exists an adapted $\mathbb{R}^J$-valued *portfolio process* $x = (x_t)_{t=0}^T$ with $x_T = 0$ such that[1]

$$x_t \in D_t, \quad S_t(x_t - x_{t-1}) + c_t \leq p_t$$

almost surely for every $t = 0, \dots, T$. Here and in what follows, we always set $x_{-1} = 0$. The vector x_t is interpreted as a portfolio that is held over the period $[t, t+1]$. At the terminal date, we require that everything is liquidated so the budget constraint becomes $S_T(-x_{T-1}) + c_T \leq 0$. The above is a numeraire-free way of writing the superhedging property; see Example 4. In the case of a stock exchange, the interpretation is that the portfolio is updated by market orders in a way that allows for delivering the claim without any investments over time. In particular, when c_t is strictly positive, the cost $S_t(x_t - x_{t-1})$ of updating the portfolio from x_{t-1} to x_t has to be strictly negative (market order of portfolio $x_t - x_{t-1}$ involves more selling than buying).

We are thus looking at situations where one sequence of payments is exchanged for another and the problem is to characterize those exchanges where residual risks can be completely hedged by an appropriate trading strategy. Much research has been devoted to the case where premium is paid only at the beginning and claims only at the end. This corresponds to the case $p = (p_0, 0, \dots, 0)$ and $c = (0, \dots, 0, c_T)$. To our knowledge, superhedging of claim processes in terms of general premium processes has not been studied before in the presence of nonlinear illiquidity effects.

Superhedging can be conveniently studied in terms of the set

$$\mathcal{C} := \{c \in \mathcal{M} \mid \exists x \in \mathcal{N}_0 : \ x_t \in D_t, \ S_t(\Delta x_t) + c_t \leq 0, \ t = 0, \dots, T\}$$

of claim processes that are freely available in the market, i.e. can be superhedged with zero cost. Here $\mathcal{N}_0$ denotes the set of all adapted portfolio processes with $x_T = 0$. A $p \in \mathcal{M}$ is a superhedging premium process for a claim process $c \in \mathcal{M}$ if and only if

$$c - p \in \mathcal{C}. \tag{1}$$

[1]Given an $\mathcal{F}_t$-measurable function $z_t : \Omega \to \mathbb{R}^J$, $S_t(z_t)$ denotes the extended real-valued random variable $\omega \mapsto S_t(z_t(\omega), \omega)$. By [42, Proposition 14.28], $S_t(z_t)$ is $\mathcal{F}_t$-measurable whenever z_t is $\mathcal{F}_t$-measurable.

Example 4 (Numeraire and stochastic integrals) *Assume that there is a perfectly liquid asset, say $0 \in J$, such that*

$$S_t(x,\omega) = x^0 + \tilde{S}_t(\tilde{x},\omega),$$
$$D_t(\omega) = \mathbb{R} \times \tilde{D}_t(\omega),$$

where $x = (x^0, \tilde{x})$ and $\tilde{S}$ and $\tilde{D}$ are the cost process and the constraints for the remaining risky assets $\tilde{J} = J \setminus \{0\}$. Given $\tilde{x} = (\tilde{x}_t)_{t=0}^T$, we can define

$$x_t^0 = x_{t-1}^0 - \tilde{S}_t(\tilde{x}_t - \tilde{x}_{t-1}) - c_t \quad t = 0, \dots, T-1,$$

so that the budget constraint holds as an equality for $t = 1, \dots, T-1$ and

$$x_{T-1}^0 = -\sum_{t=0}^{T-1} \tilde{S}_t(\tilde{x}_t - \tilde{x}_{t-1}) - \sum_{t=0}^{T-1} c_t.$$

We then get the expression

$$\mathcal{C} = \{c \in \mathcal{M} \,|\, \exists \tilde{x} : \ \tilde{x}_t \in \tilde{D}_t, \ \sum_{t=0}^{T} \tilde{S}_t(\tilde{x}_t - \tilde{x}_{t-1}) + \sum_{t=0}^{T} c_t \leq 0\}.$$

Thus, when a numeraire exists, hedging of a claim process can be reduced to hedging of cumulated claims at the terminal date. If in addition, the cost process $\tilde{S}$ is linear with $\tilde{S}_t(\tilde{x}) = \tilde{s}_t \cdot \tilde{x}$, we can write the cumulated trading costs in terms of a stochastic integral as

$$\sum_{t=0}^{T} \tilde{S}_t(\tilde{x}_t - \tilde{x}_{t-1}) = \sum_{t=0}^{T} \tilde{s}_t \cdot (\tilde{x}_t - \tilde{x}_{t-1}) = -\sum_{t=0}^{T-1} \tilde{x}_t \cdot (\tilde{s}_{t+1} - \tilde{s}_t),$$

so that

$$\mathcal{C} = \{c \in \mathcal{M} \,|\, \exists \tilde{x} : \ \tilde{x}_t \in \tilde{D}_t, \ \sum_{t=0}^{T} c_t \leq \sum_{t=0}^{T-1} \tilde{x}_t \cdot (\tilde{s}_{t+1} - \tilde{s}_t)\}.$$

This is essentially the market model studied e.g. in [16], [17, Chapter 9] and [27, Section 4], where constraints on the risky assets were considered.

The set

$$\operatorname{rc} \mathcal{C} := \{c \,|\, c' + \alpha c \in \mathcal{C} \quad \forall c' \in \mathcal{C}, \ \alpha > 0\}$$

consists of claim processes that are freely available in the market at unlimited amounts when starting at any position $c' \in \mathcal{C}$. Our subsequent analysis will be largely based on the following simple observation. Here $\mathcal{M}_-$ denotes the set of nonpositive claim processes.

Lemma 5 *The set $\mathcal{C}$ is convex and $\mathcal{M}_- \subset \operatorname{rc} \mathcal{C}$. If S is sublinear and D is conical, then $\mathcal{C}$ is a cone and $\operatorname{rc} \mathcal{C} = \mathcal{C}$.*

Proof. The fact that $\mathcal{M}_- \subset \operatorname{rc}\mathcal{C}$ is obvious from the definition of $\mathcal{C}$. The rest comes from [35, Lemma 4.1]. □

In the terminology of convex analysis, $\operatorname{rc}\mathcal{C}$ is the *recession cone* of $\mathcal{C}$; see [40, Section 8]. When $\mathcal{C}$ is *algebraically closed* (i.e. $\{\alpha \in \mathbb{R} \,|\, c + \alpha c' \in \mathcal{C}\}$ is a closed interval for every $c, c' \in \mathcal{M}$), we have the simpler expression

$$\operatorname{rc}\mathcal{C} = \bigcap_{\alpha>0} \alpha\mathcal{C}$$

and thus that $\operatorname{rc}\mathcal{C}$ is the largest convex cone contained in $\mathcal{C}$. This follows from the fact that $0 \in \mathcal{C}$ and the following result which is well-known in convex analysis.

Lemma 6 *Let C be a convex subset of a vector space. The recession cone of C is a convex cone. If C is algebraically closed, then $y \in \operatorname{rc} C$ if there exists even one $x \in C$ such that $x + \alpha y \in C$ for every $\alpha > 0$.*

Proof. It is clear that $\operatorname{rc} C$ is a cone. As for convexity, let $y_1, y_2 \in \operatorname{rc} C$ and $\lambda \in [0,1]$. It suffices to show that $x + \alpha(\lambda y_1 + (1-\lambda)y_2) \in C$ for every $x \in C$ and $\alpha > 0$. Since $y_i \in \operatorname{rc} C$, we have $x + \alpha y_i \in C$ and then, by convexity of C, $x + \alpha(\lambda y_1 + (1-\lambda)y_2) = \lambda(x + \alpha y_1) + (1-\lambda)(x + \alpha y_2) \in C$.

Let $x \in C$ and $y \neq 0$ be such that $x + \alpha y \in C$ $\forall \alpha > 0$ and let $x' \in C$ and $\alpha' > 0$ be arbitrary. It suffices to show that $x' + \alpha' y \in C$. Since $x + \alpha y \in C$ for every $\alpha \geq \alpha'$, we have, by convexity of C,

$$x' + \alpha' y + \frac{\alpha'}{\alpha}(x - x') = (1 - \frac{\alpha'}{\alpha})x' + \frac{\alpha'}{\alpha}(x + \alpha y) \in C \quad \forall \alpha \geq \alpha'.$$

Since C is algebraically closed, we must have $x' + \alpha' y \in C$. □

4 Pricing by superhedging

In many situations, a premium process $p \in \mathcal{M}$ is given and the question is what multiple of p will be sufficient to hedge a claim $c \in \mathcal{M}$. This is the case e.g. in some defined benefit pension plans where the premium process is a fraction (the contribution rate) of the salary of the insured. In swap contracts, the premium process is often defined as a multiple of a constant sequence. Given a premium process $p \in \mathcal{M}$, we define the *superhedging cost* of a $c \in \mathcal{M}$ by

$$\pi(c) := \inf\{\alpha \,|\, c - \alpha p \in \mathcal{C}\}. \tag{2}$$

In the case $p = (1, 0, \ldots, 0)$, $\pi(c)$ gives the least initial investment sufficient to superhedge $c \in \mathcal{M}$ without subsequent investments. In a pension contract, where processes c and p are the monthly pension and salary, respectively, $\pi(c)$ gives the least contribution rate sufficient for superhedging the pensions payments.

The *effective domain*

$$\operatorname{dom}\pi := \{c \in \mathcal{M} \mid \pi(c) < +\infty\} = \bigcup_{\alpha\in\mathbb{R}} (\mathcal{C} + \alpha p)$$

of π consists of the claim processes that can be superhedged with some multiple of p in a market described by a cost process S and constraints D. In general, $\operatorname{dom}\pi \neq \mathcal{M}$ but in many applications it is natural to assume that $\operatorname{dom}\pi$ contains all bounded claim processes. This holds in particular when $p = (1, 0, \ldots, 0)$ (single premium payment at the beginning) and when arbitrary long positions in cash are allowed.

Proposition 7 *The following properties are always valid.*

1. *π is convex,*
2. *π is monotone: $\pi(c) \le \pi(c')$ if $c \le c'$,*
3. *$\pi(c + \lambda p) = \pi(c) + \lambda$ for all $\lambda \in \mathbb{R}$ and $c \in \mathcal{M}$,*
4. *$\pi(0) \le 0$.*

If $\mathcal{C}$ is a cone, then

5. *π is positively homogeneous.*

Proof. Let $\lambda_i > 0$ be such that $\lambda_1 + \lambda_2 = 1$ and let $c_i \in \operatorname{dom}\pi$ and $\varepsilon > 0$ be arbitrary. If $\pi(c_i) > -\infty$ let $\alpha_i \le \pi(c_i)+\varepsilon$ be such that $c_i - \alpha_i p \in \mathcal{C}$. Otherwise, let $\alpha_i \le -1/\varepsilon$ be such that $c_i - \alpha_i p \in \mathcal{C}$. Since $\mathcal{C}$ is convex,

$$\lambda_1 c_1 + \lambda_2 c_2 - (\lambda_1\alpha_1 + \lambda_2\alpha_2)p = \lambda_1(c_1 - \alpha_1 p) + \lambda_2(c_2 - \alpha_2 p) \in \mathcal{C}$$

and thus,

$$\pi(\lambda_1 c_1 + \lambda_2 c_2) \le \lambda_1\alpha_1 + \lambda_2\alpha_2.$$

Since $\varepsilon > 0$ was arbitrary, the convexity follows. The monotonicity property follows from $\mathcal{M}_- \subset \operatorname{rc}\mathcal{C}$. The translation property is immediate from the definition of π and $\pi(0) \le 0$ holds because $0 \in \mathcal{C}$. As to the positive homogeneity, let $c \in \operatorname{dom}\pi$, $\varepsilon > 0$ and let $\alpha \le \pi(c) + \varepsilon$ be such that $c - \alpha p \in \mathcal{C}$. If $\mathcal{C}$ is a cone and $\lambda > 0$, then $\lambda c - \lambda\alpha p \in \mathcal{C}$ so that $\pi(\alpha c) \le \lambda\alpha \le \lambda(\pi(c) + \varepsilon)$ and thus, $\pi(\lambda c) \le \lambda\pi(c)$. On the other hand, since $\lambda > 0$ was arbitrary, $\pi(c) = \pi(\lambda c/\lambda) \le \pi(\lambda c)/\lambda$, so that $\pi(\lambda c) = \lambda\pi(c)$ for every $c \in \operatorname{dom}\pi$ and $\lambda > 0$. This also shows that $\operatorname{dom}\pi$ is a cone, so that $\pi(\lambda c) = \lambda\pi(c)$ holds for all $c \in \mathcal{M}$. □

We see that π has properties close to those of a convex risk measure; see e.g. [17, Chapter 4]. Consequently, we can use similar techniques in its analysis; see Section 5.

The nonpositive number $\pi(0)$ is the smallest multiple of the premium p one needs in order to find a riskless strategy in the market. If one has to deliver a claim $c \in \mathcal{M}$, one needs $\pi(c) - \pi(0)$ units more. This is analogous to [12, Definition 4.1] in the case of deterministic fixed income markets with a single premium payment at the beginning. More generally, we define the *superhedging selling price* of a $c \in \mathcal{M}$ for an agent with *initial liabilities* $\bar{c} \in \mathcal{M}$ as

$$P(\bar{c}; c) = \pi(\bar{c} + c) - \pi(\bar{c}).$$

Analogously, the *superhedging buying price* of a $c \in \mathcal{M}$ for an agent with *initial liabilities* $\bar{c} \in \mathcal{M}$ is given by $\pi(\bar{c}) - \pi(\bar{c} - c) = -P(\bar{c}; -c)$. It follows from convexity of π that

$$-P(\bar{c}; -c) \leq P(\bar{c}; c)$$

which means that agents with similar liabilities and similar market expectations should not trade with each other if they aim at superhedging their positions.

It is intuitively clear that the value an agent assigns to a claim should depend not only on the market expectations but also on the liabilities the agent might have already before the trade. For example, the selling price $P(\bar{c}^{IC}; c)$ of a home insurance contract $c \in \mathcal{M}$ for an insurance company may be lower than the buying price $-P(\bar{c}^{HO}; -c)$ for a home owner, even if the two had identical market expectations. Here $\bar{c}^{IC}$ would be the claims associated with the existing insurance portfolio of the company while $\bar{c}^{HO}$ would be the possible losses to the home owner associated with damages to the home. Another example would be the exchange of futures contracts between a wheat farmer and a wheat miller. In fact, many derivative contracts exist precisely because of the differences between initial liabilities of different parties.

A minimal condition for a pricing problem to be sensibly posed is that $p \notin \operatorname{rc} \mathcal{C}$. In other words, when looking for compensation for delivering a claim it does not make sense to ask for something that is freely available in the market at unlimited quantities.

Proposition 8 *If $\mathcal{C}$ is algebraically closed, then the conditions*

(a) $\pi(c) > -\infty$ *for some* $c \in \operatorname{dom} \pi$,

(b) $\pi(0) > -\infty$,

(c) $\pi(c) > -\infty$ *for every* $c \in \operatorname{dom} \pi$,

(d) $p \notin \operatorname{rc} \mathcal{C}$

are equivalent and imply that $\pi(c) = \inf\{\alpha \,|\, c - \alpha p \in \mathcal{C}\}$ *is attained for every* $c \in \operatorname{dom} \pi$. *If $\mathcal{C}$ is conical, (b) is equivalent to*

(e) $\pi(0) = 0$.

Proof. By definition of the recession cone, $p \in \operatorname{rc} \mathcal{C}$ means that $\pi(c) = -\infty$ for every $c \in \operatorname{dom} \pi$ so (a) and (d) are equivalent. The implication $(c) \Rightarrow (b)$ is obvious and $(b) \Rightarrow (a)$ holds by Proposition 7(4). If $\mathcal{C}$ is algebraically closed, then by Lemma 6, (c) is equivalent to (d).

The attainment of the infimum follows directly from the definition of algebraic closedness. When $\mathcal{C}$ is a cone, (b) means that $\pi(0) \geq 0$. By Proposition 7(4), this is equivalent to (d). $\square$

Thus, if $\mathcal{C}$ is algebraically closed and $\pi(0) > -\infty$, then the price $P(\bar{c}; c)$ is well defined and finite for every $\bar{c} \in \operatorname{dom} \pi$ and $c + \bar{c} \in \operatorname{dom} \pi$. In particular, if $p = (1, 0, \dots, 0)$ and arbitrary long positions in cash are allowed, then $P(\bar{c}; c)$ is well defined and finite for every bounded $\bar{c}$ and c.

In the case of perfectly liquid markets and the premium $p = (1, 0, \dots, 0)$, the condition $\pi(0) \geq 0$ means that there is no "arbitrage of the second kind" in the sense of Ingersoll [19]. In the fixed income market model of [11, 12], the condition $\pi(0) \geq 0$ was called the "weak no arbitrage" condition. When $p = (1, 0, \dots, 0)$, the condition $\pi(0) \geq 0$ is also related to the "law of one price" well known in classical perfectly liquid market models. While $\pi(0) \geq 0$ means that it is not possible to superhedge the zero claim when starting from strictly negative initial wealth, the law of one price means that it is not possible to *replicate* the zero claim when starting with strictly negative wealth; see e.g. [9].

Proposition 8 shows that the natural generalization of the condition $\pi(0) \geq 0$ to nonconical market models is the weaker requirement that $\pi(0)$ be finite. When $\mathcal{C}$ is algebraically closed, the finiteness of $\pi(0)$ is necessary and sufficient for the superhedging cost π to be a proper convex function on $\mathcal{M}$. In general, the stronger condition $\pi(0) \geq 0$ means that $-\alpha p \notin \mathcal{C}$ for all $\alpha < 0$ or equivalently that $p \notin \operatorname{pos} \mathcal{C}$, where

$$\operatorname{pos} \mathcal{C} := \bigcup_{\alpha > 0} \alpha \mathcal{C}$$

is a convex cone known as the *positive hull* of $\mathcal{C}$. Clearly, $\operatorname{rc} \mathcal{C} \subseteq \operatorname{pos} \mathcal{C}$ where equality holds iff $\mathcal{C}$ is conical. The sets $\operatorname{pos} \mathcal{C}$ and $\operatorname{rc} \mathcal{C}$ are closely related to the "marginal" and "scalable" arbitrage opportunities studied in [35].

5 Duality

There exist several pricing formulas where the value of a security is expressed as a weighted sum of its cash flows. In particular, in fixed income markets where the cash flows are deterministic, the value of an asset can be expressed in terms of future cash flows weighted according to the current term structure representing time values of cash. When valuing assets with random payouts one can often write the value as an expectation where the cash flows are weighted with a martingale density. Such martingale representations rely on the existence of a cash account (or a numeraire) which, on the other hand, means that the time value of cash is constant. When moving to illiquid markets under stochastic uncertainty one needs richer dual objects that encompass both the time value of money as well as the random nature of cash flows.

In classical perfectly liquid market models or in models with proportional transaction costs, superhedging conditions can be described in terms of the same dual variables that characterize the no arbitrage condition; see e.g. [17]

or [24]. In the presence of nonlinear illiquidity effects, this is no longer true. Instead, we obtain dual characterizations of superhedging conditions in terms of the "support function" of the set of integrable claims in $\mathcal{C}$. We then give an expression for the support function in terms of S and D, which allows for a more concrete characterizations of superhedging. In the classical case, familiar dual expressions are obtained as a special case. Our results hold under the assumption that $\mathcal{C}$ is closed in probability, a condition which is known to be satisfied under the no arbitrage condition in the case of classical perfectly liquid models; see [45]. In Section 6, we will give alternative closedness conditions that apply to general S and D.

Let $\mathcal{M}^1$ and $\mathcal{M}^\infty$ be the spaces of integrable and essentially bounded, respectively, real-valued adapted processes. Let

$$\mathcal{C}^1 := \mathcal{C} \cap \mathcal{M}^1,$$

be the set of integrable claim processes that can be superhedged with zero cost. While the elements of $\mathcal{M}^1$ represent claim processes, the elements of $\mathcal{M}^\infty$ represent *stochastic term structures* that will be used in dual representations of superhedging conditions and superhedging costs defined in Sections 3 and 4.

The bilinear form

$$(c, y) \mapsto E\sum_{t=0}^{T} c_t y_t$$

puts $\mathcal{M}^1$ and $\mathcal{M}^\infty$ in separating duality; see [41, page 13]. One can then use classical convex duality arguments to describe hedging conditions. This will involve the *support function* $\sigma_{\mathcal{C}^1} : \mathcal{M}^\infty \to \overline{\mathbb{R}}$ of $\mathcal{C}^1$ defined by

$$\sigma_{\mathcal{C}^1}(y) = \sup_{c\in\mathcal{C}^1} E\sum_{t=0}^{T} c_t y_t.$$

In the terminology of microeconomic theory, $\sigma_{\mathcal{C}^1}$ is called the *profit function* associated with the "production set" $\mathcal{C}^1$; see e.g. Aubin [3] or Mas-Collel, Whinston and Green [30]. In the present context, $\mathcal{C}^1$ consists of the integrable claim processes one can produce in the market without costs, while $\sigma_{\mathcal{C}^1}(y)$ gives the largest profit one could generate by selling an element of $\mathcal{C}^1$ at prices given by y. The following lists some basic properties of $\sigma_{\mathcal{C}^1}$.

Proposition 9 *The function $\sigma_{\mathcal{C}^1} : \mathcal{M}^\infty \to \overline{\mathbb{R}}$ is nonnegative and sublinear. Its effective domain*

$$\operatorname{dom} \sigma_{\mathcal{C}^1} = \{y \in \mathcal{M}^\infty \mid \sigma_{\mathcal{C}^1}(y) < \infty\}$$

is a convex cone in the set $\mathcal{M}^\infty_+$ of nonnegative bounded processes. If arbitrary long positions in cash are allowed, then $\operatorname{dom} \sigma_{\mathcal{C}^1}$ *is contained in the set of nonnegative supermartingales.*

Proof. The sublinearity is immediate and the nonnegativity follows from $0 \in \mathcal{C}^1$. That $\operatorname{dom} \sigma_{\mathcal{C}^1}$ is a convex cone follows from sublinearity. Since $\mathcal{C}^1$ contains all nonpositive integrable claim processes, $\operatorname{dom}\sigma_{\mathcal{C}^1}$ is contained in $\mathcal{M}^\infty_+$. If arbitrary long positions in cash are allowed, then

$$\mathcal{C}^1 \supset \{-(\Delta x^0_t)_{t=0}^T \,|\, x^0 \in \mathcal{M}^\infty_+,\ x^0_T = 0\},$$

so that

$$\begin{aligned}
\sigma_{\mathcal{C}^1}(y) &\geq \sup\{-E\sum_{t=0}^T \Delta x^0_t y_t \,|\, x^0 \in \mathcal{M}^\infty_+,\ x^0_T = 0\} \\
&= \sup\{E\sum_{t=0}^{T-1} x^0_t \Delta y_{t+1} \,|\, x^0 \in \mathcal{M}^\infty_+\} \\
&= \sup\{E\sum_{t=0}^{T-1} x^0_t E[\Delta y_{t+1} \,|\, \mathcal{F}_t] \,|\, x^0 \in \mathcal{M}^\infty_+\} \\
&= \begin{cases} 0 & \text{if } E[\Delta y_{t+1} \,|\, \mathcal{F}_t] \leq 0 \text{ for } t = 0, \dots, T-1, \\ +\infty & \text{otherwise.} \end{cases}
\end{aligned}$$

Thus, $\sigma_{\mathcal{C}^1}(y) = +\infty$ unless y is a supermartingale. □

If $\mathcal{C}$ is closed in probability then $\mathcal{C}^1$ is closed in the norm topology of $\mathcal{M}^1$ and the classical bipolar theorem (see e.g. [40, Theorem 14.5] or [3, Section 1.4.2]) says that $c \in \mathcal{C}^1$ if and only if

$$E\sum_{t=0}^T c_t y_t \leq 1$$

for every $y \in \mathcal{M}^\infty$ such that $\sigma_{\mathcal{C}^1}(y) \leq 1$. This immediately yields a dual characterization of the superhedging condition (1) for integrable claims and premiums. The following gives a dual representation for the superhedging cost (2).

Theorem 10 *Assume that $\mathcal{C}$ is closed in probability and let $p \in \mathcal{M}^1$. We have $\pi(0) > -\infty$ if and only if there is a $y \in \operatorname{dom}\sigma_{\mathcal{C}^1}$ such that $E\sum_{t=0}^T p_t y_t = 1$. In that case, π is a proper lower semicontinuous (both in norm and the weak topology) convex function on $\mathcal{M}^1$ with the representation*

$$\pi(c) = \sup_{y \in \mathcal{M}^\infty} \left\{ E\sum_{t=0}^T c_t y_t - \sigma_{\mathcal{C}^1}(y) \,\middle|\, E\sum_{t=0}^T p_t y_t = 1 \right\}.$$

Proof. When $p \in \mathcal{M}^1$, the restriction $\bar{\pi}$ of π to $\mathcal{M}^1$ can be written as $\bar{\pi}(c) = \inf\{\alpha \,|\, c - \alpha p \in \mathcal{C}^1\}$. The convex conjugate $\bar{\pi}^* : \mathcal{M}^\infty \to \overline{\mathbb{R}}$ of $\bar{\pi}$ can be expressed as

$$
\begin{aligned}
\bar\pi^*(y) &= \sup_{c\in\mathcal{M}^1}\{E\sum_{t=0}^T c_t y_t - \pi(c)\}\\
&= \sup_{c\in\mathcal{M}^1,\alpha\in\mathbb{R}}\{E\sum_{t=0}^T c_t y_t - \alpha \,|\, c-\alpha p\in\mathcal{C}^1\}\\
&= \sup_{c'\in\mathcal{M}^1,\alpha\in\mathbb{R}}\{E\sum_{t=0}^T (c'_t+\alpha p_t) y_t - \alpha \,|\, c'\in\mathcal{C}^1\}\\
&= \sup_{c'\in\mathcal{M}^1,\alpha\in\mathbb{R}}\{E\sum_{t=0}^T c'_t y_t + \left(E\sum_{t=0}^T p_t y_t - 1\right)\alpha \,|\, c'\in\mathcal{C}^1\}\\
&= \begin{cases}\sigma_{\mathcal{C}^1}(y) & \text{if } E\sum_{t=0}^T p_t y_t = 1,\\ +\infty & \text{otherwise.}\end{cases}
\end{aligned}
$$

The representation for π on $\mathcal{M}^1$ thus means that $\bar\pi$ equals the conjugate of $\bar\pi^*$. By [41, Theorem 5], this holds exactly when $\bar\pi$ is proper and lower semicontinuous. Since, by assumption, $\mathcal{C}$ is closed in probability it is also algebraically closed and then, by Proposition 8, $\bar\pi$ is proper iff $\pi(0) > -\infty$. It thus suffices to show that $\bar\pi$ is lower semicontinuous in norm, or equivalently, that the set

$$\operatorname{lev}_\gamma \bar\pi = \{c\in\mathcal{M}^1 \,|\, \bar\pi(c)\le\gamma\}$$

is norm closed for every $\gamma\in\mathbb{R}$. Lower semicontinuity in the weak topology then follows by the classical separation argument.

Let $(c^\nu)_{\nu=1}^\infty$ be a sequence in $\operatorname{lev}_\gamma\bar\pi$ that converges in norm to a $\bar c\in\mathcal{M}^1$. By Proposition 8, there are $\alpha^\nu\in\mathbb{R}$ such that $\alpha^\nu\le\gamma$ and $c^\nu-\alpha^\nu p\in\mathcal{C}^1$. If $(\alpha^\nu)_{\nu=1}^\infty$ has an accumulation point $\bar\alpha$, we get $\bar\alpha\le\gamma$ and, by closedness of $\mathcal{C}^1$, that $\bar c-\bar\alpha p\in\mathcal{C}^1$. This means that $\bar\pi(\bar c)\le\gamma$. It thus suffices to show that under the condition $\pi(0)>-\infty$, the sequence (α^ν) has to be bounded from below.

By Proposition 8, the condition $\pi(0)>-\infty$ is equivalent to $p\notin\operatorname{rc}\mathcal{C}$. Since $\mathcal{C}$ is convex, algebraically closed and $0\in\mathcal{C}$ this means that there is a $\lambda>0$ such that $p\notin\lambda\mathcal{C}$. Since $\mathcal{C}^1$ is closed, there is a neighborhood U of p such that

$$U\cap\lambda\mathcal{C}^1=\emptyset. \tag{3}$$

Assume that (α^ν) is not bounded from below. Then, since (c^ν) converges, there is a ν such that $p-c^\nu/\alpha^\nu\in U$ and $\frac{1}{-\alpha^\nu}\le\lambda$. Since $c^\nu-\alpha^\nu p\in\mathcal{C}^1$, we also have

$$p-\frac{c^\nu}{\alpha^\nu}\in\frac{1}{-\alpha^\nu}\mathcal{C}^1\subset\lambda\mathcal{C}^1,$$

where the inclusion holds since $\mathcal{C}^1$ is convex and $0\in\mathcal{C}^1$. This contradicts (3) so (α^ν) has to bounded from below. □

On one hand, the processes y in Theorem 10 generalize martingale densities beyond classical perfectly liquid markets with a cash account; see Corollary 15 below. On the other hand, they generalize term structures of interest rates beyond fixed income markets. In particular, the dual representation for π can be seen as an extension of [11, Theorem 3.2] to stochastic models and general premium processes. When arbitrary long positions in cash are allowed, then by Proposition 9, the processes y in the dual representation of π can be taken nonnegative supermartingales. Much as in [1], one can then use the Itô-Watanabe decomposition (see [15]) to write each term structure in the representation as a product $y = MA$ of a martingale M and a nonincreasing predictable process A with values in $[0, 1]$. Whereas M may be interpreted as the density process of a "pricing measure", A represents a discounting factor that accounts for the absence of a cash account.

The representation of π is analogous to the dual representation of the superhedging cost of [16, Proposition 16] in the case of classical linear models with a cash account and convex constraints on the risky assets; see also [17, Corollary 9.30]. While [16, Proposition 16] applies to claims and premiums with single payout dates, the abstract result in [20, Theorem 2] allows for general claim and premium processes like Theorem 10 but there the model is conical. Nonconical models have been studied in [47, 4] but there the premium was hidden in an abstract "ask pricing function". Theorem 10 gives a precise characterization of the premium processes for which the dual representation is valid.

The profit function $\sigma_{\mathcal{C}^1}$ plays a similar role in superhedging of claim processes as the "penalty function" does in the theory of convex risk measures; see e.g. [17, Chapter 4]. In the conical case (see Lemma 5), Theorem 10 simplifies much like the dual representation of a coherent risk measure.

Corollary 11 *Assume that $\mathcal{C}$ is conical and closed in probability. Let $p \in \mathcal{M}^1$ and*

$$\mathcal{D}^\infty = \{y \in \mathcal{M}^\infty \,|\, E\sum_{t=0}^T c_t y_t \le 0 \ \forall c \in \mathcal{C}^1\}.$$

We have $\pi(0) \ge 0$ if and only if there is a $y \in \mathcal{D}^\infty$ such that $E\sum_{t=0}^T p_t y_t = 1$. In that case, π is a proper lower semicontinuous sublinear function on $\mathcal{M}^1$ with the representation

$$\pi(c) = \sup_{y\in\mathcal{D}^\infty}\left\{ E\sum_{t=0}^T c_t y_t \,\middle|\, E\sum_{t=0}^T p_t y_t = 1\right\}.$$

Proof. By Proposition 8, the condition $\pi(0) > -\infty$ is equivalent to $\pi(0) \ge 0$ in the conical case. When $\mathcal{C}$ is a cone the set $\mathcal{C}^1$ is also a cone so that

$$\sigma_{\mathcal{C}^1}(y) = \begin{cases} 0 & \text{if } y \in \mathcal{D}^\infty,\\ +\infty & \text{otherwise.}\end{cases}$$

The claim thus follows from Theorem 10. □

For the traditional superhedging problem with a single premium payment at the beginning and single claim payment at the end, Corollary 11 can be written as follows.

Corollary 12 *Assume that $\mathcal{C}$ is conical and closed in probability, that $p = (1, 0, \ldots, 0)$ and $c = (0, \ldots, 0, c_T)$ for a $c_T \in L^1(\Omega, \mathcal{F}_T, P)$. Then $\pi(0) \geq 0$ if and only if there is a $y \in \mathcal{D}^\infty$ such that $y_0 = 1$. In that case, π is a proper lower semicontinuous (both in norm and the weak topology) sublinear function on $\mathcal{M}^1$ with the representation*

$$\pi(c) = \sup_{y \in \mathcal{D}^\infty} \{ E c_T y_T \mid y_0 = 1 \}.$$

When S is integrable (see below), we can express $\sigma_{\mathcal{C}^1}$ and thus Theorem 10 and its corollaries more concretely in terms of S and D. This will involve the space $\mathcal{N}^1$ of $\mathbb{R}^J$-valued adapted integrable processes $v = (v_t)_{t=0}^T$ and the integral functionals

$$v_t \mapsto E(y_t S_t)^*(v_t) \quad \text{and} \quad v_t \mapsto E\sigma_{D_t}(v_t)$$

associated with the normal integrands

$$(y_t S_t)^*(v, \omega) := \sup_{x \in \mathbb{R}^J} \{ x \cdot v - y_t(\omega) S_t(x, \omega) \}$$

and

$$\sigma_{D_t}(v, \omega) := \sup_{x \in \mathbb{R}^J} \{ x \cdot v \,|\, x \in D_t(\omega) \}.$$

The first one gives the maximum value of a position in the underlying assets and cash when the assets are priced by v and cash by $y(\omega)$. The function $v \mapsto \sigma_{D_t}(v, \omega)$ gives the maximum value of a position in the underlying asset over the feasible set. Since $S_t(0, \omega) = 0$ and $0 \in D_t(\omega)$ for every t and ω, the functions $(y_t S_t)^*$ and σ_{D_t} are nonnegative. That $(y_t S_t)^*$ and σ_{D_t} do define normal integrands follows from [42, Theorem 14.50].

We say that a cost process $S = (S_t)_{t=0}^T$ is *integrable* if the functions $S_t(x, \cdot)$ are integrable for every $t = 0, \ldots, T$ and $x \in \mathbb{R}^J$. In the classical linear case $S_t(x, \omega) = s_t(\omega) \cdot x$, integrability means that the marginal price s is integrable in the usual sense. The following is from [35].

Lemma 13 *If S is integrable, then for $y \in \mathcal{M}^\infty_+$,*

$$\sigma_{\mathcal{C}^1}(y) = \inf_{v \in \mathcal{N}^1} \left\{ \sum_{t=0}^{T} E(y_t S_t)^*(v_t) + \sum_{t=0}^{T-1} E\sigma_{D_t}(E[\Delta v_{t+1} | \mathcal{F}_t]) \right\},$$

while $\sigma_{\mathcal{C}^1}(y) = +\infty$ for $y \notin \mathcal{M}^\infty_+$. The infimum is attained for every $y \in \mathcal{M}^\infty_+$.

In the conical case, Lemma 13 yields the following expression for the polar cone of $\mathcal{C}^1$ in Corollary 11.

Corollary 14 *If S is sublinear and integrable and if D is conical, then the polar of $\mathcal{C}^1$ can be expressed as*

$$\mathcal{D}^\infty = \{ y \in \mathcal{M}^\infty_+ \,|\, \exists s \in \mathcal{N} : \; ys \in \mathcal{M}^1, \; s_t \in Z_t, \; E[\Delta(y_t s_t) \,|\, \mathcal{F}_{t-1}] \in D_t^* \},$$

where $Z_t(\omega) = \{ s \in \mathbb{R}^J \,|\, s \cdot x \leq S_t(x, \omega) \; \forall x \in \mathbb{R}^J \}$ and $D_t^*(\omega)$ is the polar cone of $D_t(\omega)$.

Proof. If S is sublinear and D is conical, we have, by Theorems 13.1 and 13.2 of [40], that

$$(y_t S_t)^*(v,\omega) = \begin{cases} 0 & \text{if } v \in y_t(\omega)Z_t(\omega), \\ +\infty & \text{otherwise} \end{cases}$$

and

$$\sigma_{D_t(\omega)}(v,\omega) = \begin{cases} 0 & \text{if } v \in D_t(\omega)^*, \\ +\infty & \text{otherwise.} \end{cases}$$

By Lemma 13, the polar cone $\mathcal{D}^\infty = \{y \in \mathcal{M}^\infty \mid \sigma_{\mathcal{C}^1}(y) \le 0\}$ can thus be written

$$\mathcal{D}^\infty = \{y \in \mathcal{M}^\infty_+ \mid \exists v \in \mathcal{N}^1 : \ v_t \in y_t Z_t, \ E[\Delta v_t \mid \mathcal{F}_{t-1}] \in D^*_t \quad t = 1, \dots T\},$$

so it suffices to make the substitution $v_t = y_t s_t$. □

In classical perfectly liquid models where $S_t(x,\omega) = s_t(\omega) \cdot x$ and $D_t(\omega) = \mathbb{R}^J$, we have $Z_t(\omega) = \{s_t(\omega)\}$ and $D^*_t(\omega) = \{0\}$ so Corollary 14 says that

$$\mathcal{D}^\infty = \{y \in \mathcal{M}^\infty_+ \mid (y_t s_t) \text{ is a martingale}\}$$

as long as s is integrable. In particular, if one of the assets has nonzero constant price then every $y \in \mathcal{D}^\infty$ is a martingale. In this case, Theorem 10 can be written in the following more familiar form; see e.g. Theorem 2 on page 55 of Ingersoll [19] for the case of finite probability spaces.

Corollary 15 *Consider the classical linear model with a cash account and an integrable price process s. If $\mathcal{C}$ is closed, then the existence of a martingale density for s is equivalent to the condition $\pi(0) \ge 0$ with the premium $p = (1, 0, \dots, 0)$. In this case,*

$$\pi(c) = \sup_{Q \in \mathcal{P}} E^Q \sum_{t=0}^{T} c_t,$$

where $\mathcal{P}$ is the set of martingale measures that are absolutely continuous with respect to P.

Proof. As noted above, the elements $\mathcal{D}^\infty$ are martingales y such that ys is also a martingale. The existence of a martingale density is thus equivalent to the existence of a $y \in \mathcal{D}^\infty$ such that $y_0 = 1$. By Corollary 12, this is equivalent to $\pi(0) \ge 0$ with the premium process $p = (1, 0, \dots, 0)$. The representation then follows from the correspondence $dQ/dP = y_T$ between absolutely continuous martingale measures and terminal values of the term structures $y \in \mathcal{D}^\infty$. □

In the classical linear model with a cash account, the closedness of $\mathcal{C}$ and the condition $\pi(0) \ge 0$ with $p = (1, 0, \dots, 0)$ both hold under the no arbitrage condition; see Schachermayer [45]. The stronger no arbitrage condition also implies the existence of a strictly positive martingale density. The relationships between no arbitrage conditions and the existence of strictly positive stochastic term structures in general convex models have been studied in [35]. However, in nonconical models, neither arbitrage nor the associated strictly positive term structures are relevant when it comes to superhedging. In particular, the no arbitrage condition does not imply the closedness of $\mathcal{C}$ in general; see Example 17 below.

The condition $\pi(0) \geq 0$ is like the "law of one price" except that it allows for throwing away of money. On one hand, the condition $\pi(0) \geq 0$ yields the nonnegativity of the dual variables while the law of one price only yields the existence of a martingale which could take negative values; see [9]. On the other hand, allowing for throwing away of money in the definition of π we have to impose the closedness of $\mathcal{C}$ explicitly. In classical perfectly liquid models, the set of claims that can be *exactly replicated* with zero cost is always closed; see [25].

6 Closedness of $\mathcal{C}$

In light of the above results, the closedness of $\mathcal{C}$ in probability becomes an interesting issue. It was shown by Schachermayer [45] that when S is a linear cost process with a cash account (see Example 4) and $D = \mathbb{R}^J$, the closedness is implied by the classical no arbitrage condition. This section, gives sufficient conditions for other choices of S and D that guarantee that $\mathcal{C}$ is closed in probability.

In classical linear models, the finiteness of Ω is known to be sufficient for closedness even when there is arbitrage. More generally, we have the following.

Example 16 *If S and D are polyhedral and Ω is finite then $\mathcal{C}$ is closed.*

Proof. By [40, Theorem 19.1] it suffices to show that $\mathcal{C}$ is polyhedral. The set $\mathcal{C}$ is the projection of the convex set

$$E = \{(x, c) \in \mathcal{N}_0 \times \mathcal{M} \mid x_t \in D_t,\ S_t(\Delta x_t) + c_t \leq 0,\ t = 0, \dots, T\}.$$

When S and D are polyhedral, we can describe the pointwise conditions $x_t \in D_t$ and $S_t(\Delta x_t)+c_t \leq 0$ by a finite collection of linear inequalities. When Ω is finite, the set E becomes an intersection of a finite collection of closed half-spaces. The set $\mathcal{C}$ is then polyhedral since it is a projection of a polyhedral convex set; see [40, Theorem 19.3]. □

In a general nonlinear model, however, the set $\mathcal{C}$ may fail to be closed already with finite Ω and even under the no arbitrage condition.

Example 17 *Consider Example 4 in the case $T = 1$, so that*

$$\mathcal{C} = \{c \in \mathcal{M} \mid \exists \tilde{x}_0 \in \tilde{D}_0 :\ c_0 + c_1 \leq \tilde{x}_0 \cdot (s_1 - s_0)\}.$$

Let $\Omega = \{\omega^1, \omega^2\}$, $\tilde{J} = \{1, 2\}$, $\tilde{D}_0 = \{(x^1, x^2) \mid x^j \geq -1,\ (x^1 + 1)(x^2 + 1) \geq 1\}$, $\tilde{s}_0 = (1, 1)$ and

$$\tilde{s}_1(\omega) = \begin{cases} (1, 2) & \text{if } \omega = \omega^1, \\ (1, 0) & \text{if } \omega = \omega^2. \end{cases}$$

Since $\tilde{s}^1$ is constant, we get

$$\mathcal{C} = \{c \in \mathcal{M} \,|\, \exists \tilde{x}_0^2 \in \tilde{D}_0^2 : \; c_0 + c_1 \le \tilde{x}_0^2(s_1^2 - s_0^2)\},$$

where $\tilde{D}_0^2$ is the projection of $\tilde{D}_0$ on the second component. Since $\tilde{D}_0^2 = (-1, +\infty)$, $s_1^2(\omega^1) - s_0^2 = 1$ and $s_1^2(\omega^2) - s_0^2 = -1$, we get

$$\begin{aligned}\mathcal{C} &= \{c \in \mathcal{M} \,|\, \exists \tilde{x}_0 > -1 : \; c_0 + c_1(\omega^1) \le \tilde{x}_0^2, \; c_0 + c_1(\omega^2) \le -\tilde{x}_0^2\} \\ &= \{c \in \mathcal{M} \,|\, c_0 + c_1(\omega^1) + c_0 + c_1(\omega^2) \le 0, \; c_0 + c_1(\omega^2) < 1\},\end{aligned}$$

which is not closed even though the no arbitrage condition $\mathcal{C} \cap \mathcal{M}_+ = \{0\}$ is satisfied.

In order to find sufficient conditions for nonlinear models with general Ω, we resort to traditional closedness criteria from convex analysis; see [40, Section 9]. Given an $\alpha > 0$, it is easily checked that

$$(\alpha \star S)_t(x, \omega) := \alpha S_t(\alpha^{-1} x, \omega)$$

defines a convex cost process in the sense of Definition 1. If S is positively homogeneous, we have $\alpha \star S = S$, but in general, $\alpha \star S$ decreases as α increases; see [40, Theorem 23.1]. The upper limit

$$S_t^\infty(x, \omega) := \sup_{\alpha > 0} \alpha \star S_t(x, \omega),$$

known as the *recession function* of $S_t(\cdot, \omega)$, describes the behavior of $S_t(x, \omega)$ infinitely far from the origin; see [40, Section 8]. Analogously, if D is conical, we have $\alpha D = D$, but in general, αD gets smaller when α decreases. Since $D_t(\omega)$ is closed and convex, the intersection

$$D_t^\infty(\omega) = \bigcap_{\alpha > 0} \alpha D_t(\omega),$$

coincides with the *recession cone* of $D_t(\omega)$; see [40, Corollary 8.3.2].

An $\mathbb{R}^J$-valued adapted process $s = (s_t)$ will be called a *market price process* if $s_t \in \partial S_t(0)$ almost surely for every $t = 0, \dots, T$; see [35]. Here,

$$\partial S_t(0, \omega) := \{v \in \mathbb{R}^J \,|\, S_t(x, \omega) \ge S_t(0, \omega) + v \cdot x \; \forall x \in \mathbb{R}^J\}$$

is the *subdifferential* of S_t at the origin. If $S_t(\cdot, \omega)$ happens to be smooth at the origin, then $\partial S_t(0, \omega) = \{\nabla S_t(0, \omega)\}$.

Theorem 18 *The set $\mathcal{C}$ is closed in probability if*

$$D_t^\infty(\omega) \cap \{x \in \mathbb{R}^J \,|\, S_t^\infty(x, \omega) \le 0\} = \{0\}$$

almost surely for every $t = 0, \dots, T$. This holds, in particular, if there exists a componentwise strictly positive market price process and if $D^\infty \subset \mathbb{R}_+^J$.

Proof. Let $(c^\nu)_{\nu=1}^\infty$ be a sequence in $\mathcal{C}$ converging to a c. By passing to a subsequence if necessary, we may assume that $c^\nu \to c$ almost surely. Let $x^\nu \in \mathcal{N}_0$ be a superhedging portfolio process for c^ν, i.e.

$$x_t^\nu \in D_t, \quad S_t(x_t^\nu - x_{t-1}^\nu) + c_t^\nu \le 0$$

almost surely for $t = 0, \dots, T$ and $x_{-1}^\nu = x_T^\nu = 0$. We will show that the sequence $(x^\nu)_{\nu=1}^\infty$ is almost surely bounded.

Assume that x_{t-1}^ν is almost surely bounded and let $a_{t-1} \in L^0$ be such that $x_{t-1}^\nu \in a_{t-1}\mathbb{B}$ almost surely for every ν. Defining $\underline{c}_t(\omega) = \inf c_t^\nu(\omega)$ we then get that

$$\begin{aligned} x_t^\nu(\omega) &\in D_t(\omega) \cap \{x \in \mathbb{R}^J \,|\, S_t(x - x_{t-1}^\nu(\omega), \omega) + c_t^\nu(\omega) \le 0\} \\ &\subset D_t(\omega) \cap \left[\{x \in \mathbb{R}^J \,|\, S_t(x, \omega) + c_t^\nu(\omega) \le 0\} + a_{t-1}(\omega)\mathbb{B}\right] \\ &\subset D_t(\omega) \cap \left[\{x \in \mathbb{R}^J \,|\, S_t(x, \omega) + \underline{c}_t(\omega) \le 0\} + a_{t-1}(\omega)\mathbb{B}\right]. \end{aligned}$$

By [40, Theorem 8.4], this set is bounded exactly when its recession cone consists only of the zero vector. By Corollary 8.3.3 and Theorems 9.1 and 8.7 of [40], the recession cone can be written as

$$D_t^\infty(\omega) \cap \{x \in \mathbb{R}^J \,|\, S_t^\infty(x, \omega) \le 0\},$$

which equals $\{0\}$, by assumption. It thus follows that $(x_t^\nu)_{\nu=1}^\infty$ is almost surely bounded and then, by induction, the whole sequence $(x^\nu)_{\nu=1}^\infty$ has to be almost surely bounded.

By Komlos' principle of subsequences (see e.g. [17, Lemma 1.69]), there is a sequence of convex combinations

$$\bar{x}^\mu = \sum_{\nu=\mu}^\infty \alpha^{\mu,\nu} x^\nu$$

that converges almost surely to an x. Since $c^\nu \to c$ almost surely, we also get that

$$\bar{c}^\mu := \sum_{\nu=\mu}^\infty \alpha^{\mu,\nu} c^\nu \to c \quad P\text{-a.s.}.$$

By convexity, of D and S,

$$\bar{x}_t^\mu \in D_t, \quad S_t(\bar{x}_t^\mu - \bar{x}_{t-1}^\mu) + \bar{c}_t^\mu \le 0$$

and then, by closedness of $D_t(\omega)$ and lower semicontinuity of $S_t(\cdot, \omega)$,

$$x_t \in D_t, \quad S_t(x_t - x_{t-1}) + c_t \le 0.$$

Thus, $c \in \mathcal{C}$ and the first claim follows.

If $s \in \partial S(0)$ is a market price process, then $s_t(\omega) \cdot x \le S_t(x, \omega)$ for every $x \in \mathbb{R}^J$ and thus $s_t(\omega) \cdot x \le S_t^\infty(s, \omega)$ for every $x \in \mathbb{R}^J$. If we also have $D^\infty \subset \mathbb{R}_+^J$, then

$$D_t^\infty(\omega) \cap \{x \in \mathbb{R}^J \,|\, S_t^\infty(x, \omega) \le 0\} \subset \mathbb{R}_+^J \cap \{x \in \mathbb{R}^J \,|\, s_t(\omega) \cdot x \le 0\},$$

which reduces to the origin when s is strictly positive. ☐

The set $D_t^\infty(\omega)$ consists of portfolios that can be scaled by arbitrarily large positive numbers without ever leaving the set $D_t(\omega)$ of feasible portfolios. By [40, Theorem 8.6], the set

$$\{x \in \mathbb{R}^J \mid S_t^\infty(x,\omega) \leq 0\}$$

gives the set of portfolios x such that the cost $S_t(\alpha x, \omega)$ is nonincreasing as a function of α. Since $S_t(0,\omega) = 0$, we also have $S_t(x,\omega) \leq 0$ for every x with $S_t^\infty(x,\omega) \leq 0$.

The existence of a strictly positive market price process in Theorem 18 is a natural assumption in many situations. In double auction markets, for example, it means that ask prices of all assets are always strictly positive. The condition $D_t^\infty(\omega) \subset \mathbb{R}_+^J$ means that if a portfolio $x \in \mathbb{R}^J$ has one or more negative components then αx leaves the set $D_t(\omega)$ for large enough $\alpha > 0$. This holds in particular if portfolios are not allowed to go infinitely short in any of the assets.

Example 17 shows that the no arbitrage condition does not imply the conditions of Theorem 18 (in Example 17, $D_0^\infty(\omega) = \mathbb{R}\times\mathbb{R}_+^2$ and $S_t^\infty(x,\omega) = s_t(\omega)\cdot x$). On the other hand, the conditions of Theorem 18 may very well hold (and thus, $\mathcal{C}$ be closed) even when the no arbitrage condition is violated.

Example 19 *Let $S_t(x,\omega) = s_t(\omega)\cdot x$ where $s = (s_t)$ is a componentwise strictly positive marginal price process. It is easy to construct examples of s that allow for arbitrage in an unconstrained market. Let $\bar{x} \in \mathcal{N}_0$ be an arbitrage strategy in such a model and consider another model with constraints defined by $D_t(\omega) = \{x \in \mathbb{R}^J \mid x \geq \bar{x}_t(\omega)\}$. In this model, $\bar{x}$ is still an arbitrage strategy but now the conditions of Theorem 18 are satisfied so $\mathcal{C}$ is closed.*

Sufficient conditions for closedness of $\mathcal{C}$ can also be derived from the results of Schachermayer [46], Kabanov, Rásonyi and Stricker [22] as well as the forthcoming paper Pennanen and Penner [36]. Whereas [46] and [22] deal with conical models, [36] allows for more general convex models. In these papers, closedness is obtained for the larger set of portfolio-valued claims. However, this is not necessary when studying claims with cash delivery as in this paper. More importantly, none of the above papers allows for portfolio constraints.

7 Conclusions

This paper extended some classical dual characterizations of superhedging to illiquid markets with general claim and premium processes. The characterizations were given in terms of stochastic term structures which generalize term structures of interest rates beyond fixed income markets as well as martingale densities beyond stochastic markets with a cash account. The characterizations are valid whenever the set of freely available claim processes is closed in probability and the superhedging cost of the zero claim is finite. In the special case of classical perfectly liquid markets with a single premium payment at the beginning, both conditions are implied by the no arbitrage condition. Section 6 gives alternative closedness conditions for general convex cost functions and convex constraints. They apply, in particular, in double auction markets when one is not allowed to go infinitely short in any of the traded assets.

Most of the results in this paper were stated in terms of the set $\mathcal{C}$ of claim processes hedgeable with zero cost. This means that the results are not tied to the particular market model presented in Section 2 but apply to any model where the set $\mathcal{C}$ is closed in probability and has the properties in Lemma 5. In particular, one could look for conditions that yield convexity in market models with long terms price impacts; see e.g. Alfonsi, Schied and Schulz [2].

In reality, one rarely looks for superhedging strategies when trading in practice. Instead, one (more or less quantitatively) sets bounds on acceptable levels of "risk" when taking positions in the market and when quoting prices. Risk based pricing has been extensively studied in the case of classical perfectly liquid market models; see e.g. [17, Chapter 8]. Allowing for risky positions takes us beyond the completely riskless superhedging formulations studied in this paper. Nevertheless, closedness and duality results such as the ones derived here will be in an important role when extending risk based pricing to illiquid market models with general claim and premium processes. This will be studied in a separate paper.

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