# Risk Analysis and Valuation of Life Insurance Contracts: Combining Actuarial and Financial Approaches

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**Abstract**

In this paper, we analyze traditional (i.e. not unit-linked) life insurance contracts with a guaranteed interest rate and surplus participation. We consider three different surplus distribution models and an asset allocation that consists of money market, bonds with different maturities and stocks. In this setting, we combine actuarial and financial approaches by selecting a risk minimizing asset allocation (under the real world measure $\mathbf{P}$) and distributing terminal surplus such that the contract value (under the pricing measure $\mathbf{Q}$) is fair. We prove that this strategy is always possible unless the insurance contracts introduce arbitrage opportunities in the market. We then analyze differences between the different surplus distribution models and investigate the impact of the selected risk measure on the risk minimizing portfolio.

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# 1 Introduction

Interest rate guarantees are a very common product feature within traditional participating life insurance contracts in many markets. There are two major types of interest rate guarantees:

The simplest interest rate guarantee is a so-called point-to-point guarantee, i.e. a guarantee that is only relevant at maturity of the contract. Another type is called cliquet-style (or year-by-year) guarantee. This means that the policy holders have an account to which every year a certain rate of return has to be credited. Usually, life insurance companies provide the guaranteed rate of interest plus some surplus on the policy holder's account every year.

Cliquet-style guarantees of course may cause significant problems in years of low interest rates and plunging stock markets. They may force insurers to provide relatively high guaranteed rates of interest to accounts to which a big portion of past years' surplus has already been credited. Adverse capital market scenarios of recent years appeared to have caused significant problems for insurers offering this type of guarantee. Therefore, the analysis of traditional life insurance contracts with cliquet-style guarantees has become a subject of increasing concern for the academic world as well as for practitioners.

There are financial and actuarial approaches to handling financial guarantees within life insurance contracts. The financial approach is concerned with risk-neutral valuation and fair pricing and has been researched by various authors such as Bryis and de Varenne (1997), Grosen and Jørgensen (2000), Grosen and Jørgensen (2002) or Bauer et al. (2006). Note that the concept of risk-neutral valuation is based on the assumption of a perfect (or super-) hedging strategy, which insurance companies normally do not or cannot follow (cf. e.g. Bauer et al. (2006)). If the insurer does or can not invest in a portfolio that replicates the liabilities, the company remains at risk and should therefore additionally perform some risk analyses. The actuarial approach focuses on quantifying this risk with suitable risk-measures under an objective ‘real-world’ probability-measure, cf. e.g. Kling, Richter and Russ (2007a) or Kling, Richter and Russ (2007b). Such approaches also play an important role e.g. in financial strength ratings or under the new Solvency II approach. Amongst others, Gatzert (2005) investigates parameter combinations that yield fair contracts and analyzes the risk imposed by fair contracts for various insurance contract models, starting with a simple point-to-point guarantee and afterwards analyzing more sophisticated Danish- and UK-style contracts. Kling (2007) focuses on traditional German insurance contracts where the interdependence of various parameters concerning the risk exposure of fair contracts is studied. Gatzert (2007) extends the work from Gatzert (2005) where an approach to 'risk pricing' is introduced using the 'fair value of default' to determine contracts with the same risk exposure. However, this 'risk measure' neglects real-world scenarios and is only concerned with the (risk-neutral) value of the introduced default put option. Whereas Gatzert (2007) analyzes some real-world risk generated by the determined contracts, the risk exposure is not incorporated in the pricing procedure. Barbarin and Devolder (2005) introduce a methodology that allows for combining the financial and actuarial approach. They consider a contract, similar to Bryis and de Varenne’s (1997), with a point-to-point guarantee and terminal surplus participation.

To integrate both approaches, they use a two-step method of pricing life insurance contracts: First, they determine a guaranteed interest rate such that certain solvency requirements are

satisfied, using value at risk and expected shortfall risk measures. Second, to obtain fair contracts, they use risk-neutral valuation and adjust the participation rate accordingly.

In the present work we extend Barbarin and Devolder's (2005) methodology which then allows the pricing of life insurance contracts in a more general liability framework including in particular typical features of the German insurance market. We then propose a methodology that allows us to find parameter combinations that minimize the real world risk without changing the fair value of the contract. We show that the proposed methodology works in general as long as the insurance contract design does not introduce arbitrage-opportunities.

The remainder of this paper is organized as follows. After an introduction of the considered financial market, the insurer's asset allocation and different liability models in Section 2, Section 3 presents the methodology of combining the actuarial and financial approach and the theoretical result that the strategy we propose is always possible unless the insurance contracts introduce arbitrage opportunities in the market. In Section 4, we show various numerical results for the introduced liability models, emphasizing on both, the risk a specific contract design and asset allocation imposes on the insurance company and the valuation of the contract from the client's perspective. We further investigate the difference resulting from different risk measures. Section 5 concludes.

# 2 Model framework

## 2.1 Insurance company

Following Kling, Richter and Russ (2007a), we consider a simplified 'balance sheet' of the insurance company as follows:

| **Assets** | **Liabilities** |
|---|---|
| | $L(t)$ |
| $A(t)$ | $B(t)$ |
| | $R(t)$ |
| $A(t)$ | $A(t)$ |

Here, $A(t)$ denotes the market value of the company's assets. $L(t)$ represents the insurer's liabilities measured by the actuarial reserve for the insurance contracts. Every year $L(t)$ has to earn at least a fixed guaranteed interest rate $i$, thus $L(t+1) \geq L(t)(1+i)$. The insured can participate in the insurer's asset return exceeding the guaranteed rate in two ways: By regular surplus participation if in any year more than guaranteed interest rate $i$ is credited to the account $L$ and by terminal surplus participation. $B(t)$ models a collective terminal surplus account, which is used to provide additional surplus participation at the maturity of a client's contract. This account may be reduced at any time in order to ensure the company's liquidity

which leaves $B(t)$ to be an optional bonus payment and $B(t) \geq 0$ for all $t$. The residual value $R(t) = A(t) - (L(t) + B(t))$ denotes the (hidden) reserves of the life insurer.

## 2.2 Financial market

We now introduce the model for the financial market and the financial instruments in the insurer's asset portfolio. We allow investment in money market, bonds and stocks. We use the Vasiçek (1977) model for stochastic interest rates and a Geometric Brownian Motion (cf. Black and Scholes (1973)) for a reference stock or stock index.

We first specify our asset model under the real-world probability measure $\mathbf{P}$ and then switch to the risk-neutral measure $\mathbf{Q}$ which will be used for valuation purposes. We consider a probability space $(\Omega, F, \mathbf{F}, \mathbf{P})$ with the natural filtration $\mathbf{F} = F_t = \sigma((W_1(s), W_2(s)), s \leq t)$ generated by independent $\mathbf{P}$ – Brownian Motions $W_1(t)$ and $W_2(t)$ and let $r(t)$ denote the short-rate and $S(t)$ the value of the stock at time *t*.

The asset model is then given by the stochastic differential equations (SDEs)

$$dr(t) = a(b - r(t))dt + \sigma_r dW_1(t)$$
$$dS(t) = S(t)\left(\mu dt + \sigma_S\left(\rho dW_1(t) + \sqrt{1-\rho^2}dW_2(t)\right)\right)$$

with $\rho \in [-1,1]$ denoting the coefficient of correlation. To simplify notation, we let $W_3(t) := \rho W_1(t) + \sqrt{1-\rho^2}W_2(t)$ [2]. Thus, for $s \leq t$, a closed form solution of the above SDEs is given by[3]

$$r(t) = e^{-a(t-s)}r(s) + b\left(1 - e^{-a(t-s)}\right) + \sigma_r e^{-at}\int_s^t e^{au}dW_1(u)$$
$$S(t) = S(s)e^{\left\{\left(\mu - \frac{1}{2}\sigma_S^2\right)(t-s) + \sigma_S(W_3(t) - W_3(s))\right\}}$$

A money market investment is then modelled by a continuous investment in the short rate which introduces the so-called *bank account* or risk free investment $\beta(t) = e^{\int_0^t r(s)ds}$.

We further consider a bond portfolio consisting of different zero-bonds. Hence we need to determine $p(t,T)$, the price at time *t* of a zero-bond with maturity $T$. We assume that $p(t,T) = F(t, r(t))$ holds for some smooth function $F(t, r(t))$. Since the short rate is not observable on the market we may not be able to hedge derivatives on the short rate (e.g. zero-bonds) by investing in the underlying itself as it could be done e.g. in a Black-Scholes framework. Investing in the bank account instead would result in an incomplete market.

By constructing a portfolio with no instantaneous risk (e.g. consisting of two zero-bonds with different maturities) and applying no arbitrage arguments, one arrives at the so-called *market*

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[2] From Lévy's theorem it follows that $W_3(t)$ is a $\mathbf{P}$ – Brownian Motion as well.

[3] Compare Bjørk (2005) for further details.

*price of risk* $\lambda(t,r(t))$ and hence at a partial differential equation for zero-bond prices[4], the so-called *term structure equation*.

$$F_t(t,r(t))+\left(a(b-r(t))-\lambda(t,r(t))\sigma_r\right)F_r(t,r(t))+\frac{1}{2}\sigma_r^2F_{rr}(t,r(t))-r(t)F(t,r(t))=0$$

with terminal condition $F(T,r(T))=1$.

The Feynman-Kaç[5] formula then allows for a probabilistic interpretation of above partial differential equation by

$$p(t,T)=F(t,r(t))=\mathbf{E}_{\tilde{\mathbf{Q}}}\left(e^{-\int_t^T r(s)ds}\middle| r(t)\right)$$

with a probability measure $\tilde{\mathbf{Q}}$ and a stochastic process $r(t)$ with $\tilde{\mathbf{Q}}-$dynamics $dr(t)=\left(a(b-r(t))-\lambda(t,r(t))\sigma_r\right)dt+\sigma_r d\tilde{W}_1$. Note that observed zero-bond prices induce the market price of risk $\lambda(t,r(t))$ and therefore no obvious form or parameterization of $\lambda(t,r(t))$ exists ad hoc. However, if and only if we assume $\lambda(t,r(t))=\lambda$, the short rate process under $\tilde{\mathbf{Q}}$ remains of the Vasiçek-type. From standard interest rate theory (cf. e.g. Bjørk (2005)) it follows that $p(t,T)=e^{A(t,T)-B(t,T)r(t)}$ with $A(t,T)=\left(\frac{\sigma_r^2}{2a^2}-b+\frac{\lambda\sigma_r}{a}\right)\left((T-t)-B(t,T)\right)-\frac{\sigma_r^2}{4a}B(t,T)^2$ and $B(t,T)=\frac{1}{a}\left(1-e^{-a(T-t)}-1\right)$.

Therefore, $p(t,T)$ follows a log-normal distribution for $t<T$ and by applying Itô's Lemma the zero-bond's real world dynamics are consequently derived as $dp(t,T)=p(t,T)\left((r(t)-\lambda\sigma_r B(t,T))dt-\sigma_r B(t,T)dW_1(t)\right)$. Defining a probability measure $\mathbf{Q}$ equivalent to $\mathbf{P}$ by the Radon-Nikodym density

$$\left.\frac{d\mathbf{Q}}{d\mathbf{P}}\right|_{F_t}=\exp\left\{-\lambda W_1(t)-\frac{1}{2}\lambda^2 t-\int_0^t\frac{\mu-r(s)-\rho\lambda\sigma_S}{\sigma_S\sqrt{1-\rho^2}}dW_2(s)-\frac{1}{2}\int_0^t\left(\frac{\mu-r(s)-\rho\lambda\sigma_S}{\sigma_S\sqrt{1-\rho^2}}\right)^2 ds\right\}$$

yields $\left(e^{-\int_0^t r(s)ds}S(t)\right)_{t\geq 0}$ and $\left(e^{-\int_0^t r(s)ds}p(t,T)\right)_{t\geq 0,t<T}$ being $\mathbf{Q}-$martingales.

Therefore, a complete setup for risk-analysis under the real-world measure $\mathbf{P}$ and valuation using the risk-neutral measure $\mathbf{Q}$ is provided.

## 2.3 Insurer's investment strategy

Now, we introduce the insurer's investment strategy consisting of above investment vehicles and let $T$ denote the considered time horizon. We assume the insurer invests in the money

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[4] For a complete derivation cf. Graf (2008).

[5] Cf. e.g., Bjørk (2005).

market with a constant ($F_0$-measurable) proportion $x_\beta$, in the stock market with a constant proportion $x_S$ and in the bond market with a constant proportion $x_B$. These proportions are kept stable by continuous rebalancing and fulfil $x_\beta + x_S + x_B = 1$.

To simplify notation, we assume that a 'restructuring' of the bond portfolio occurs only at anniversary dates $0\ldots,T-1$ and assume there exist zero-bonds with time to maturities $1,\ldots,T^*$ at each anniversary date, where $T^*$ denotes the maximum duration of a bond, the insurer invests in. For $t \in [i,i+1)$ let the $F_i$-measurable random variable $x_{ij}$ denote the proportion of the bond with time to maturity *j* within the bond portfolio. This proportion is kept constant over the period $[i,i+1)$ by continuous rebalancing. Naturally, we require $\sum_{j=1}^{T^*} x_{ij} = 1$ for all $i$.

We let the $F_t$-measurable random variable $c_\beta(t)$ denote the number of shares the insurance company holds of the money market account $\beta(t)$. Analogously, $c_S(t)$ is the number of shares of the stock market $S(t)$ at time *t* and $c_{ij}(t)$ denotes the number of bonds maturing at time $i+j$ the company holds at time *t*. This yields $\frac{c_\beta(t)\beta(t)}{A(t)} = x_\beta$, $\frac{c_S(t)S(t)}{A(t)} = x_S$ and $\frac{c_{ij}(t)p(t,j+i)}{\sum_{j=1}^{T^*} c_{ij}(t)p(t,j+i)} = x_{ij}$ for $t \in [i,i+1)$ .Finally, we get $A(t) = c_\beta(t)\beta(t) + c_S(t)S(t) + \sum_{j=1}^{T^*} c_{ij}(t)p(t,i+j)$.

***Self-financing portfolio***

We assume the reference portfolio to be self-financing. Hence, for $t \in [i,i+1)$, we obtain $dA(t) = c_\beta(t)d\beta(t) + c_S(t)dS(t) + \sum_{j=1}^{T^*} c_{ij}(t)dp(t,i+j)$. Thus, the dynamics of the insurer's asset portfolio can then be written as

$$\frac{dA(t)}{A(t)} = x_\beta(t)r(t)dt + x_S(t)\left(\mu dt + \sigma_S dW_3(t)\right) + \sum_{j=1}^{T^*} x_{ij} x_B \left(\left(r(t) - \lambda\sigma_r B(t,i+j)\right)dt - \sigma_r dW_!(t)\right)$$

implying a lognormal distribution of $A(t)$ given $F_i$ for $t \in [i,i+1)$ (for a formal proof, see Appendix A). If we further assume the $x_{ij}$ to be $F_0-$measurable (i.e. deterministic) the insurer's asset portfolio essentially follows a Geometric Brownian Motion.

At this stage, one might wonder why we introduced different stochastic processes for interest rates and stocks if we finally arrive at an asset portfolio following a simple Geometric Brownian Motion (GBM) (under the above assumptions). First, we would like to stress that this justifies the assumption (made e.g. by Gatzert (2007), Kling, Richter and Russ (2007a) or Kling, Richter and Russ (2007b) and many other papers on the subject) that an insurer's

asset portfolio develops according to a GBM. On the other hand, our model framework gives us more flexibility: If we price derivatives that depend on the assets contained in the portfolio we can work with the corresponding processes $r(t)$ and $S(t)$ to obtain appropriate results. Finally, by relaxing some of the assumptions above, our model allows for time- and path-dependent modelling of the composition of the asset portfolio which then – of course – results in a more complicated stochastic process, and loses some of the analytical tractability that can be exploited in the GBM-setting.

## 2.4 Liability model

***Point-to-point model***

We start by introducing a simple liability model similar to Bryis and de Varenne (1997) and Barbarin and Devolder (2005) by modelling a term-fix contract with single premium $P$, guaranteed interest rate $i$ and a terminal bonus participation rate $\eta$. Hence, every year the guaranteed interest is credited to the actuarial reserve and at contract's maturity an additional surplus participation is provided. Therefore, we obtain for $t = 1,\ldots,T$

$$L(t) = P(1+i)^t$$
$$B(t) = \begin{cases} 0 & ,t \neq T \\ \eta \max\{A(T) - L(T), 0\} & ,t = T \end{cases}$$

We further assume the lump sum $P$ to be invested in the insurer's reference portfolio as modelled in the section above which then gives $A(0) = P$.

As mentioned, $\eta$ has no influence on the company's shortfall-risk during the term of the contract. Thus, the risk exposure depends only on the asset allocation parameters and the guaranteed rate. Then, $\eta$ can be chosen independently to achieve the desired contract value.

***Cliquet-style guarantee: The MUST-Case***

Now we expand the previous model by including annual surplus participation as legally required e.g. in Germany. Here, following Bauer et al. (2006), we distinguish between the so-called MUST-case, explained in this section (i.e. the case of an insurer distributing just enough profit to satisfy legal requirements) and the so-called IS-case, explained in the next section that tries to model observed behaviour of German insurers, that is influenced also by competition. For more details, see Bauer et al. (2006).

Due to prudent product pricing, insurance companies usually achieve returns on their assets that exceed guaranteed rates. In many countries, the insured are legally entitled to participate in the resulting surplus. E.g. in Germany, at least $\delta = 90\%$ of the company's return exceeding the guaranteed interest rate has to be distributed to the insured.[6] However,

[6] Compare the Ordinance on Minimum Premium Refunds in Life Insurance ('Mindestzuführungsverordnung -MindZV') for more details concerning the required annual surplus participation of the German life insurance market.

the achieved surplus is calculated on *book values* of assets denoted by $A_b(t)$. Since accounting rules give insurers certain freedoms in managing book values, they can also manage at least parts of the surplus distribution.

Book values under German accounting rules are modelled as follows: We assume stocks to follow the so-called 'lower-value principle'[7] meaning that a stock's book value can not exceed the initial value. Further, if the market value falls below the book value, under certain circumstances the insurer can avoid write-offs.[8] For the sake of simplicity, we therefore assume the stock's book value to always coincide with the initial value. Market price fluctuations are only shown in the difference between market and book value of assets, the so-called hidden reserves (that can also be negative). Concerning the bond portfolio, we distinguish between bearer and registered bonds, and denote with $y_B$[9] the fraction of registered bonds in our bond portfolio. According to German legislation, we also use the lower-value principle for bearer bonds and let the book value coincide with the market value for money market and registered bond investments. This yields $A_b(t) = x_\beta A(t) + y_B x_B A(t) + (x_S + (1-y_B)x_B)A(0)$.[10]

Summarizing, we get the following development of the liabilities including surplus for $t = 1,\ldots,T$:

$$L(t) = L(t-1)(1+i) + \max\{\delta(A_b(t) - A_b(t-1)) - iL(t-1), 0\}$$
$$B(t) = \begin{cases} 0 & ,t \neq T \\ \eta \max\{A(T) - L(T), 0\} & ,t = T \end{cases}$$

starting with $L(0) = P$. Note that granting surplus to the actuarial reserve $L$ implies that past surplus is also entitled to earning the guaranteed rate in the future.

***Cliquet-style guarantee: The IS-Case***

We now describe how German insurance companies typically allow for surplus participation. In order to signal financial stability to the market, they try to keep the surplus participation rather stable over time. Thus, following Kling, Richter and Russ (2007b), we assume that the insurer uses the following management rule:

As long as the company has 'sufficient' reserves, some target surplus is distributed (resulting in some target total interest $z$, which is the sum of guaranteed interest and surplus). In case the company's reserves fall below a certain lower boundary, the surplus is reduced and in case the reserves increase above a certain upper boundary, the surplus is increased. We let

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[7] Following §253 German commercial code.

[8] Following §341(b) German commercial code.

[9] Within the Results section, we use $y_B$ = 75% estimated from GDV (2007).

[10] Since, in Section 4, we only consider the shortfall probability and the expected shortfall at the end of the time horizon, funding gaps on the insurer's balance sheet during the contract's term do not influence the results. Therefore, we do not need to model necessary changes in book values if $L(t)>A_b(t)$.

$q(t) := \frac{R(t)}{L(t)}$ denote the so-called *reserve quota*. Additionally, we denote with $q_l$ and $q_u$ the lower respectively upper boundary for the reserve quota.

The surplus distribution policy is then given by the following $F_t$ – measurable management rule:

- If $A(t) \le (1+q_u)((1+z)L(t-1))$ and $A(t) \ge (1+q_l)((1+z)L(t-1))$, we credit the target interest rate $z$ to the actuarial reserve (implying $q(t) \in [q_l, q_u]$ afterwards).
- If $A(t) > (1+q_u)((1+z)L(t-1))$ a higher rate $z^*$ is used that ensures $q(t) = q_u$ after surplus distribution. This is achieved by setting $z^* = \frac{A(t) - (1+q_u)L(t-1)}{(1+q_u)L(t-1)}$.
- If $A(t) < (1+q_l)((1+z)L(t-1))$ we analogously use the value $z^*$ that makes $q(t) = q_l$ after surplus distribution: $z^* = \frac{A(t) - (1+q_l)L(t-1)}{(1+q_l)L(t-1)}$.
- If the compulsory surplus as explained above exceeds the surplus calculated here, the compulsory surplus is distributed.

# 3 Methodology

We will now explain the analyses we will perform within our model. Obviously, we are able to analyze the ('real-world') risk the insurer is exposed to. In more detail, for some suitable risk-measure under $\mathbf{P}$, we can then analyze the impact of varying asset allocations for given parameters (e.g. the guaranteed interest rate or the target interest rate) or investigate the effect of different guaranteed interest rates or target interest rates for a given asset allocation. We can analyze if and how the choice of risk-measure affects the corresponding results. Further, a risk-neutral valuation under $\mathbf{Q}$ of the insurance contract can be performed. And finally, both techniques can be combined.

In our analysis, we use the following risk measures: the *probability of shortfall* $\mathbf{P}(A(T) < L(T))$ and the *expected shortfall* $\mathbf{E}_P\left((L(T) - A(T))\mathbf{1}_{\{A(T) \le L(T)\}}\right)$.

***Combining the actuarial and financial approach***

The concrete choice of the terminal bonus participation rate $\eta$ does obviously not affect the insurer's risk as defined above. Therefore, one strategy could be to choose a value of $\eta$ that makes the contract fair, i.e. $\mathbf{E}_\mathbf{Q}\left(e^{\int_0^T -r(s)ds}(L(T) + B(T))\right) = P$ at time 0. However, for practical purposes only values of $\eta \in [0,1]$ are suitable. The following theoretical result shows under what circumstances an appropriate value of $\eta$ yields a fair contract.

**Proposition**

For all asset allocations and an arbitrary liability structure independent of the terminal bonus payment, we obtain a fair contract with $\eta \in (-\infty,1]$ if the condition $\mathbf{E_Q}\left(e^{\int_0^T -r(s)ds} \max\{A(T)-L(T),0\}\right) \neq 0$ holds.

*Proof:*

The 'fair' terminal participation rate is a root of the continuous function $F(\eta) = \mathbf{E_Q}\left(e^{\int_0^T -r(s)ds}\left(L(T)+\eta \max\{A(T)-L(T),0\}\right)\right) - P$. From the above condition, we obtain $\mathbf{E_Q}\left(e^{\int_0^T -r(s)ds} \max\{A(T)-L(T),0\}\right) > 0$ and thus $\lim_{\eta\to-\infty} F(\eta) = -\infty$. In addition, we get

$$\begin{aligned} F(1) &= \mathbf{E_Q}\left(e^{\int_0^T -r(s)ds}\left(L(T)+\max\{A(T)-L(T),0\}\right)\right) - P \\ &\geq \mathbf{E_Q}\left(e^{\int_0^T -r(s)ds}\left(L(T)+A(T)-L(T)\right)\right) - P = 0 \end{aligned}$$

since $\left(e^{\int_0^t -r(s)ds} A(t)\right)_{t>0}$ is a $\mathbf{Q}-$martingale. Using the intermediate value theorem completes the proof.

*Remarks*

- $\eta < 0$ implies that the value of the insurance contract exceeds the initial premium payment $P$. Therefore, if the insurance contract doe not introduce arbitrage opportunities in the market a fair terminal bonus participation rate in $[0,1]$ can be found under the above condition.
- The condition above would only be violated if $L(T) > A(T)$ $\mathbf{Q}-$a.s., which in our framework is equivalent to $A(T)$ having zero volatility and the guaranteed rate exceeding the risk free rate.

In the next sections, we analyze the effects on the insurer's risk situation of a pricing strategy that makes use of this result: First, determine affected parameters – e, g, asset allocation or target interest rate – consistent with a pre-specified (real-world) tolerable risk or such that the respective risk is minimized. Then, compute the associated terminal bonus participation rate that makes the contract fair. If the resulting rate is below zero, the contract should not be offered because it would introduce arbitrage opportunities to the market. Of course, the results will depend on the selected risk measure.

# 4 Results

Within this section, we used following parameters:

| Interest rate model | | | | | Stock market model | | Correlation |
|---|---|---|---|---|---|---|---|
| $a$ | $b$ | $r(0)$ | $\sigma_r$ | $\lambda$ | $\mu$ | $\sigma_S$ | $\rho$ |
| 30% | 4,50% | 1,15% | 2,00% | -23% | 9% | 20% | 15% |

Additionally, the bond portfolio consists of equally balanced zero bonds with time to maturities of $1,\ldots,10$ years, meaning $x_{ij} := \frac{1}{10}, i = 1,\ldots,T, j = 1,\ldots,10$. In the following, we investigate a contract with lump sum $P$=1,000 and a time horizon of $T = 10$ years.

## 4.1 Point-to-point guarantee

First, we consider the point-to-point model, consisting of a guaranteed interest rate $i$ and terminal bonus participation rate $\eta$ in more detail.

***Risk analysis***

In the point-to-point model, there are closed form solutions for both, the shortfall probability and the expected shortfall.[11]

$$\mathbf{P}\big(A(T) < L(T)\big) = \Phi\left(\frac{\ln L(T) - \mu_{A(T)}}{\sigma_{A(T)}}\right)$$

and

$$\mathbf{E}_P\Big(\big(L(T) - A(T)\big)\mathbf{1}_{\{A(T)<L(T)\}}\Big) = L(T)\mathbf{P}\big(A(T) < L(T)\big) - e^{\mu_{A(T)} + \frac{\sigma^2_{A(T)}}{2}}\Phi\left(\frac{\ln L(T) - \mu_{A(T)} - \sigma^2_{A(T)}}{\sigma_{A(T)}}\right)$$

Since $\mu_{A(T)}$ and $\sigma_{A(T)}$ depend on the asset allocation, we first analyze how the risk depends on the asset allocation. Figure 1 shows the shortfall probability as a function of the insurer's asset allocation for a guaranteed interest rate of $i = 2.25\%$, the current guaranteed rate in Germany.

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[11] $\mu_{A(T)} = \mathbf{E}_P \ln A(T), \sigma^2_{A(T)} = \mathbf{Var}_P \ln A(T)$, compare Appendix A for a more detailed calculation. Further, $\Phi(.)$ denotes the cumulative distribution function of a standard normal random variable.

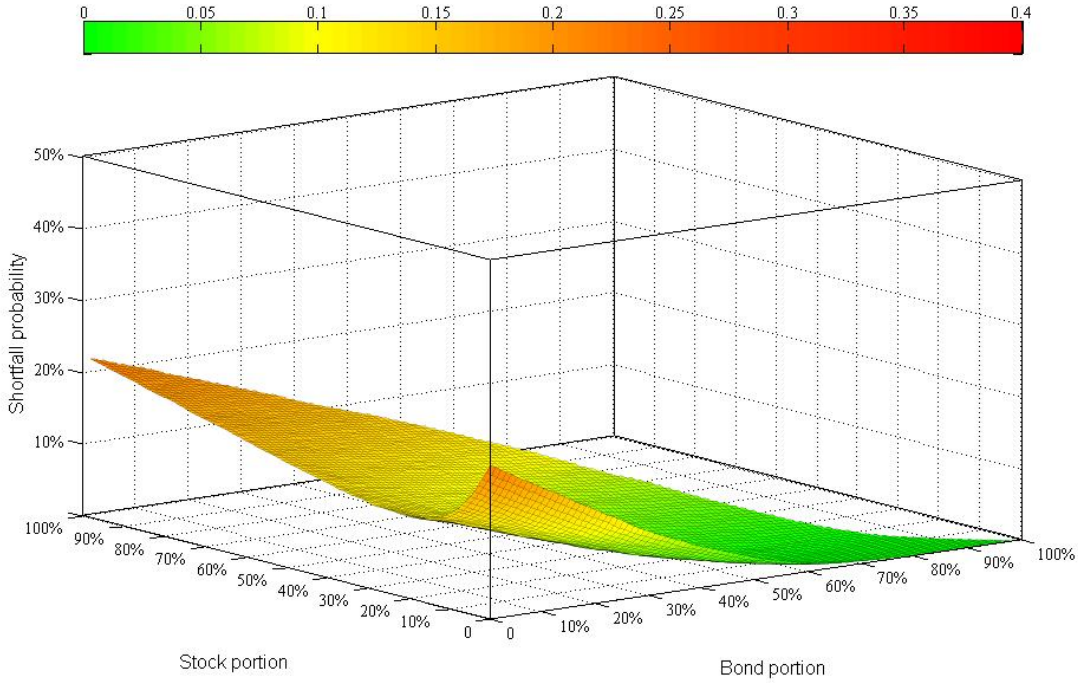


**Figure 1: Shortfall probability as a function of the insurer's asset allocation**

As described above, the asset allocation is given by a stock portion, a bond portion, and a money market portion that add up to 1. Thus, the asset allocation in the very front corner where stock portion and bond portion are 0 is the one where 100% of the assets are invested into the money market. The corresponding shortfall probability is 21% and thus almost as high as the riskiest which is reached if 100% stocks are held (22%). It is quite intuitive that a high stock portion corresponding with high volatility leads to high risk and thus a high shortfall probability but for a complete money market investment, this needs further explanation: The initial short rate is given by 1.15% and is thus significantly below the guaranteed rate of interest. Although the long term expectation of the short rate is 4.5%, the rather short time horizon of 10 years still leads to a significant probability of the money investment not achieving the guaranteed return.

For any fixed money market portion, the shortfall probability is first decreasing with increasing stock portions, reaches a local minimum between 2% and 20% stocks (depending on the bond portion) and is then increasing in the stock portion. Thus, our results show a clear diversification effect between stocks and other assets. In other words, the risk minimizing portfolio is not one with 0% stocks if the shortfall probability is the considered risk measure. The smallest shortfall probability is achieved for a 2% investment in stocks and 98% in bonds.

Now, we use the expected shortfall as our risk measure. Figure 2 displays the relative expected shortfall, i.e. the expected shortfall as a percentage of the initial premium $P$.

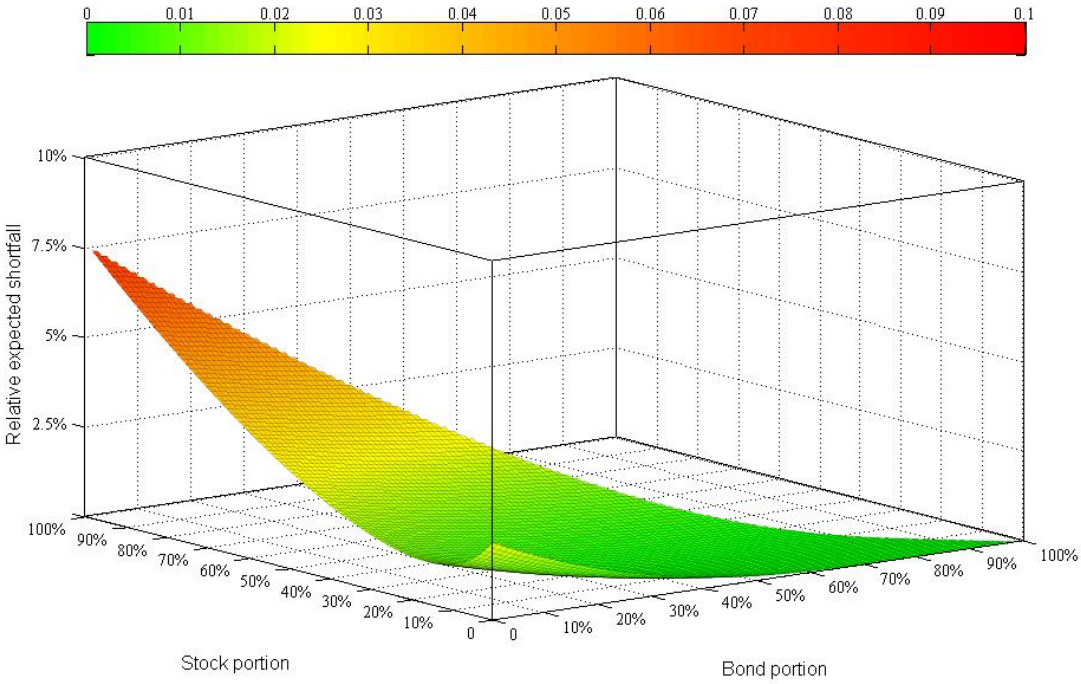


**Figure 2: Relative expected shortfall as a function of the insurer's asset allocation**

Using the expected shortfall as a risk measure also considers the extent of the shortfall. Thus, under this risk measure, rather high stock portions and thus rather high volatilities lead to higher risk. The highest risk is therefore achieved for a pure stock market investment. This significantly exceeds the expected shortfall related to a pure money market investment although the shortfall probabilities for the two asset allocations were very similar. The expected shortfall for a pure money market investment is equal to that of a 40% stock and 60% bond portion.

The risk minimizing strategy is very similar under both risk measures. Risk is still minimal for a 2% stock and 98% bond investment. Also, the diversification effect described above can also be observed: For any fixed money market portion, 0% stocks is not the risk minimizing strategy,

***Fair contracts***

In this section, the focus of our analysis is on the value of a contract from a client's point of view. As described in Section 3, we call a contract fair if the value of the payoff equals the premium paid. Figure 3 shows the terminal participation rates that make the contract fair.

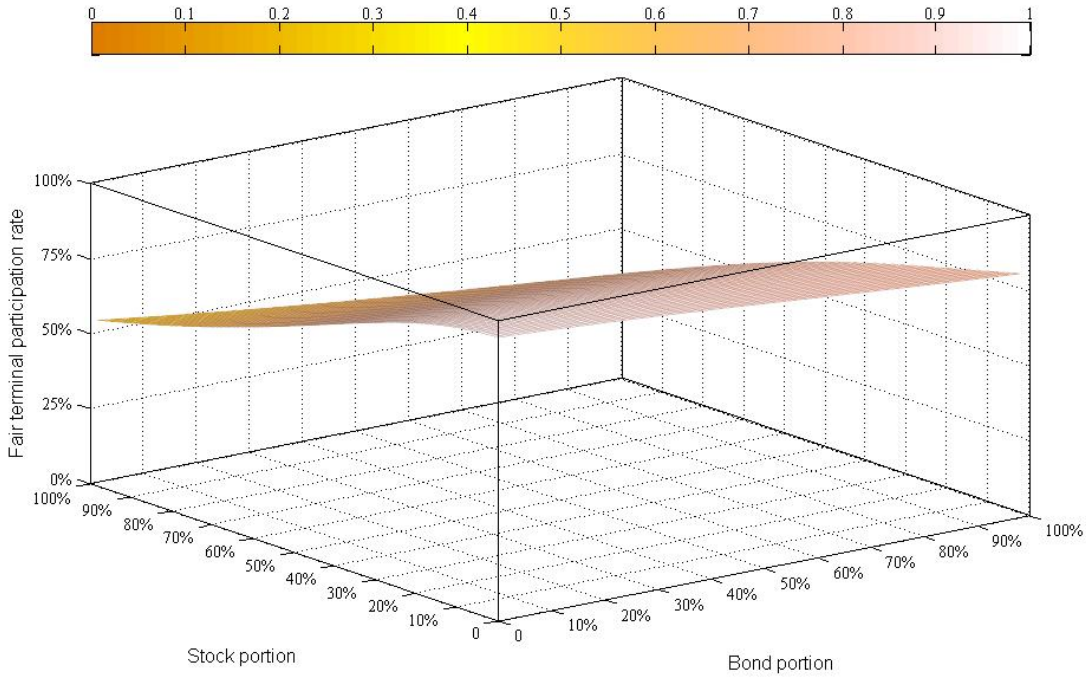


**Figure 3: Fair terminal participation rate as a function of the insurer's asset allocation**

All participation rates obtained are between 55% and 95% meaning that all possible asset allocations can be combined with an admissible terminal participation rate that makes the contract fair. Thus, for the considered point-to-point guarantee, the insurance company may first determine its asset allocation according to any given risk constraints and can then determine the terminal participation rate to make the contract fair in value.

If the insurance company mainly invests in money market instruments (which yields a fairly high shortfall probability and a medium expected shortfall) it needs to provide a very high terminal participation rate. In the extreme case of a pure money market investment, the fair terminal participation rate equals 94% which means that such an asset allocation would at the same time produce fairly high risk for the insurance company and (before terminal bonus participation) low expected returns for the client.

For different asset allocations the company can face significantly lower risk and at the same time needs less terminal bonus participation to make the contract fair.

***Optimal asset allocations***

Now, we will study risk-minimizing asset allocations for a given guaranteed interest rate $i$. For solving the corresponding optimization problem, we used a heuristic search algorithm based on Evolution Strategies[12].

The following figure shows the asset allocation that minimizes the shortfall probability for any guaranteed rate of interest $i$ as well as the corresponding shortfall probability.

[12] Compare e.g. Rechenberg (1994).

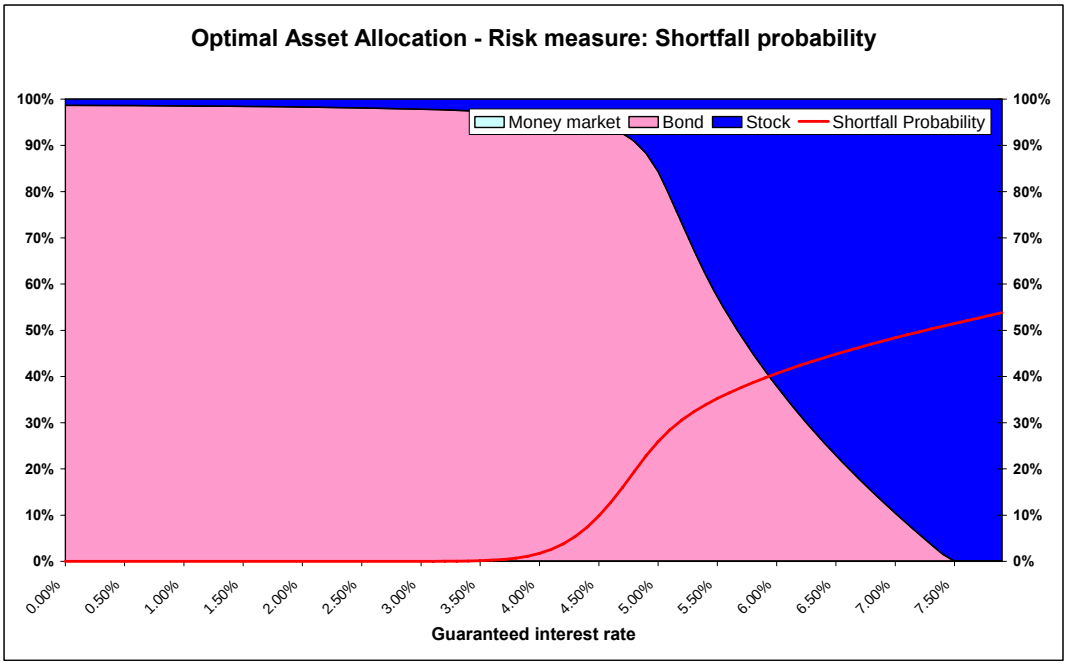


**Figure 4: Risk minimizing asset allocation as a function of the guaranteed rate of interest; Risk measure: Shortfall probability**

We observe that for any guaranteed rate of interest within the considered range, the risk minimizing portfolio contains stocks and bonds only.

As long as the guaranteed rate of interest is below 4% p.a., the insurance company is basically able to avoid shortfall risk. Even for a guarantee of 4%, the shortfall probability is below 2%. Of course, shortfall probabilities significantly increase for higher guarantees.

While for guaranteed rates of interest up to 4%, the risk minimizing asset allocation is fairly stable at a very low level of stocks (1% - 3%) and a very high portion of bonds, stock portions significantly increase for guaranteed rates of interest at or above the long term expectation of interest rates (4.5%). In the extreme case of a guaranteed rate of interest of 7.5% or greater, 100% stocks would result in the lowest shortfall probability.

At first glance, it seems to be somehow counterintuitive, that the most volatile investment strategy leads to the lowest risk. The reason for this, however, is quite obvious. If guaranteed rates are rather high (in particular higher than the long term expectation of interest rates), the probability of bond or money market investment reaching the guaranteed liability is much smaller than for a stock investment. Therefore, the corresponding shortfall probability is lower for the stock investment.

This brings up the question whether the shortfall probability is an adequate risk measure since it neglects the amount of shortfall that can be significantly higher for volatile stock investments.

Figure 5 therefore provides the same analysis where the relative expected shortfall is used as a risk measure.

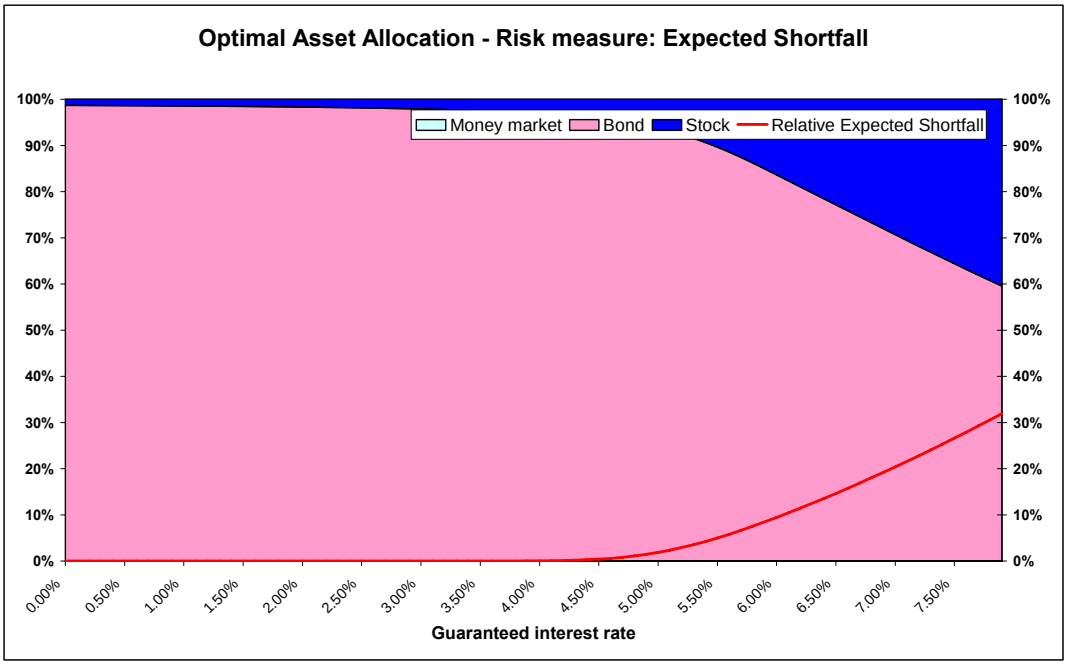


**Figure 5: Risk minimizing asset allocation as a function of the guaranteed rate of interest; Risk measure: Expected shortfall**

For guaranteed rates below 4%, the results do not differ significantly from those above. Since, however, the expected shortfall risk measure takes into account possible high shortfall amounts, the increase of the stock portion with increasing guaranteed rates of interest is significantly smaller. Risk minimizing stock portions stay below 35% for all considered values of *i*.

This shows how a risk management strategy based on shortfall probabilities only (such as for example a pure value at risk measure) can provide wrong incentives.

## 4.2 Must-Case

In the more complex setting of cliquet-style guarantees, no closed form solutions exist for the relevant distributions. Therefore, we need to rely on numerical methods to derive our results. Appendix A shows that the distribution of $\ln A(i+1) - \ln A(i)$ depends on the realization of $r(i)$ and therefore realizations of the multivariate normal distributed random variable $\left(\ln A(i+1) - \ln A(i), r(i+1)\right)_{i=0,\ldots,T-1}$ are required for the numerical analyses. We generate a normally distributed random sample using a Box-Muller transformation, cf. e.g. Fishman (1996). For each combination of parameters, 10,000 simulations were performed to calculate the Monte Carlo estimate for the shortfall probability and expected shortfall.

***Risk analysis***

Figure 6 shows the shortfall probability as a function of the insurer's asset allocation for a guaranteed interest rate of $i = 2.25\%$ in the MUST-Case, i.e. the case where only the legally required surplus is paid on top of the guaranteed rate of interest.

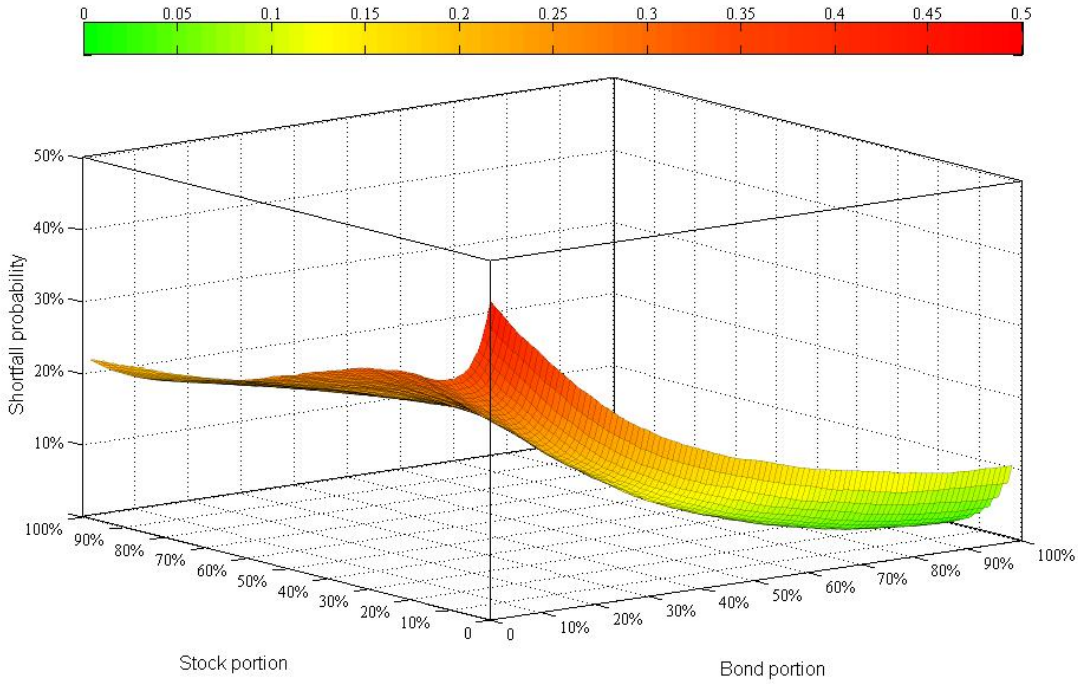


**Figure 6: Shortfall probability as a function of the insurer's asset allocation**

First, it is worth noting that compared to the point-to-point guarantee, shortfall probabilities are c.p. significantly higher if surplus is provided on an ongoing basis, even if the insurer only provides the surplus that is legally required. Since the annual surplus highly depends on book values and thus accounting rules, the influence of the asset allocation is different than in the point to point case.

The asset class with the highest degree of 'freedom in accounting' are stocks. An insurer investing 100% in stocks can postpone surplus participation even if stocks perform well. Therefore, capital market fluctuations can be somewhat "smoothed". In our model, for a pure buy-and-hold strategy in stocks, book value earnings are always 0. This leads to the special case where the must-case coincides with the point-to-point guarantee above and thus results in a shortfall probability of 22%.

On the other hand, if the insurer invests 100% in the money market where no freedom of accounting is available, the shortfall probability reaches its maximum at 44%, twice the value for a pure stock investment and also twice the value in the point-to-point case.

The risk minimizing asset allocation contains slightly more stocks than in the point-to-point case and is given by 90% bonds and 10% stocks.

Figure 7 shows the same results if the relative expected shortfall is used as a risk measure.

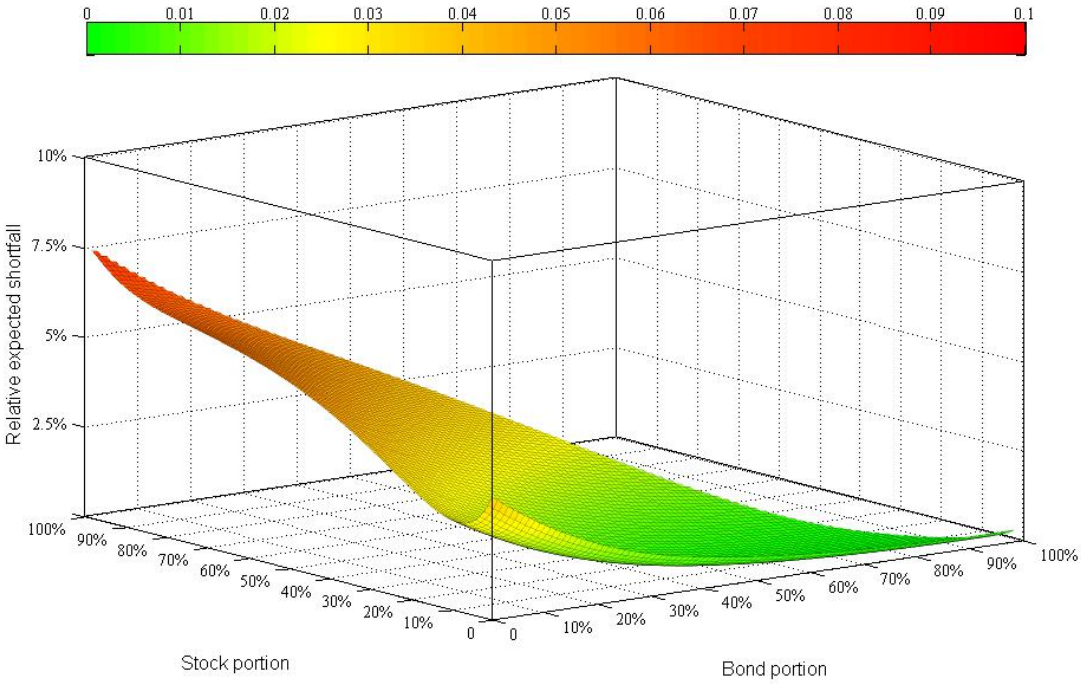


**Figure 7: Relative expected shortfall as a function of the insurer's asset allocation**

The results under the expected shortfall as a risk measure look rather similar to those from the point-to-point case. For the reason explained above, for a stock ratio of 100% the MUST-case and the point-to-point case coincide. Furthermore, 100% stocks turns out to be the riskiest asset allocation under this risk measure.

For a 100% money market investment, the expected shortfall increases by 60% as compared to the point-to-point case but is still not as risky as the pure stock investment. The risk-minimizing strategy is very similar to that achieved under the shortfall probability risk measure, roughly 10% stocks and 90% bonds.

***Fair contracts***

Similar to the point-to-point case, Figure 8 shows the terminal participation rates that make the contract fair.

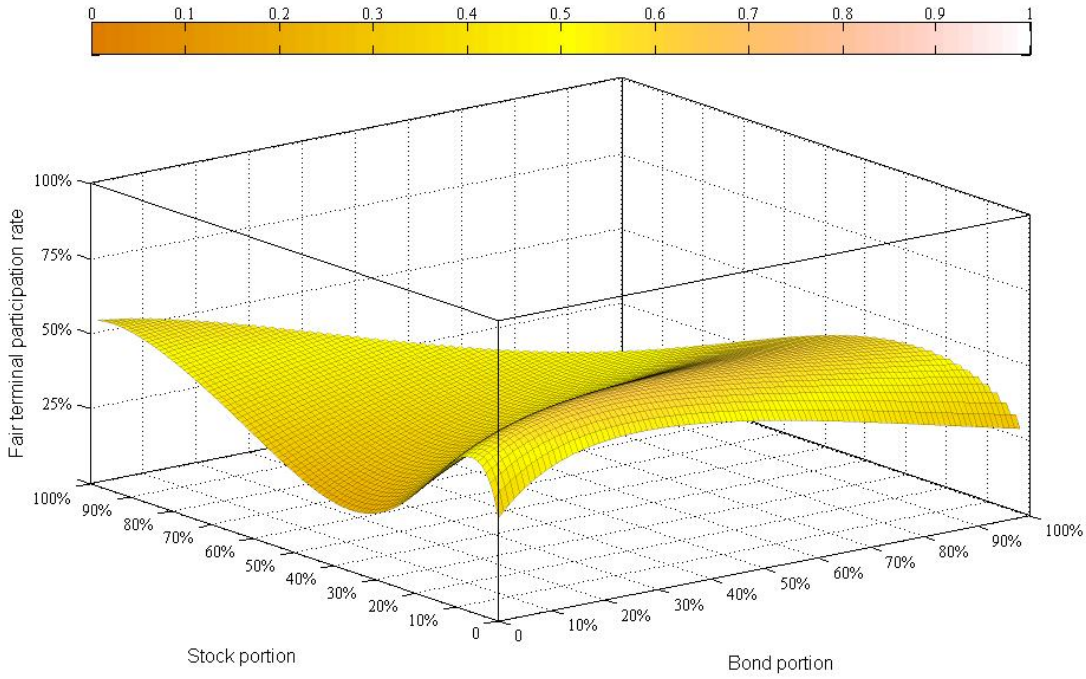


**Figure 8: Fair terminal participation rate as a function of the insurer's asset allocation**

All participation rates are between 20% and 63% and basically all are lower than in the point-to-point case. This on the one hand means that all possible asset allocations can be combined with an admissible terminal participation rate that makes the contract fair and on the other hand confirms that ongoing surplus in the MUST-case leads to some customer value which should be compensated by a lower terminal participation rate to provide the same value to customers.

Whereas a complete investment in the money market required the highest terminal participation rate in the point-to-point model, a portfolio consisting of 30% bonds, 8% stocks and 62% money market now – ceteris paribus – generates the lowest contract value before terminal surplus and therefore needs the highest participation rate to make the contract fair. In contrast, the risk-minimizing asset allocation, i.e. 90% bonds and 10% stocks, yields a rather high terminal surplus rate.

***Optimal asset allocations***

Figure 9 shows the risk minimizing asset allocation as a function of the guaranteed rate of interest if the shortfall probability is used as a risk measure.

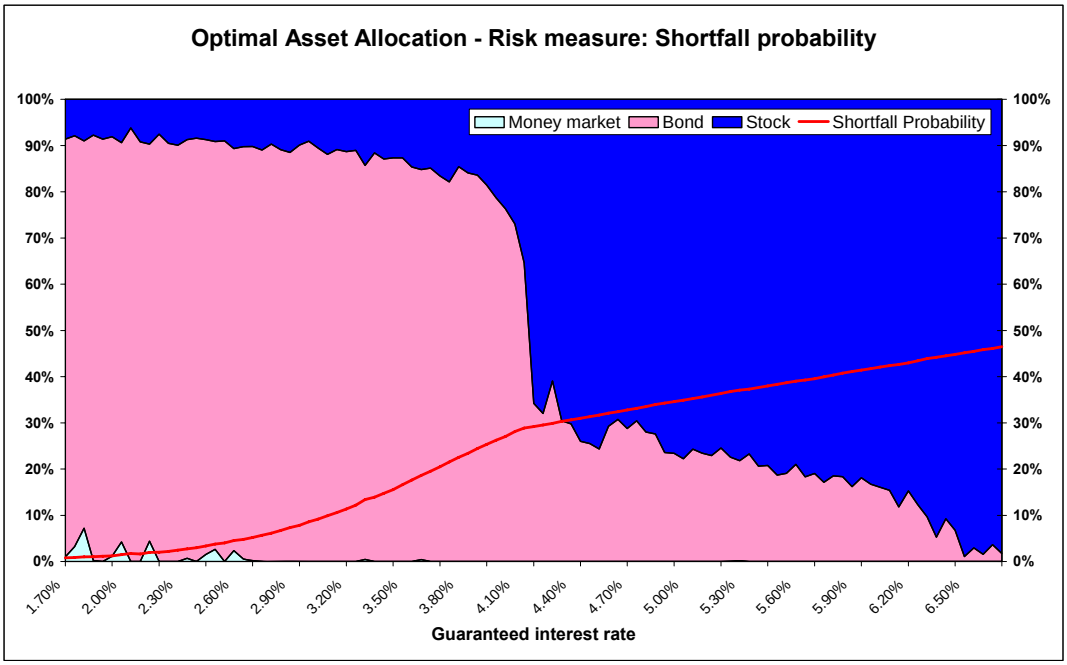


**Figure 9: Risk minimizing asset allocation as a function of the guaranteed rate of interest; Risk measure: Shortfall probability**

First of all, we observe that the results look less smooth when compared to similar results in the point-to-point model (cf. Figure 4). This is due to volatility of the Monte-Carlo estimate as well as the heuristic search algorithm. By increasing computational time any desired accuracy could be achieved.

For rather low guaranteed interest rates, the optimal asset allocation consists – similar to the point-to-point model – mainly of bonds and a fairly little stock exposure. However, whereas the insurer was basically able to eliminate the probability of shortfall following the risk-minimizing strategy in the previous section, here, even for low guaranteed interest rates some shortfall probability remains. This shows the additional risk cliquet-style guarantees induce.

As soon as the guaranteed interest rate exceeds 4% p.a. the stock portion of the risk minimizing portfolio increases heavily. Similar with the point-to-point model, bond and money market can not provide sufficient returns to cope with the rather high guaranteed rate and therefore a more risky portfolio is necessary to minimize the shortfall probability. Finally, for extremely high guaranteed interest rates, the risk minimizing portfolio consists entirely of stocks which then results in no regular surplus participation at all (since for 100% stocks the point-to-point model and the MUST-Case coincide).

However, note again that the shortfall probability neglects the amount of shortfall. Therefore similar analyses using the expected shortfall as target risk measure are shown in Figure 10.

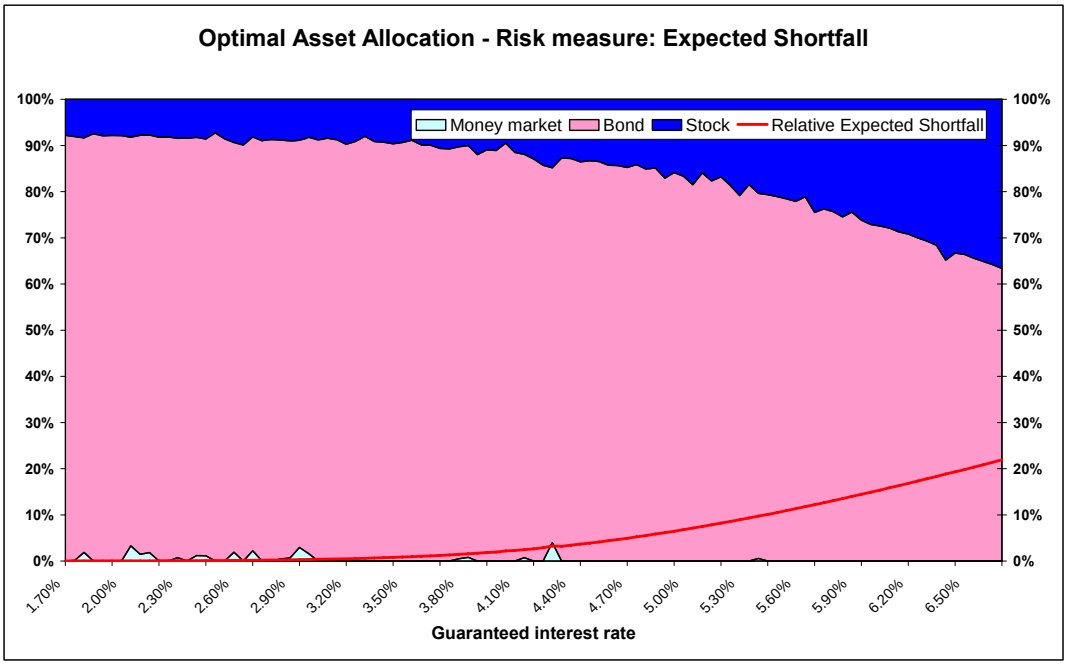


**Figure 10: Risk minimizing asset allocation as a function of the guaranteed rate of interest; Risk measure: Expected shortfall**

For guaranteed rates below 4% p.a., the risk minimizing asset allocation is very similar under both risk measures. However, for guaranteed interest rates above the long-term average interest rate, the stock portion of the risk minimizing asset allocation increases significantly slower if the expected shortfall is used as a risk measure, for the same reasons that have been explained in the point-to-point model.

## 4.3 Is-case

Finally, we show the results for the Is-Case as described in Section 2.4. For the following calculations we assumed a target interest rate of $z = 4{,}5\%$ and a lower respectively upper boundary for the reserve quota of $q_l = 5\%$ and $q_u = 30\%$. These parameters mean that the insurance company will keep surplus stable at 4.5% as long as its reserves stay within 5% and 30%.

***Risk analysis***

As above, we start with an investigation of the insurance company's risk exposure given different asset allocations using the shortfall probability as the relevant risk measure.

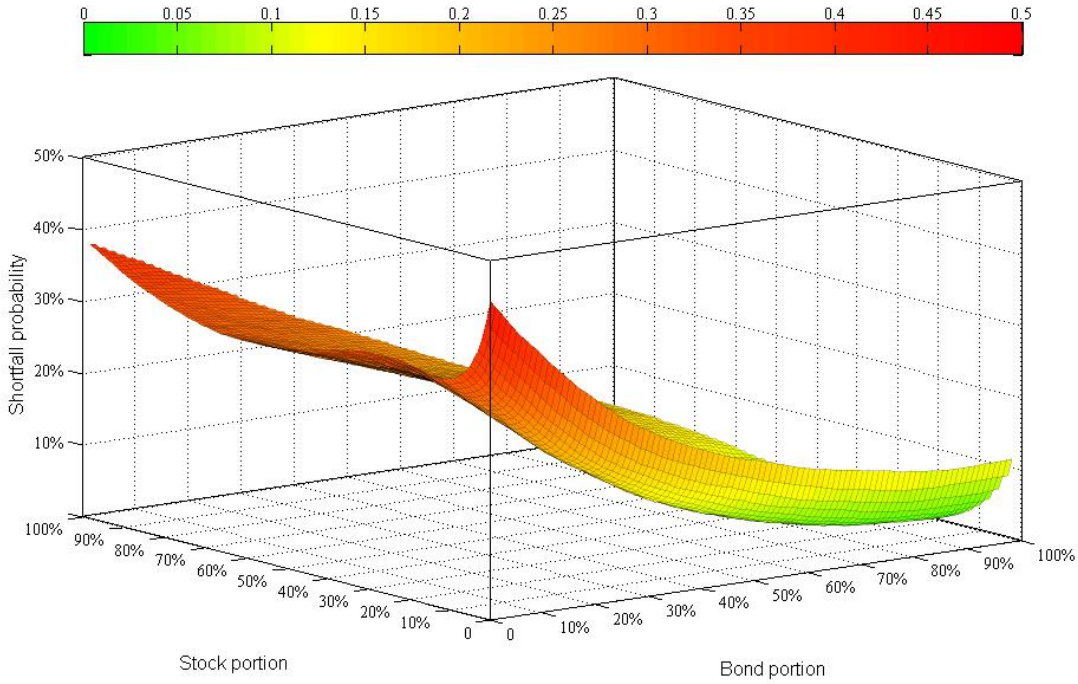


**Figure 11: Shortfall probability as a function of the insurer's asset allocation**

It is worth noting that, compared to the Must-Case, shortfall probabilities do not significantly increase for asset allocations with low stock ratios. This is once more due to accounting rules that already in the Must-case force the insurer to provide significant surplus participations. Consequently, the risk-minimizing strategy only changes slightly to 91% bonds and 9% stocks.

In contrast, for asset allocations with high stock ratios that potentially create rather high hidden reserves the insurer's risk tremendously increases under the Is-case. Giving away a larger portion of the reserves (e.g. during well-performing stock markets) than legally required, may afterwards jeopardize the insurance company's solvency since the guaranteed rate has to be credited on previous years' surplus, as well. In consequence, the higher the stock ratio, the less generous should the insurer be in giving away more regular surplus than legally required. This confirms findings of Kling, Richter and Russ (2007b) who state that high stock portions are not sustainable under cliquet-style guarantees.

Again, we further study the company's risk using the expected shortfall (cf. Figure 12).

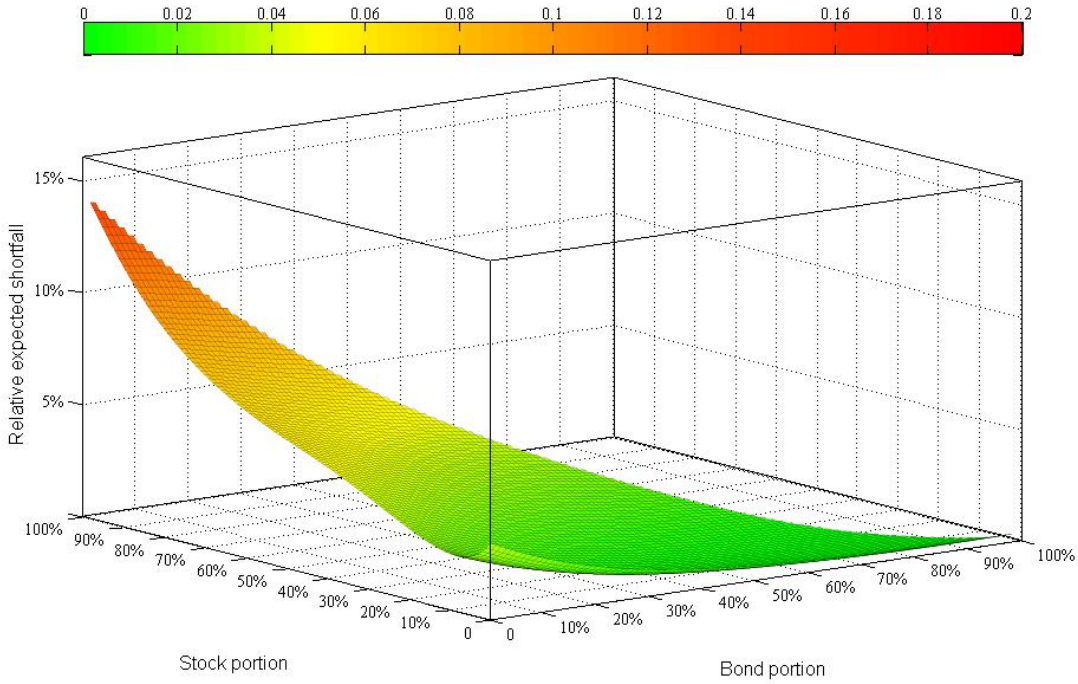


**Figure 12: Relative expected shortfall as a function of the insurer's asset allocation**

The risk for a complete investment in stocks increases similarly when compared to the Must-Case under for both risk measures. Interestingly, even in the IS-case, risk practically vanishes if the insurance company holds high bond portions and low equity portions. The risk minimizing strategy once more is given by roughly 90% bonds and 10% stocks.

While under the shortfall probability, 100% money market and 100% stock is equally risky, using the expected shortfall as a risk measure, clearly identifies a complete investment in stocks as the riskiest asset allocation . These findings are rather similar to those observed for the point-to-point guarantee or the MUST-case.

***Fair contract***

Similar to the previous models, Figure 13 shows terminal participation rates that make the considered contracts fair.

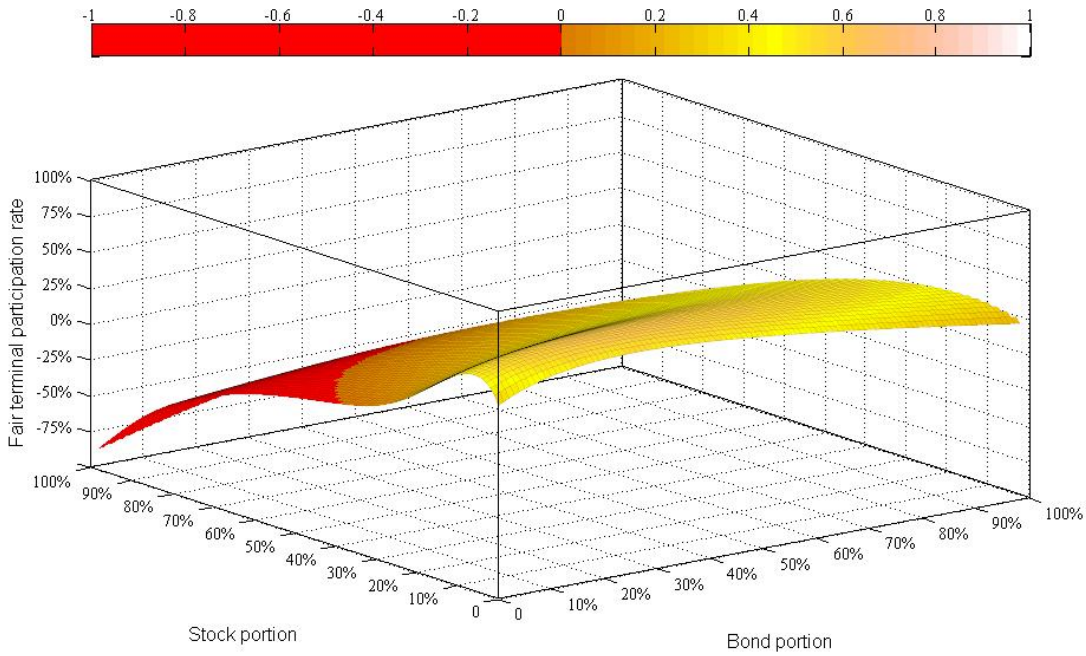


**Figure 13: Fair terminal participation rate as a function of the insurer's asset allocation**

Since ongoing surplus participation in the IS-case is already very valuable, terminal participation rates for all asset allocations stay below 60%. Further, in the Is-Case, there are asset allocations for which only a negative terminal participation rate would make the contract fair. This means that the value of the contract without terminal bonus already exceeds the single premium paid. These parameter settings therefore would create arbitrage opportunities in the market and should hence not be offered. The respective asset allocations are coloured in red in Figure 13 and are characterized by rather high stock ratios. Since the risk measure expected shortfall identified those to be particularly risky, both approaches, the actuarial and the financial one, lead to similar management decisions in this case.

### *Optimal asset allocations*

Finally, we investigate optimal asset allocations within the IS-Case. However, at this stage we focus on the effects of changing the target (as opposed to the guaranteed) interest rate. Figure 14 and Figure 15 show the risk-minimizing asset allocation as a function of the target interest rate with for a guaranteed interest rate $i = 2.25\%$, lower reserve boundary $q_l = 5\%$ and upper reserve boundary $q_u = 30\%$ using the shortfall probability and the expected shortfall, respectively.

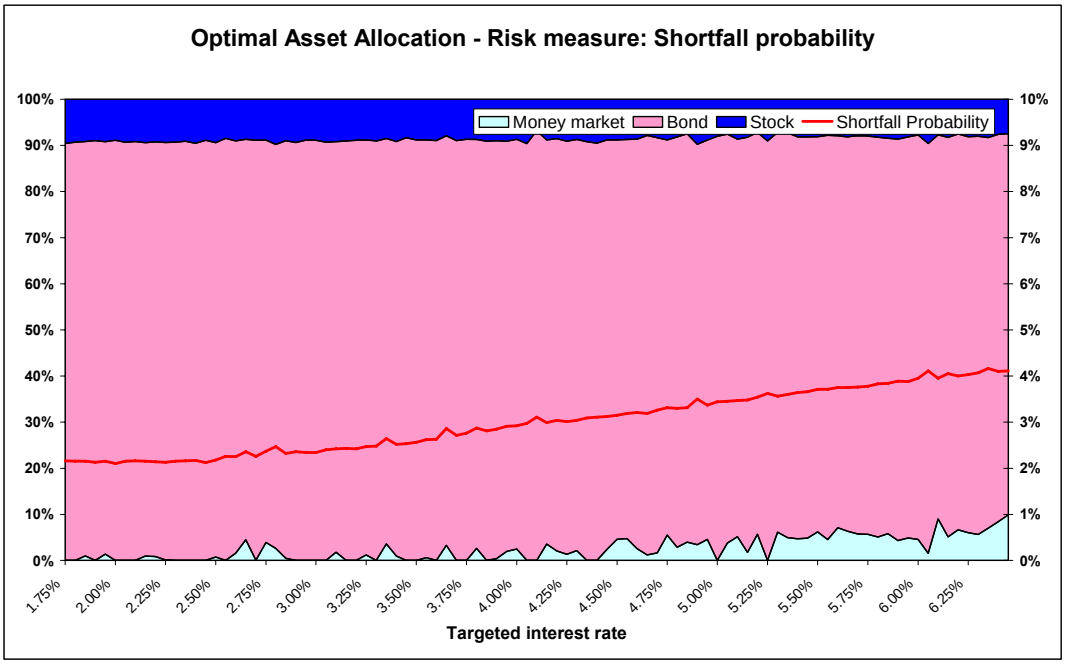


**Figure 14: Risk minimizing asset allocation as a function of the targeted rate of interest; Risk measure: Shortfall probability**

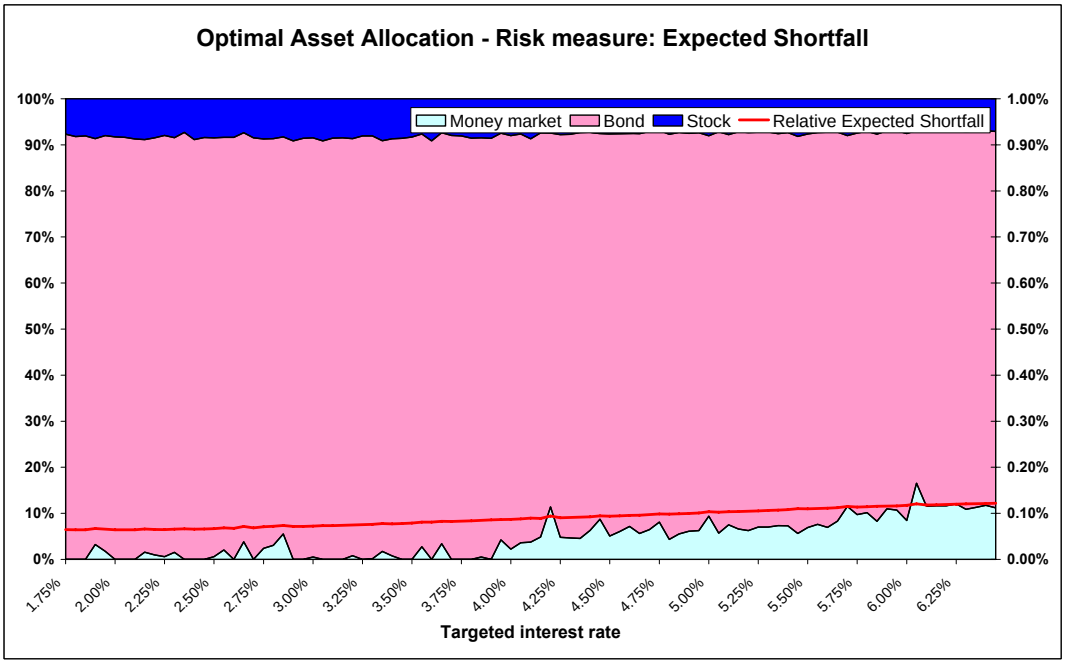


**Figure 15: Risk minimizing asset allocation as a function of the targeted rate of interest; Risk measure: Expected shortfall**

First, we note that – compared to the previous results – the different risk measures now yield very similar optimal asset allocations. The shortfall probability increases only slightly for increasing target interest rate. Therefore, comparing with the results above, the guaranteed rate (and not the target interest rate) is identified as being the main risk driver within our considered liability models.

For the chosen guaranteed rate of interest of 2.25% risk can basically be avoided independent of the target rate of interest. Relative expected shortfall remains almost constant around 0.2%. This shows that – if suitable asset allocations are chosen – the target rate of interest does not really have a practical impact on risk. However, the risk-minimizing asset allocation changes with a changing target rate. With increasing target rate, while the stock portion remains constant around 10%, the bond portion slightly decreases and the money market portion increases to 10%.

# 5 Conclusions and outlook

In this paper, we have analyzed three different types of participating life insurance contracts. The theoretical result from Section 3 shows, that – unless contract design introduces arbitrage to the market, it is always possible to combine actuarial and financial approaches such that a management of the insurer's asset-liability mismatch risk and a desired contract pricing can be achieved simultaneously. In our numerical analyses, we found that optimal, i.e. risk-minimizing, asset allocations as well as the amount of risk depend heavily on the selected liability modes (i.e. surplus distribution mechanism). Also, the results depend very strongly on the chosen risk measure. Our results indicate that under many circumstances, using the shortfall-probability as the sole risk measure can lead to wrong incentives. This should be of interest to practitioners as well as regulators when implementing value at risk-based regulation.

Of course, our model and analyses can and should be refined in future research. It would be particularly worthwhile including management rules that would allow for path dependent asset allocation strategies. Also, the question what an optimal bond portfolio under a given liability model would look like would be of great interest. Finally, the model could be made more realistic by including surrender and mortality and considering more that just one insurance contract on the insurer's balance sheet.

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# A Appendix

We will now proof that under the assumptions of Section 2.3 the insurer's asset portfolio follows a Geometric Brownian Motion. We will further briefly sketch computations of $\mu_{A(t)} = \mathbf{E}_P \ln A(t)$ and $\sigma^2_{A(t)} = \mathbf{Var}_P \ln A(t)$ for $t \in \{1,\ldots,T\}$.

Due to the self-financing property, the insurer's asset portfolio's dynamics can be written as $dA(t) = c_\beta(t)d\beta(t) + c_S(t)dS(t) + \sum_{j=1}^{T^*} c_{ij}(t)dp(t,i+j)$ at time $t \in [i,i+1)$. Thus,

$$\begin{aligned} dA(t) &= c_\beta(t)\beta(t)dr(t) + c_S(t)S(t)(\mu dt + \sigma_S dW_3(t)) + \sum_{j=1}^{T^*} c_{ij}(t)dp(t,i+j) \\ &= c_\beta(t)\beta(t)dr(t) + c_S(t)S(t)(\mu dt + \sigma_S dW_3(t)) + \\ &\quad \sum_{j=1}^{T^*} c_{ij}(t)p(t,T)((r(t) - \lambda\sigma_r B(t,T))dt - \sigma_r B(t,T)dW_1(t)) \end{aligned}$$

which yields

$$\begin{aligned} \frac{dA(t)}{A(t)} &= x_\beta(t)r(t)dt + x_S(t)(\mu dt + \sigma_S dW_3(t)) + \\ &\quad \sum_{j=1}^{T^*} x_{ij}x_B((r(t) - \lambda\sigma_r B(t,i+j))dt - \sigma_r B(t,i+j)dW_!(t)). \end{aligned}$$

Applying Itô-Formula for multiple processes[13] then gives

$$\begin{aligned} d\ln A(t) &= x_\beta(t)r(t)dt + x_S(t)(\mu dt + \sigma_S dW_3(t)) + \\ &\quad \sum_{j=1}^{T^*} x_{ij}x_B((r(t) - \lambda\sigma_r B(t,i+j))dt - \sigma_r B(t,i+j)dW_!(t)) \\ &\quad -\frac{1}{2}\left(\left(\sum_{j=1}^{T^*} x_{ij}x_B\sigma_r B(t,i+j)\right)^2 - 2x_S x_B \rho\sigma_S\sigma_r \sum_{j=1}^{T^*} x_{ij}B(t,j+i) + (x_S\sigma_S)^2\right)dt \end{aligned}$$

which implies

[13] For Itô processes $X_1(t), X_2(t)$ and a sufficiently smooth function $f(t,x_1,x_2)$ one obtains $df(t,X_1,X_2) = f_t dt + \sum_{j=1}^{2} f_{x_j} dX_j + \frac{1}{2}\sum_{i,j=1}^{2} f_{x_j,x_j} dX_j dX_j$, cf. e.g., Shreve (2004).

$$
\begin{aligned}
&\ln A(i+1)-\ln A(i)\\
&=\int_i^{i+1} d\ln A(t)\\
&=\underbrace{\int_i^{i+1}\left(x_\beta r(t)+x_S\mu+\sum_{j=1}^{T^*}x_{ij}x_B\left(r(t)-\lambda\sigma_r B(t,i+j)\right)\right)dt}_{(I_i)}\ +\\
&\underbrace{\int_i^{i+1}x_S\sigma_S dW_3(t)}_{(II_i)}-\underbrace{\int_i^{i+1}\left(\sum_{j=1}^{T^*}x_{ij}x_B\sigma_r B(t,i+j)\right)dW_1(t)}_{(III_i)}\ -\\
&\underbrace{\frac{1}{2}\left(\left(\sum_{j=1}^{T^*}x_{ij}x_B\sigma_r B(t,i+j)\right)^2-2x_Sx_B\rho\sigma_S\sigma_r\sum_{j=1}^{T^*}x_{ij}B(t,j+i)+\left(x_S\sigma_S\right)^2\right)dt.}_{(IV_i)}
\end{aligned}
$$

$(I_i)$ is a normal distributed integral, since $r(t)$ is normal distributed and the integral evolves as a limit of Riemann sums. $(II_i)$ and $(III_i)$ are standard stochastic integrals with respect to a Brownian Motion and with deterministic coefficients and therefore normal distributed as well. Since $(IV_i)$ is deterministic, $\ln A(i+1)-\ln A(i)$ follows a normal distribution. Hence, (since we assumed $x_{ij}\in F_0$), it follows that $\ln A(T)-\ln A(0)=\sum_{i=1}^{T}\ln A(i)-\ln A(i-1)$ is normal distributed as well which completes the proof.

We now briefly sketch the computation of the expectation $\mu_{A(t)}$ and the variance $\sigma^2_{A(t)}$ of $\ln A(t)$ for $t\in\{1,\ldots,T\}$. For the expectation and variance of $(I_i)$ defined above, we get

$$
\begin{aligned}
\mathbf{E}_P(I_i)&=\int_i^{i+1}\left(x_S\mu+\sum_{j=1}^{T^*}x_{ij}x_B\left(r(t)-\lambda\sigma_r B(t,i+j)\right)\right)dt+\mathbf{E}_P\left(\int_i^{i+1}\left(x_\beta r(t)\right)dt\right)\\
&=\int_i^{i+1}\left(x_S\mu+\sum_{j=1}^{T^*}x_{ij}x_B\left(r(t)-\lambda\sigma_r B(t,i+j)\right)\right)dt\\
&\quad+\mathbf{E}_P\left(x_\beta\int_i^{i+1}\left(e^{-at}r(0)+b\left(1-e^{-at}\right)+\int_0^t\sigma_r e^{-a(t-u)}dW_1(u)\right)dt\right)\\
&=\int_i^{i+1}\left(x_S\mu+\sum_{j=1}^{T^*}x_{ij}x_B\left(r(t)-\lambda\sigma_r B(t,i+j)\right)\right)dt+x_\beta\int_i^{i+1}\left(e^{-at}r(0)+b\left(1-e^{-at}\right)\right)dt
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{Var}_P(I_i)&=\mathbf{Var}_P\left(x_\beta\int_i^{i+1}\left(\int_0^t\sigma_r e^{-a(t-u)}dW_1(u)\right)dt\right)\\
&=x_\beta^2\mathbf{E}_P\left(\int_i^{i+1}\left(\int_0^t\sigma_r e^{-a(t-u)}dW_1(u)\right)dt\right)^2\\
&=x_\beta^2\int_i^{i+1}\int_i^{i+1}\mathbf{E}_P\left(\sigma_r e^{-at}\left(\int_0^t e^{au}dW_1(u)\right)\sigma_r e^{-as}\left(\int_0^s e^{au}dW_1(u)\right)\right)dtds\\
&=x_\beta^2\int_i^{i+1}\int_i^{i+1}\sigma_r e^{-a(t+s)}\mathbf{E}_P\left(\left(\int_0^{\min\{s,t\}}e^{au}dW_1(u)\right)^2\right)dtds\\
&=x_\beta^2\int_i^{i+1}\int_i^{i+1}\sigma_r e^{-a(t+s)}\left(\int_0^{\min\{s,t\}}e^{2au}du\right)dtds.
\end{aligned}
$$

The covariance of $(I_i,III_i)$ is given by

$$\begin{aligned}
\mathbf{Cov}_P\left(I_i, III_i\right) &= \mathbf{Cov}_P\left(x_\beta \int_i^{i+1}\left(\int_0^t \sigma_r e^{-a(t-u)} dW_1(u)\right)dt, \int_i^{i+1} x_S \sigma_S dW_3(u)\right) \\
&= x_\beta \sigma_r x_S \sigma_S \int_i^{i+1} \mathbf{Cov}_P\left(\int_0^t e^{-a(t-u)} dW_1(u), \int_i^{i+1} dW_3(u)\right)dt \\
&= x_\beta \sigma_r x_S \sigma_S \int_i^{i+1} \mathbf{Cov}_P\left(\int_i^t e^{-a(t-u)} dW_1(u), \int_i^t dW_3(u)\right)dt \\
&= x_\beta \sigma_r x_S \sigma_S \int_i^{i+1}\left(\rho \int_i^t e^{-a(t-u)} du\right), dt
\end{aligned}$$

After some tedious calculations of the expectations and the covariance matrix of $\left((I_i),(II_i),(III_i),(IV_i)\right)$ which follow essentially the same pattern, one can similarly achieve closed form solutions for $\mu_{A(t)} = \sum_{i=1}^t \mu_i + \ln A(0)$ and $\sigma^2_{A(t)} = \sum_{i=1}^t \sigma_i^2 + \sum_{\substack{i,j=1 \\ i \neq j}}^t \mathbf{Cov}_P\left(\left(\ln A(i) - \ln A(i-1)\right), \left(\ln A(j) - \ln A(j-1)\right)\right)$.