

# OPTIMAL STRATEGIES OF HEDGING PORTFOLIO OF UNIT-LINKED LIFE INSURANCE CONTRACTS WITH MINIMUM DEATH GUARANTEE

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## Abstract

In this paper, we are interested in hedging strategies allowing insurer to reduce risk of portfolio of unit-linked life insurance contracts with minimum death guarantee. Hedging strategies are developed in Black and Scholes model and in Merton jumps model. According to the new frameworks (IFRS, Solvency II and MCEV), risk premium is integrated in our valuations. We study optimality of hedging strategies by comparing risk indicators (Expected loss, volatility, VaR and CTE) related to costs generated by error of re-hedging and costs of transaction. We analyze robustness of hedging strategies by stress-testing future mortality and unit-linked.

**KEYWORDS:** Unit-linked, Death guarantee, Hedging strategies, Costs of transaction and error of re-hedging, risk indicators, stress-testing.

## Résumé

Dans ce papier, nous nous intéressons à la couverture des contrats en unités de compte avec garanties décès. Nous présentons des stratégies de couverture opérationnelles permettant de réduire de façon significative les coûts futurs liés à ce type de contrats. Suivant les recommandations des nouveaux référentiels (IFRS, Solvabilité 2 et MCEV), la prime de risque est introduite dans les évaluations. L'optimalité des stratégies est constatée au moyen de la comparaison des indicateurs de risque (Pertes espérée, écart type, VaR, CTE et perte Maximale) des stratégies dans le modèle standard de Black-Scholes et dans le modèle à sauts de Merton. Nous analysons la robustesse des stratégies à une hausse brutale de la mortalité future et à une forte dépréciation du prix de l'actif sous-jacent.

**MOTS-CLEFS :** Unités de comptes, Garanties décès, Stratégies de couverture, Coûts de transaction et erreur de couverture, indicateurs de risque, stress-testing.

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## 1. Introduction

The news frameworks (Accountant: IFRS/IAS, Prudential: Solvency II, and financial communication: Market Consistent Embedded Value) encourages the insurance companies to adopt an economic approach in the evaluations of their liabilities (Thérond (2007)). On this subject, the concept of “Fair Value” is fundamental. Fair Value of an asset or a liability is the amount in which two interested parties and also informed would be exchanged this asset or this liability. Fair values are usually taken to mean arbitrage free values, or values consistent with pricing in efficient markets. The arbitrage free valuation of an item is one which makes it impossible to guarantee riskless profits by buying or selling the item. This leads to the concept that if two portfolios have identical cash flows, and the portfolios can be priced in an efficient market, the two portfolios will have the same price. Otherwise, an investor can sell one portfolio, buy the other and make free money. So the Fair value is the price that the market naturally assigns to any tradable asset.

The risk neutral valuation produces a fair value of any liability. As call back in Milliman Consultants and actuaries (2005) the main reason for using risk neutral or fair valuations is because it represents the objective market cost of purchasing a replicating portfolio for the liability, thus ensuring that the company will have sufficient resources to meet the liability over all possible market movements. Risk-neutral valuation effectively translates the risky, market-dependent costs of the guarantee into a fixed cost item for the insurance company.

Thus, in logic of fair valuation, purchasing a replicating portfolio is essential in the evaluation of liabilities. Accordingly, in case of unit-linked for example, Frantz et al. (2003) show that fair valuation is only valid if the underlying hedging is actually applied<sup>1</sup>. In such contracts, the return obtained by the policyholder on its savings is linked to some financial asset. The policyholder thus supports the risk of the investment. The investment can be made on one asset support or on a portfolio of assets. Various types of guarantees can be added to the pure unit-linked contract. In our study, we shall concentrate on the minimum death benefit guarantee. In this case, the insurer’s liability in case of death of policyholder will be:  $\max(K, V) = V + [K - V]^+$ , where  $V$  is value of unit-linked and  $K$  the guarantee. If  $V < K$  then the insurer will pay the additional amount  $K - V$ . We can thus notice that the risk related to these contracts is real. However, this risk is often underestimated by the insurance companies, which so expose themselves to massive losses connected to a market in strong decline.

Frantz et al. (2003) analyze delta hedging in framework of model of Black & Scholes (1973). Black and Scholes model supposes the process of the returns is continuous, distributed according to a normal law and its volatility is constant during time. However, the empirical reports show that all these assumptions are not always true on markets as attest of the works of Cont (2001). Moreover, the classic valuation of unit-linked supposes a perfect mutualisation of the deaths in the insurance portfolio. We can wonder thus logically about the quality of the strategy of hedging setting-up by the insurer in case of future abnormally high death rate in the portfolio.

Moreover, other hedging strategies exist. Hedging strategies which we can develop come primarily from the methods used for the hedging derivatives. In practice, hedging a portfolio

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<sup>1</sup> Same if the application of the strategy is not always desirable or even feasible in practice.

of derivatives is typically done through matching of different sensitivities between the given portfolio and the hedging portfolio. As an alternative, a hedging portfolio can be chosen to minimize a measure of the hedge risk for a given time horizon.

The object of this paper is to analyze the optimality of some strategies of hedging being offered to the insurer to cover risk related to the unit-linked life insurance contracts with minimum death benefit guarantee. These contracts are subjected to two types of risk: financial risk and mortality risk.

Financial risk represents possibility of poorless evolution of Unit-linked and mortality risk results from possibility of strong fluctuation in sampling. In this last case, the future mortality of the portfolio is stronger than foreseen, may be due to not validity of the assumption of mutualisation of the deaths retained during the evaluation of the contract.

## 2. Risk-neutral valuation

Except the reasons already noted in the introduction, another reason for the using of the risk neutral valuation comes from the microeconomic theory of the uncertain. Indeed, remind two of the assumptions founders:

- The individuals prefer strictly more income unless income; (or that is equivalent less loss to more loss);
- The individuals are risk adverse.

The consequences of these assumptions on the choices of the agent are strong. Indeed, a risk adverse individual prefers to have the expectation of the random variable with probability 1 rather than the random variable it even. It means that between two games with identical expectation of earnings, the agent will choose the least risky. However, it will be inclined to change its choice if an additional amount is proposed to him. This amount is the risk premium.

Fair value must integrate this risk premium. This one reflects the risk adverse character of investors on markets. The incorporation of this risk premium allows the passage of the initial environment to a risk neutral environment. The valuation of financial assets is generally made in this risk neutral framework. The passage in this universe is made by means of the formulae of change of probability, the same justified by the Theorem of Radon-Nikodym.

The theory of deflators is an alternative in the risk neutral valuation. Generally used in Assets-Liabilities Management (ALM), the deflators are stochastic factors of actualization which make it possible to bring up the future flows of the liability. They allow obtaining a “Market Consistent” valuation of projected flows i.e. to find the initial value of risky assets.

Really, we have equivalence between the use of the deflators and the risk neutral valuation. Indeed, the deflator is nothing else than the density of the risk neutral measure according to the historic measure. The existence of this density results from the Radon-Nikodym Theorem.

The density risk neutral (or the deflator) depends on the nature of the studied risk. Within the framework of our study we shall be confronted with two risks: mortality risk and financial risk.

At first, we shall accept the assumption collectively admitted as for the risk of mortality namely: “the perfect mutualization of the deaths”<sup>2</sup>. Accordingly, the mortality risk

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<sup>2</sup> We will reconsider this assumption in a later study.

“disappears”, in the sense that we can foresee certainly the future number of deaths. But a mortality risk premium can be introduced by modeling mortality prudentially. In that case, the question of the level of prudence to adopt is open.

For the financial risk of managing, we will restrict ourselves to the Markovian models and in an efficient environment. Thus, we make the assumption of absence of arbitrage opportunity. Within this framework, one of the standard results of the financial theory is that all the assets are martingales under the risk neutral probability.

This specific character of assets under the risk neutral probability, besides the fact of simplifying the calculations, allows resolving the problem of actualization of generated future flows. To have Fair value it will be enough to generate the asset under the risk neutral probability and to make actualization by using of the free-risk rate.

### 3. Insurance portfolio

We suppose that portfolio is constituted by  $N$  policyholders who invest on the single financial asset support. Policyholder  $i$  aged  $x_i$  invest into a single risky asset  $(S_t)_{t \geq 0}$ . The insurer gives a guarantee of  $K_i$  in case the insured  $i$  dies before retirement. In case policyholder  $i$  dies at time  $T$  the insurer will pay  $S_T + [K_i - S_T]^+$  to the beneficiaries of insured  $i$ . Note that  $[K_i - S_T]^+$  is payoff of a European put option with maturity  $T$  and strike price  $K_i$  on the underlying asset  $(S_t)_{0 \leq t \leq T}$ .

The engagement of the insurer with respect to the policyholder  $i$  is written:  $e^{-rT_{x_i}} [K_i - S_{T_{x_i}}]^+ 1_{T_{x_i} \leq \tau_i}$ . Where  $T_{x_i}$  is time to death of policyholder,  $\tau_i$  is maturity of contract and  $r$  is risk-free interest rate<sup>3</sup>.

We note  $P$  the physical probability measure and  $Q$  the risk-neutral measure. We assume that physical and risk-neutral measures are independent. We also assume that in case of death in a period, insurer makes the payments at the end of the corresponding period<sup>4</sup>.

Now, we can write expression of single pure premium  $\Pi_i$  related of the contract:

$$\Pi_i = E_{P \times Q} \left[ e^{-rT_{x_i}} [K_i - S_{T_{x_i}}]^+ 1_{T_{x_i} \leq \tau_i} \right].$$

Using the assumption of independence between physical measure and risk-neutral measure, and using property of conditional expectation<sup>5</sup>, we can write:

$$\Pi_i = \sum_{t \leq \tau_i} \Pr(x_i, t) \times E_Q \left[ e^{-rt} [K_i - S_t]^+ \right].$$

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<sup>3</sup> We suppose constant.

<sup>4</sup> For example, for annual payments, the insurer pays the value of the Unit-linked with a guarantee on December 31 of each year.

<sup>5</sup> We are conditioning by the time to death.

With  $\Pr(x_i, t)$  is the probability, under the physical measure, that an individual aged  $x_i$  today dies exactly  $t$  years later.

Single pure premium  $\Pi$  about portfolio is:  $\Pi = \sum_{i=1}^N \Pi_i$ .

$$\Pi = \sum_{i=1}^N \sum_{t \leq \tau_i} \Pr(x_i, t) \times E_Q \left[ e^{-rt} [K_i - S_t]^+ \right] \Leftrightarrow \Pi = \sum_{i=1}^N \sum_{t \leq \tau_i} \Pr(x_i, t) \times P_0(t, S_0, K_i), \quad (3.1)$$

where  $P_0(t, S_0, K_i)$  is the price at times 0, of put on the underlying asset of strike  $K_i$  (guarantee of policyholder  $i$ ) of maturity  $t$ .

#### 4. Hedging Strategies

In the following, we investigate different kinds of hedging strategies allowing us to hedge optimally risk related of our insurance portfolio. Optimal strategy consists for the insurer to buy European options puts on the market. In that case, the value of the hedging portfolio is equal, all the time, to value of insurance portfolio. Hedging is perfect.

Cost of this strategy is:

$$L_{per} = \sum_{i=1}^N \sum_{t \leq \tau_i} \Pr(x_i, t) \times P_0(t, S_0, K_i) = \Pi.$$

Let us point out that  $\Pr(x_i, t)$  represents the probability, under the physical measure, that an individual aged  $x_i$  today dies exactly  $t$  years later. It is also the optimal quantity of option of maturity  $t$  to be held in the hedging portfolio. This optimal quantity is relevant only under the assumption of perfect mutualisation of the deaths between the policyholders and good anticipation of future mortality of insurance portfolio.

If insurer makes a good forecast of future mortality and the market has effectively all these options, this strategy allows the insurer to hedge optimally risk of portfolio. But it is not possible since the corresponding put options are hardly found, mainly due to the very long maturities involved. Moreover, the insurer is subjected to the risk related to a bad estimation of future mortality. It's necessary to think about others strategies which are taking account of these remarks.

We are going to adapt the traditional strategies of hedging derivatives (matching sensitivities and risk minimisation) to hedge our portfolio. We will build hedging strategies of option and extend it to the insurance portfolio assuming a perfect mutualisation of the deaths.

We will compare the results with the semi-static strategy of hedging of Carr & Wu (2004). This strategy supposes that it is possible to hedge an option of long run by a portfolio of options short maturity.

We study the relevance of hedging strategies by analyzing characteristics of discounted future costs  $L$ . The investors support costs during each of their deals. These costs reduce of so much the profitability of their operations. The consideration of these costs is essential in the evaluation of the performances of the stock-exchange portfolios.

On the market of share, the costs of transaction are generally decomposed into two components: the implicit component and the explicit component. The explicit costs correspond to expenses, commissions, taxes supported during the passage of an order, while the implicit costs return to the price ranges or to the impact of the deals on the prices for the large-sized transactions. The total cost of transaction appears as a sum of heterogeneous components which it is difficult to estimate. Deville (2001) shows that, the costs vary according to the level of capitalization of the underlying asset. He also shows that, except certain exceptional years, the total cost of transaction on the Paris Stock Exchange varies between 0.02 % and 1.20 % of exchanged value. In our study, we assume that costs of transaction represent a proportion  $c$  of the exchanged value.

Let  $P_T^H$  terminal value of the hedging portfolio and  $P_0^H$  initial value of the hedging portfolio. Let  $F$  total amount of friction i.e. sum of costs of transaction and errors of re-hedging. The expectation of discounted future costs is expectation of future payments<sup>6</sup>  $\Pi^*$  minus net value of the hedging portfolio ( $P_T^H - P_0^H$ ) corrected by costs of transaction and errors of re-hedging  $F$  :

$$L = \Pi^* - (P_T^H - P_0^H - F) \quad (4.1)$$

#### 4.1. Delta hedging

We want to match the sensitivity of underlying asset between the insurance portfolio and the hedging portfolio. It means that during an infinitely small time, hedging portfolio constituted by matching sensitivities is risk free. Our portfolio of hedging will be constituted by the underlying asset and the risk-free asset. This approach is linear, it is easy to extend hedging portfolio of an option to insurance portfolio.

##### 4.1.1. Frequently rebalancing

In the first time we are going to consider a strategy consisting in modifying the portfolio of hedging in a periodic way (of period  $h$ <sup>7</sup>). This technique requires frequent buying/selling assets in portfolio, and hence may incur significant transaction costs.

###### 4.1.1.1 Option hedging

Now consider option put  $P(n, S_0, K_i)$  with maturity  $n$  and strike  $K_i$ . Assuming that value of hedging portfolio at period  $t \leq n$  is written  $\alpha_t^{i,n} S_t + \beta_t^{i,n}$ .

Where:  $\alpha_t^{i,n}$  is the quantity of underlying asset in portfolio and  $\beta_t^{i,n}$  is quantity of risk-free asset in portfolio. We can write the expression of the error related to the hedging:

$$W_t(i, n) = -P(n-t, S_t, K_i) + \alpha_t^{i,n} S_t + \beta_t^{i,n}.$$

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<sup>6</sup>  $\Pi$  is the expectation of  $\Pi^*$  under risk-neutral measure,  $\Pi = E_Q(\Pi^*)$

<sup>7</sup> Year, 6 months, 3 months, 1 month, week, day, hour...

We want to immunize the error with fluctuations of the underlying asset. The composition of the portfolio must be such as:

$$\frac{\partial W_t}{\partial S_t} = 0 \text{ And } W_t = 0 \Leftrightarrow \begin{cases} \alpha_t^{i,n} = \frac{\partial P(n-t, S_t, K_i)}{\partial S_t} \\ \beta_t^{i,n} = P(n-t, S_t, K_i) - \frac{\partial P(n-t, S_t, K_i)}{\partial S_t} S_t \end{cases} \quad (4.2)$$

This hedging strategy could result in high costs: costs associated with the transactions and the errors of hedging. We can write values of costs of transaction and errors of hedging.

#### 4.1.1.2 Errors of hedging

The error of hedging,  $\widetilde{W}_k$  is the difference between the new portfolio made up at time  $k$  and the value of the portfolio made up at the previous period. This difference represents the amount exchanged at time  $k$ . It is also the cost of recombining of the portfolio of hedging.

$$\begin{aligned} \widetilde{W}_k(i, n) &= P(n-k, S_k, K_i) - \alpha_{k-1}^{i,n} S_k - \beta_{k-1}^{i,n} e^{rh} \\ \Leftrightarrow \widetilde{W}_k(i, n) &= (\alpha_k^{i,n} - \alpha_{k-1}^{i,n}) S_k + (\beta_k^{i,n} - \beta_{k-1}^{i,n} e^{rh}) \end{aligned}$$

#### 4.1.1.3 Costs of transaction

We add the cost of transaction  $\widetilde{C}_k$  in total costs of hedging strategy at time  $k$ , which constitutes a proportion  $c$  of the exchanged value:

$$\widetilde{C}_k(i, n) = c \left| \alpha_k^{i,n} - \alpha_{k-1}^{i,n} \right| S_k + c \left| \beta_k^{i,n} - \beta_{k-1}^{i,n} e^{rh} \right|$$

#### 4.1.1.4 Total frictions

Frictions are the total costs of transaction and errors of re-hedging associated to option hedging portfolio. We notice  $\widetilde{W}_{DY,1}$ <sup>8</sup> frictions of dynamic delta hedging with frequently rebalancing:

$$\widetilde{W}_{DY,1}(i, n) = c \left( \alpha_0^{i,n} S_0 + \beta_0^{i,n} \right) + \sum_{k=1}^{n/h-1} \left( \widetilde{W}_k(i, n) + \widetilde{C}_k(i, n) \right) e^{-k \times h \times r} \quad (4.3)$$

We can estimate discounted future costs of this strategy for put  $P(n, S_0, K_i)$ . We make  $N_s$  simulations of trajectories of underlying asset for maturity  $n$ . For simulation  $j$  we find friction  $\widetilde{W}_{DY,1}^j$  using (4.3). Discounted future costs,  $L_{DY,1}(i, n)$ , of dynamic delta hedging with frequently rebalancing of put  $P(n, S_0, K_i)$  is written:

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<sup>8</sup> *DY* indicates dynamic delta hedging and 1 frequently rebalancing, 2 rebalancing according to interval of error.

$$L_{DY,1}(i,n) = E \left( \left( \left[ K_i - S_n \right]^+ - (1-c) \left( \alpha_{n/h-1}^{i,n} S_n + \beta_{n/h-1}^{i,n} e^{h \times r} \right) \right) e^{-m} + \widetilde{W}_{DY,1}(i,n) + (1+c) \left( \alpha_0^{i,n} S_0 + \beta_0^{i,n} \right) \right)$$

#### 4.1.1.5 Insurance portfolio hedging

Let us remind that the insurer makes payments in a periodic time, according to the period  $h$ . Thus, his portfolio consists of  $N_d = \frac{T}{h}$  options of maturity  $1 \times h, \dots, i \times h, \dots, N_d \times h$ . We can easily extend the preceding results in the whole of the insurance portfolio.

At time  $n$ , for the simulation  $j$  and the insurer  $i$ , the amount to be paid in time  $n$  is:

$$M_j^{i,n} = \left[ K_i - S_n^j \right]^+ 1_{T_{x_i}^j = n}.$$

To extend this result to the whole portfolio, we are going to assume, like subsequently, a perfect mutualisation of the deaths between the policy-holders. So, we can write an estimation of discounted future costs of insurance portfolio  $L_{DY,1}$ :

$$\Rightarrow L_{DY,1} = \Pi_{Dyn,1} - \left( P_{Dyn,1}^H - P_{Dyn,1}^{H,0} - F_{Dyn,1} \right) \quad (4.4)$$

With:

$$\left\{ \begin{array}{l} \Pi_{Dyn,1} = \frac{1}{N_s} \sum_{i=1}^N \sum_{n=1}^{\tau_i} \sum_{j=1}^{N_s} M_j^{i,n} e^{-rn} \\ P_{Dyn,1}^H = \frac{1}{N_s} \sum_{i=1}^N \sum_{n=1}^{\tau_i} \sum_{j=1}^{N_s} (1-c) \Pr(x_i, n) \left( \alpha_{j,n/h-1}^{i,n} S_n^j + \beta_{j,n/h-1}^{i,n} e^{h \times r} \right) e^{-rn} \\ P_{Dyn,1}^{H,0} = \frac{(1+c)}{N_s} \sum_{i=1}^N \sum_{n=1}^{\tau_i} \Pr(x_i, n) \left[ \alpha_0^{i,n} S_0 + \beta_0^{i,n} \right] \\ F_{Dyn,1} = \frac{1}{N_s} \sum_{i=1}^N \sum_{n=1}^{\tau_i} \sum_{j=1}^{N_s} \Pr(x_i, n) \widetilde{W}_{DY,1}^j(i, n) \end{array} \right.$$

where:

- $N_s$  is the number of trajectories of underlying asset simulated,
- $\Pi_{Dyn,1}$  is expectation of future payments provide by dynamic delta hedging,
- $P_{Dyn,1}^H$  is The “final” value of hedging portfolio provide by dynamic delta hedging,
- $P_{Dyn,1}^{H,0}$  is The initial value of hedging portfolio of delta hedging,
- $F_{Dyn,1}$  costs of transaction and errors of re-hedging of delta hedging.



#### 4.1.2. Rebalancing according to error of hedging

An alternative of the first strategy consists in rebalancing the portfolio of hedging at the times  $k$  if the error of hedging is higher than a threshold.

##### 4.1.2.1 Hedging option

The idea is to recompose the portfolio hedging when the error of hedging in the period  $k$  goes out of an interval  $[a, b]$ . In the first time, we built hedging portfolio matching sensitivities of option and hedging portfolio at time  $k = 0$  :

$$\begin{cases} \alpha_0^{i,n} = \frac{\partial P(n, S_0, K_i)}{\partial S} \\ \beta_0^{i,n} = P(n, S_0, K_i) - \frac{\partial P(n, S_0, K_i)}{\partial S} S_0 \end{cases}$$

At time  $k$ , we make estimation of cost of re-hedging of portfolio of hedging

$$\widetilde{W}_k^2(i, n) = P(n-k, S_k, K_i) - \alpha_0^{i,n} S_k - \beta_0^{i,n} e^{rhk}.$$

If  $\widetilde{W}_k^2(i, n) \notin [a, b]$ , then we modify our hedging portfolio. Concretely, we build the meter  $(r_l)_{1 \leq l \leq l_{\max}}$  which identifies the moment of rebalancing of our hedging portfolio.

Then, if  $\widetilde{W}_k^2(i, n) \notin [a, b]$ ,  $r_1 = k$  and

$$\begin{cases} \alpha_{r_1}^{i,n} = \frac{\partial P(n-r_1, S_{r_1}, K_i, r)}{\partial S} \\ \beta_{r_1}^{i,n} = P(n-r_1, S_{r_1}, K_i, r) - \frac{\partial P(n-r_1, S_{r_1}, K_i, r)}{\partial S} S_{r_1} \end{cases}.$$

In the same way, for  $k > r_1$ , we calculate

$$\widetilde{W}_k^2(i, n) = P(n-k, S_k, K_i) - \alpha_{r_1}^{i,n} S_k - \beta_{r_1}^{i,n} e^{rh(k-r_1)},$$

$$\text{if } \widetilde{W}_k^2(i, n, j) \notin [a, b], \text{ then } r_2 = k, \text{ and } \begin{cases} \alpha_{r_2}^{i,n} = \frac{\partial P(n-r_2, S_{r_2}, K_i)}{\partial S} \\ \beta_{r_2}^{i,n} = P(n-r_2, S_{r_2}, K_i) - \frac{\partial P(n-r_2, S_{r_2}, K_i)}{\partial S} S_{r_2} \end{cases}$$

We continue until the payment of the pay-off.

By analogy with the first strategy, we can write total costs associated with the frictions, necessary to hedge put  $P(n, S_0, K_i)$  :

$$\widetilde{W}_{DY,2}(i, n) = c \left( \alpha_0^{i,n} S_0 + \beta_0^{i,n} \right) + \sum_{l=1}^{l_{\max}} \left( \widetilde{W}_{r_l}^2(i, n) + \widetilde{C}_{r_l}^2(i, n) \right) e^{-r \times h \times \eta},$$

where:

$$\widetilde{C}_\eta^2(i, n) = c \left| \alpha_\eta^{i, n} - \alpha_{\eta-1}^{i, n} \right| S_\eta + c \left| \beta_\eta^{i, n} - \beta_{\eta-1}^{i, n} \right|,$$

and

$$\widetilde{W}_\eta^2(i, n) = \left( \alpha_\eta^{i, n} - \alpha_{\eta-1}^{i, n} \right) S_\eta + \left( \beta_\eta^{i, n} - \beta_{\eta-1}^{i, n} \right).$$

Discounted future costs of this strategy for put  $P(n, S_0, K_i)$  are written:

$$L_{DY,1}(i, n) = E \left\{ \left( \left[ K_i - S_n \right]^+ - (1-c) \left( \alpha_{\eta_{\max}}^{i, n} S_n + \beta_{\eta_{\max}}^{i, n} e^{(\eta_{\max} - \eta_{\max-1})h \times r} \right) \right) e^{-rn} \right. \\ \left. + \widetilde{W}_{DY,2}(i, n) + \alpha_0^{i, n} S_0 + \beta_0^{i, n} \right\}$$

In this study, we choose the threshold like a percentage of the maximum pay-off. For example, for one put option with strike 100, the maximum pay-off is 100. This situation would arrive so the underlying asset was worth 0. A threshold with 1% of maximum pay-off be worth 1, and a re-hedging interval is  $[-1, 1]$ .

Another possibility consists in choosing the threshold according to total frictions. The threshold can be variable, related to the price of underlying asset.

#### 4.1.2.2 Insurance portfolio hedging

By analogy of frequently rebalancing, an estimation of discounted future costs  $L_{DY,2}$  for the insurance portfolio is:

$$L_{DY,2} = \Pi_{Dyn,2} - \left( P_{Dyn,2}^H - P_{Dyn,2}^{H,0} - F_{Dyn,2} \right), \quad (4.5)$$

where:

$$\left\{ \begin{array}{l} \Pi_{Dyn,2} = \frac{1}{N_s} \sum_{i=1}^N \sum_{n=1}^{\tau_i} \sum_{j=1}^{N_s} M_j^{i,n} e^{-rn} \\ P_{Dyn,2}^H = \frac{1}{N_s} \sum_{i=1}^N \sum_{n=1}^{\tau_i} \sum_{j=1}^{N_s} \left[ (1-c) \Pr(x_i, n) \left( \alpha_{j, r_{k \max}}^{i, n} S_n^j + \beta_{j, r_{k \max}}^{i, n} e^{(r_{k \max} - r_{k \max-1})h \times r} \right) e^{-rn} \right] \\ P_{Dyn,2}^{H,0} = \frac{(1+c)}{N_s} \sum_{i=1}^N \sum_{n=1}^{\tau_i} \Pr(x_i, n) \left[ \alpha_0^{i, n} S_0 + \beta_0^{i, n} \right] \\ F_{Dyn,2} = \frac{1}{N_s} \sum_{i=1}^N \sum_{n=1}^{\tau_i} \sum_{j=1}^{N_s} \Pr(x_i, n) \widetilde{W}_{DY,2}^j(i, n) \end{array} \right.$$

## 4.2. Risk minimisation strategies

The hedging strategy by minimization consists in optimizing hedging portfolio by checking the residual error. The main idea here is to build a portfolio such as the risk between

portfolio hedging and engagements of the insurer are minimal. In this approach, we also use the underlying asset and risk free asset to form a hedge portfolio.

In risk-neutral environment, it means minimizing economic value of the residual risk of cover. The economic indicators retained are numerous. The most used are the utility and the variance. The inconvenience of the first one is that it corresponds has a non-linear rule of pricing and hedging. We cannot extend it easily to our insurance portfolio. This strategy is not adapted. Second strategy is called quadratic minimisation, in spite of it penalizes the profits and the losses in the same way, and gives a linear ratio of cover. It can be prolonged thus easily has the insurance portfolio.

#### 4.2.1. Static hedging

Firstly, we will find static portfolio. It consists in building a hedging portfolio at the beginning of period and not modifying it until at the maturity.

##### 4.2.1.1 Option hedging

We assume that the hedging portfolio of put  $P(n, S_0, K_i)$  is constituted<sup>9</sup> by  $a$  ( $a \in \mathbb{N}^*$ ) assets, its value at  $n$  of is:  $V_n(\beta^{i,n}, a) = \sum_{l=1}^a \beta_0^{i,n,l} A_n^l$ . With  $\beta_0^{i,n,l}$  is quantity of asset  $A_n^l$  in portfolio. Error of this strategy is written:  $W_n(i, n) = -[K_i - S_n]^+ + V_n(\beta^{i,n}, a)$

If we consider only the risk-free asset, underlying asset and puts option on this hedging portfolio, we can write:

$$W_n(i, n) = -[K_i - S_n]^+ + \beta^{i,n,0} + \beta^{i,n,1} \times S_n + \sum_{l=2}^a \beta_0^{i,n,l} \times [K^l - S_n]^+ \quad (4.6)$$

$$W_n(i, n) = -[K_i - S_n]^+ + {}^t(\beta^{i,n}) P_n$$

Where  $\beta^{i,n} = \begin{bmatrix} \beta^{i,n,0} \\ \beta^{i,n,1} \\ \beta^{i,n,2} \\ \cdot \\ \cdot \\ \beta^{i,n,a} \end{bmatrix}$  and value at maturity  $n$  of our hedging portfolio  $P_n = \begin{bmatrix} 1 \\ S_n \\ [K^2 - S_n]^+ \\ \cdot \\ \cdot \\ [K^a - S_n]^+ \end{bmatrix}$

Optimal composition of static Portfolio hedging put is define by:

$$\min_{\beta^{i,n}} Risk(W_n(i, n)) \quad (4.7)$$

If we assume that measure of risk is the variance then (4.7) is equivalent to

<sup>9</sup> Note: we have the same assets in portfolio for all options.

$$\begin{cases} \min_{\beta^{i,n}} Risk(W_n(i,n)) \\ sc \ E^Q(W_n(i,n)) = 0 \end{cases} \quad (4.8)$$

The constraint,  $E^Q(W_n(i,n)) = 0$ , means that the error of hedging must be null on average.

If  $H(i,n) = [K_i - S_n]^+$ , it's equivalent making a regression of  $H(i,n)$  on  $P_n$ . If we simulate  $N_S$   $\bar{H}(i,n) = (H^j(i,n))_{1 \leq j \leq N_S}$  and  $\bar{P}_n = (P_n^j)_{1 \leq j \leq N_S}$ .

$$\begin{aligned} H^j(i,n) &= {}^t(\beta^{i,n}) \times P_n^j + e^j, \quad j = 1, \dots, N_S \\ \Leftrightarrow \bar{H}(i,n) &= {}^t\bar{P}_n \times \beta^{i,n} + E, \quad \text{Where, } E = [e^j]_{j=1, \dots, N_S} \\ \Rightarrow (\beta^{i,n})^* &= \left( (\bar{P}_n \times {}^t\bar{P}_n)^{-1} \times \bar{P}_n \right) \times \bar{H}(i,n) \end{aligned} \quad (4.9)$$

$W_n(i,n) = e$ , where  $e$  is a white noise.

Discounted future costs of this strategy for put  $P(n, S_0, K_i)$  are written:

$$L(i,n) = \frac{1}{N_S} \sum_{j=1}^{N_S} \left( \left( [K_i - S_n^j]^+ - (1-c)(\beta^{i,n})^* \times P_n^j \right) e^{-rm} + (1+c)(\beta^{i,n})^* \times P_0 \right). \quad (4.10)$$

#### 4.2.1.2 Insurance portfolio hedging

If we sum on the insurance portfolio, we can writing value of hedging portfolio in time 0:

$$\begin{aligned} P_{stl}^{H,0} &= \sum_{i=1}^N \sum_{n=1}^{\tau_i} \Pr(x_i, n) {}^t(\beta^{i,n}) P \\ &= \sum_{i=1}^N \sum_{n=1}^{\tau_i} \left( \Pr(x_i, n) \times \beta^{i,n,0} \right) e^{-nr} + \sum_{i=1}^N \sum_{n=1}^{\tau_i} \left( \Pr(x_i, n) \times \beta^{i,n,1} \right) S_0 \\ &\quad + \sum_{i=1}^N \sum_{n=1}^{\tau_i} \Pr(x_i, n) \sum_{l=2}^A \beta_0^{i,n,l} \times P_0(n, S_0, K^l) \\ &= B_0 + B_1 \times S_0 + \sum_{l=2}^A \sum_{i=1}^N \sum_{n=1}^{\tau_i} B^{i,n,l} \times P_0(n, S_0, K^l) \end{aligned} \quad (4.11)$$

where:

$$\left\{ \begin{array}{l} B_0 = \sum_{i=1}^N \sum_{n=1}^{\tau_i} \left( \Pr(x_i, n) \times \beta^{i,n,0} \right) e^{-nr} \\ B_1 = \sum_{i=1}^N \sum_{n=1}^{\tau_i} \left( \Pr(x_i, n) \times \beta^{i,n,1} \right) \\ B^{i,n,l} = \Pr(x_i, n) \times \beta_0^{i,n,l} \end{array} \right.$$

$B_0$  (resp.  $B_1$ ) is the quantity of risk-free asset (resp. underlying asset) necessary to cover risk of insurance portfolio. Our hedging portfolio will be only constituted by underlying asset and risk-free asset.

We can write an estimation of discounted future costs of insurance portfolio  $L_{ST}$  :

$$L_{ST} = \Pi_{Stat} - \left( P_{Stat}^H - P_{Stat}^{H,0} \right), \quad (4.12)$$

where:

$$\left\{ \begin{array}{l} \Pi_{Stat} = \frac{1}{N_s} \sum_{i=1}^N \sum_{n=1}^{\tau_i} \sum_{j=1}^{N_s} M_j^{i,n} e^{-rn} \\ P_{Stat}^H = \frac{1}{N_s} \sum_{i=1}^N \sum_{n=1}^{\tau_i} \sum_{j=1}^{N_s} (1-c) \Pr(x_i, n) \times \left( \beta^{i,n,0} e^{nr} + \beta^{i,n,1} S_n^j \right) \\ P_{Stat}^{H,0} = (1+c) P_{st}^{H,0} \end{array} \right.$$

The static hedging does not make it possible to the manager to integrate future extra information provided by the market. The dynamic hedging allows staging this lack. Dynamic minimisation consists in rebalancing continuously the hedging portfolio.

#### 4.2.2. Dynamic minimisation

We will improve the previous strategy. Instead of maintaining hedging portfolio unchanged until maturity we will modify it frequently.

We notice  $P_{Min,Dyn}^{H,t}(i, n)$  the value at  $t$  of hedging portfolio of option put  $P(n, S_0, K_i)$ .

$$P_{Min,Dyn}^{H,t}(i, n) = Cap + \beta_t^{i,n} e^{-r(n-t)} + \alpha_t^{i,n} S_t, \quad (4.13)$$

where  $Cap$  is initial capital.

Our portfolio is self-financing if we have:

$$\begin{aligned}
P_{Min,Dyn}^{H,t}(i,n) &= Cap + \beta_t^{i,n} e^{-r(n-t)} + \alpha_t^{i,n} S_t \\
&= Cap + \int_0^t \beta_s^{i,n} d\left(e^{-r(n-s)}\right) + \int_0^t \alpha_s^{i,n} d(S_s) \\
\Leftrightarrow \widehat{P}_{Min,Dyn}^{H,t}(i,n) &= Cap + \int_0^t \alpha_s^{i,n} d\widehat{S}_s. \text{ With } \beta_t^{0,i,n} = \widehat{P}_{Min,Dyn}^{H,t}(i,n) - Cap e^{-rt} - \alpha_t^{i,n} \widehat{S}_s
\end{aligned}$$

Valuation is making under the risk-neutral measure. Error of hedging is written:

$$W_n(i,n) = Cap + \int_0^n \alpha_s^{i,n} d\widehat{S}_s - e^{-rn} H(i,n), \text{ where } H(i,n) = [K_i - S_n]^+.$$

The optimal portfolio is given using the program of minimization of the variance:

$$\left(\beta_t^{i,n}, \alpha_t^{i,n}\right)_{0 \leq t \leq n} = \text{Arg} \left( \min E^Q (W_n(i,n))^2 \right). \quad (4.14)$$

This approach makes it possible to obtain the optimal composition of the portfolio at each period. Rebalancing is frequently. It is similar to the delta approach in neutral with frequently rebalancing, but the composition of the hedging portfolio changes according to model of underlying asset (see section 5)<sup>10</sup>.

By analogy with the dynamic delta hedging, we can write an estimation of discounted future costs:

$$\Rightarrow L_{Min,dyn} = \Pi_{Min,dyn} - \left( P_{Min,dyn}^H - P_{Min,dyn}^{H,0} - F_{Min,dyn} \right), \quad (4.15)$$

with:

$$\left\{ \begin{aligned}
\Pi_{Min,dyn} &= \frac{1}{N_s} \sum_{i=1}^N \sum_{n=1}^{\tau_i} \sum_{j=1}^{N_s} M_j^{i,n} e^{-rn} \\
P_{Min,dyn}^H &= \frac{1}{N_s} \sum_{i=1}^N \sum_{n=1}^{\tau_i} \sum_{j=1}^{N_s} (1-c) \Pr(x_i, n) \left( \alpha_{j,n/h-1}^{i,n} S_n^j + \beta_{j,n/h-1}^{i,n} e^{hxr} \right) e^{-rn} \\
P_{Min,dyn}^{H,0} &= \frac{(1+c)}{N_s} \sum_{i=1}^N \sum_{n=1}^{\tau_i} \Pr(x_i, n) \left[ \alpha_0^{i,n} S_0 + \beta_0^{i,n} \right] \\
F_{Min,dyn} &= \frac{1}{N_s} \sum_{i=1}^N \sum_{n=1}^{\tau_i} \sum_{j=1}^{N_s} \Pr(x_i, n) \widetilde{W}_{DY,1}(i, n, j)
\end{aligned} \right.$$

<sup>10</sup> The optimal ratio of hedging integrates the jumps risk, to see the comparison with the delta and the optimal ratio; we can see Tankov (2008).

### 4.3. Semi-Static hedging using short-term options

This strategy of hedging finds its relevance in the results of Breeden & Litzenberger (1978), who showed that the risk-neutral density relates to the second strike derivative of the put pricing function by:

$$f(S,t,K,T) = e^{r(T-t)} \frac{\partial^2 P}{\partial K^2}(S,t,K,T). \quad (4.16)$$

Using this result Carr P. & Wu L. (2004) stated the following theorem:

**Theorem:** Under no arbitrage and the Markovian assumption, the time- $t$  value of a European put option maturing at a fixed time  $T \geq t$  relates to the time- $t$  value of a continuum of European put options at a shorter maturity  $u \in [t, T]$  by:

$$P(S,t,K,T) = \int_0^\infty w(k) P(S,t,k,u) dk, \quad u \in [t, T] \quad (4.17)$$

For all possible nonnegative values of  $S$  and at all time  $t \leq u$ , the weighting function  $w(k)$  is

given by  $w(k) = \frac{\partial^2}{\partial k^2} P(k,u,K,T)$  (for proof see Carr & Wu (2004)).

This Theorem states the spanning relation in terms of put options. The spanning relation holds for all possible values of the spot price  $S$  and at all times up to the expiry of the options in the spanning portfolio. The option weights  $w(k)$  are independent of  $S$  and  $t$ . This property characterizes the static nature of the spanning relation. Under no arbitrage, once we form the spanning portfolio, no rebalancing is necessary until the maturity date of the options in the spanning portfolio.

In practice, investors can neither rebalance a portfolio continuously, nor can they form a static portfolio involving a continuum of securities. Both strategies involve an infinite number of transactions. In the presence of discrete transaction costs, both would lead to financial ruin. The number of put options used in the portfolio is chosen to balance the cost from the hedging error with the cost from transacting in these options. We approximate the spanning integral in equation by a weighted sum of a finite number  $a$  of put options at strike  $K_j$ ,  $j = 1, 2, \dots, a$ .

$$\int_0^\infty w(k) P(S,t,k,u) dk \approx \sum_{j=1}^a w(K_j) P(S,t,K_j,u). \quad (4.18)$$

Where we choose the strike points  $K_j$  and their corresponding weights based on the Gauss-Hermite quadrature rule<sup>11</sup>. To apply the quadrature rules, we need to map the quadrature nodes and weights  $\{x_j, w_j\}_{j=1}^a$  to our choice of option strikes and the portfolio weights  $W_j$ . One reasonable choice of a mapping function between the strikes and the quadrature nodes is given by:

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<sup>11</sup>See appendix D.

$$\kappa(x) = \text{Ke}^{\left( x\sigma\sqrt{2(T-u)} - \left( r + \frac{\sigma^2}{2} \right) (T-u) \right)},$$

where  $\sigma^2$  denotes the annualized variance of the log asset return. This choice is motivated by the gamma weighting function under the Black & Scholes model, which is given by:

$$q(\kappa) = \frac{\partial^2 P(\kappa, u, K, T)}{\partial \kappa^2} = \frac{\varphi(d_2)}{\kappa\sigma\sqrt{T-u}},$$

where  $\varphi(\cdot)$  denotes the probability density of a standard normal and the standardized variable  $d_1$  is defined as:

$$d_2 = \frac{\ln(\kappa/K) + \left( r - \frac{\sigma^2}{2} \right) (T-u)}{\sigma\sqrt{T-u}}.$$

We can then obtain the mapping in (9) by replacing  $d_1$  with  $\sqrt{2}x$ , which can also be regarded as a standard normal variable. Thus, given the Gauss-Hermite quadrature  $\{x_j, w_j\}_{j=1}^a$ , we choose the strike points as

$$K_j = \text{Ke}^{\left( x_j\sigma\sqrt{2(T-u)} - \left( r + \frac{\sigma^2}{2} \right) (T-u) \right)}$$

The portfolio weights are then given by

$$q_j = \frac{q(K_j)K_j'(x_j)}{e^{-x_j}} w_j = \frac{q(K_j)K_j\sigma\sqrt{2(t-u)}}{e^{-x_j}} w_j.$$

Conceivably, we can use different methods for the finite approximation. The Gauss-Hermite quadrature method chooses both the optimal strike levels and the optimal weight under each strike. This method is applicable to a market where options at many different strikes are available. Carr and Lu notes that more maturity of the options of cover is close to that covered more the cover is effective. The error of cover decrease with the number of hedging options selected. Our Hedging portfolio will be covered by options of maturity lower than  $u$  years. Discounted future costs of this strategy for put  $P(n, S_0, K_i)$ <sup>12</sup> are written:

$$L_{Carr}(i, n) = \frac{1}{N_s} \sum_{j=1}^{N_s} \left( \left( [K_i - S_n^j]^+ e^{-rn} - (1-cs) \sum_{j=1}^a q_j^{i,n} [K_j^{i,n} - S_u]^+ e^{-ru} \right) + (1+cs) \sum_{j=1}^a q_j^{i,n} P(u, S_0, K_j^{i,n}) \right)$$

Now we can write our portfolio hedging:

$$P_{Carr}^{H,0} = \sum_{i=1}^N \left( \sum_{n=1}^u \Pr(x_i, n) \times P_0(n, S_0, K_i) + \sum_{n=u+1}^{\tau_i} \left( \Pr(x_i, n) \times \sum_{j=1}^a q_j^{i,n} P(u, S_0, K_j^{i,n}) \right) \right), \quad (4.19)$$

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<sup>12</sup> where  $n > u$ .



where  $q_j^{i,n}$  and  $K_j^{i,n}$  represent weights and strike points necessities to hedge put  $P_0(n, S_0, K_i)$  with  $n > u$ . We can write an estimation of discounted future costs of insurance portfolio  $L_{Carr}$ :

$$\Rightarrow L_{Car} = \Pi_{Carr} - (P_{Carr}^H - P_{Carr}^{H,0}), \quad (4.20)$$

where:

$$\left\{ \begin{array}{l} \Pi_{Carr} = \frac{1}{N_s} \sum_{i=1}^N \sum_{n=1}^{\tau_i} \sum_{j=1}^{N_s} M_j^{i,n} e^{-rn} \\ P_{Carr}^H = \frac{1}{N_s} \left\{ \sum_{i=1}^N \sum_{n=1}^u \sum_{j=1}^{N_s} (1-cs) \Pr(x_i, n) [K_i - S_n]^+ e^{-rn} \right. \\ \quad \left. + \sum_{i=1}^N \sum_{n=u+1}^{\tau_i} \sum_{j=1}^{N_s} (1-cs) \Pr(x_i, n) \times \sum_{j=1}^a q_j^{i,n} [K_j^{i,n} - S_u]^+ e^{-ru} \right\} \\ P_{Carr}^{H,0} = \frac{(1+cs)}{N_s} P_{Carr}^{H,0} \end{array} \right.$$

## 5. Implementation of hedging

Now, we will implement the hedging strategies which we have just developed. For that, we will model the return of the underlying asset and mortality related to our insurance portfolio.

### 5.1. Mortality risk

Now we want to estimate the function of probability  $\Pr(x, n)$ . It is a standard problem in insurance life. A standard solution consists to estimate this probability using the mortality tables. Indeed, the risk of mortality is generally modeled through the mortality tables. These tables can be built starting from the data on mortality of the portfolio or with lawful mortality tables. The approach adopted here is adapted for one or the other of these methods.

In this case, we can write  $\Pr(x, n) = q_{x+n} \times {}_n p_x$ , with:

$$\left\{ \begin{array}{l} q_{x+n} = \frac{L_{x+n} - L_{x+n+1}}{L_{x+n}}, \text{ The instantaneous rate of mortality} \\ {}_n p_x = \frac{L_{x+n+1}}{L_{x+n}}, \text{ probability of survival until the old } x+n, \text{ for an insurer aged } x \end{array} \right.$$

Where  $(L_x)_{0 \leq x \leq X_{end}}$  is the mortality table which described the mortality in portfolio population.  $L_x$  is the number of survivors to each old.

We assume that evaluation is made at the end of the period of the death and company makes the payment this time. Here, we made an implicit approximation on the old one of the

policyholder. Indeed, the date of evaluation can not correspond to the date of birthday of the policyholder. But Planchet et al (2005) affirm that the precision brought by this adjustment is modest taking into consideration inaccuracy on the modelling underlying asset or the imperfect mutualisation of mortality.

In first time, we retain mortality table TH 00-02. This table was drawn up on the basis of observation INSEE of the French population between 2000 and 2002 and was smoothed. It is recommended as well for the evaluation of the contracts of the types death guarantee as for the life guarantee.

## 5.2. Modelling financial risk: Black and Scholes model

The standard modelling of the underlying asset is based on the model of Black & Scholes. Assumptions of the Black and Scholes Model are:

- The stock pays no dividends during the option's life
- Markets are efficient
- No commissions are charged
- Returns are log normally distributed
- Interest rates remain constant and known
- European exercise terms are used

We will reconsider these assumptions further.

$$\text{Dynamic of risk-free asset: } \begin{cases} S_0^0 = 1 \\ \frac{dS_t^0}{S_t^0} = rdt \end{cases}, \text{ where } r, \text{ the risk-free interest rate is constant.}$$

In this model, the underlying asset is assumed following classical geometric brownian motion, described by the following stochastic differential equation (SDE) under the physical probability measure:

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t$$

The solution of this SDE is:

$$S_t = S_0 \exp\left(\left(r - \frac{\sigma^2}{2}\right)t + \sigma W_t\right) \quad (5.1)$$

Under the risk neutral probability measure and the price of european put at t, with price strike K and maturity T is:

$$P^{BS}(T-t, S_0, K, \sigma) = Ke^{-r(T-t)}\Phi(-d_2) - S_0\Phi(-d_1), \quad (5.2)$$

where:

$$\begin{cases} d_1(S_0, K, \sigma, T-t) = d_2 + \sigma\sqrt{T-t} \\ d_2(S_0, K, \sigma, T-t) = \frac{\log\left(\frac{S_0}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \end{cases} \quad (5.3)$$

The delta is given by:  $\Delta^{BS}(T-t, S, K, \sigma) = \frac{\partial P^{BS}(T-t, S, K, \sigma)}{\partial S} = -\Phi(-d_1)$ .

In the environment of Black Scholes, the hedging strategy by minimization of the variance is identical to delta neutral hedging (See Appendix D).

Given the Gauss-Hermite quadrature  $\{x_j, w_j\}_{j=1}^a$ , the static portfolio strikes and weights, deduced by the Carr static hedging, are given by,

$$\begin{cases} K_j = \text{Ke}^{\left(x_j \sigma \sqrt{2(T-u)} - \left(r + \frac{\sigma^2}{2}\right)(T-u)\right)} \\ q_j = \frac{q(K_j) K_j \sigma \sqrt{2(t-u)}}{e^{-x_j}} w_j = \frac{1}{\sqrt{\pi}} w_j \end{cases} \quad j = 1, \dots, a$$

### 5.3. New underlying asset model: Merton's jump model

Black and Scholes assume ideal conditions in the market for the underlying asset and option. They suppose that the process of the underlying asset return belongs to the subfamily of the processes of diffusion, which rests on the Brownian movement. The problem of this family of process is that it supposes the continuity of the trajectories of price, which seems not very realistic when the reality of the markets is observed. Thus the observation of the prices asset reveals the presence of jumps, which can seem discontinuities in the trajectories of price. Other empirical reports made starting from the data of markets show that, contrary to what is envisaged by this model, implicit volatility is not constant. Its curve has even, in several cases, a convexity compared to the strike, a phenomenon known classically under the name of "smile of volatility". In addition, others empirical studies show that the distribution of return present an asymmetry on the left and tails of distribution thicker than those of a normal law, as attest of work of Cont (2001).

For all these reasons, like Merton (1976), we are going to introduce jumps in Black and Sholes model. The new model, Merton jump-diffusion model, obtained is not perfect since it does not make it possible to reproduce the asymmetrical character of the distribution of the underlying asset, but this new model provides closed semi formulas and less difficult to calibrate than Kou model (see Kou & Wang (2004)) or its extension (see Randrianarivony (2006)). We will analyze them what occurs within the framework of the jumps models. More exactly, we will be interested in the risk indicators within the framework of the Merton jumps model.

The Merton jump-diffusion model assumes the following risk-neutral dynamics for the underlying security price movement,

$$\frac{dS_t}{S_t} = dZ_t, \quad \text{with} \quad Z_t = rt + \sigma W_t + \sum_{j=0}^{N_t} L_j \quad (5.4)$$

where  $\sum_{j=0}^{N_t} L_j$  is a Poisson compound characterized by processes:  $(N_t)_{t>0}$  a Poisson process

which intensity  $\lambda$  and  $L_j$  is distributed according to a lognormal law of probability.

$\ln(1+L) \sim N(m, \sigma_u)$ . It is also assumed that  $N$ ,  $L$  and  $W$  are independent.

The jump risk is an idiosyncratic risk, it is the risk associated with individual assets. This specific risk can be reduced through diversification within a portfolio of assets (Markowitz (1952)).

Under a risk neutral probability measure solution of this stochastic differential equation is (Merton (1976)):

$$S_t = S_0 \exp \left\{ \left( r - \frac{\sigma^2}{2} \right) t + \sigma W_t + \sum_{j=1}^{N_t} \text{Log}(1+L_j) \right\} \quad (5.5)$$

And price of european option put at  $t$ , with price strike  $K$  and maturity  $T$  is written:

$$P^M(T-t, S_0, K, \sigma) = \sum_{n \geq 0} \frac{e^{-\lambda(T-t)} (\lambda(T-t))^n}{n!} P^{BS}(T-t, S_n, K, \sigma_n) \quad (5.6)$$

$$\text{where: } \begin{cases} -S_n = S_0 e^{nm + \frac{n\sigma_u^2}{2} - \lambda(T-t) \exp\left(m + \frac{\sigma_u^2}{2}\right) + \lambda(T-t)} \\ -\sigma_n = \sqrt{\sigma^2 + \frac{n\sigma_u^2}{2}} \end{cases}$$

### 5.3.1. Delta Hedging

Delta of this option is given by following expression:

$$\Delta^M = \frac{\partial P^M(T-t, S, K, \sigma)}{\partial S} = \sum_{n \geq 0} \frac{e^{-\lambda(T-t)} (\lambda(T-t))^n}{n!} \frac{S_n}{S} \Delta^{BS}(T-t, S_n, K, \sigma_n). \quad (5.7)$$

### 5.3.2. Dynamic minimisation

In framework of Merton, optimally hedging portfolio is given by (See Tankov & Voltchkova (2006) or Gabriel & Surlas (2006)):  $(\beta_t, \alpha_t)$  where,

$$\left\{ \begin{array}{l} \hat{c} = E_Q(e^{-rT} H) \\ \alpha_t = \frac{\sigma^2 \Delta^M + \frac{\lambda}{S} \left[ \left( e^{m + \frac{\sigma_u^2}{2}} P^M \left( t, S_t e^{m + \frac{3\sigma_u^2}{2}}, \sqrt{\sigma^2 + \frac{\sigma_u^2}{T-t}} \right) - P^M \left( t, S_t e^{m + \frac{\sigma_u^2}{2}}, \sqrt{\sigma^2 + \frac{\sigma_u^2}{T-t}} \right) \right) - P^M(t, S_t) \left( e^{m + \frac{\sigma_u^2}{2}} - 1 \right) \right]}{\sigma^2 + \lambda \left[ e^{2m + 2\sigma_u^2} - 2e^{m + \frac{\sigma_u^2}{2}} + 1 \right]} \\ \beta_t = e^{-rt} \left( P_{Min, Dyn}^{H,t}(t, T) - \hat{c} - \alpha_t S_t \right) \end{array} \right.$$

### 5.3.3. Carr hedging

The strike weighting function is given by:

$$\begin{aligned} q(\kappa) &= \frac{\partial^2 P^M(T-u, S, \kappa, \sigma)}{\partial \kappa^2} = \sum_{n \geq 0} \frac{e^{-\lambda(T-u)} (\lambda(T-u))^n}{n!} \frac{\partial^2 P^{BS}(T-u, S_n, \kappa, \sigma_n)}{\partial \kappa^2} \\ &= \sum_{n \geq 0} \frac{e^{-\lambda(T-u)} (\lambda(T-u))^n}{n!} \frac{e^{-r(T-u)} \varphi(d_2(T-u, S_n, \kappa, \sigma_n))}{\kappa \sigma_n \sqrt{T-u}} \end{aligned}$$

We define the strike price points based on the Gauss-Hermite quadrature  $\{x_j, w_j\}_{j=1}^a$  as follows:

$$K_j = Ke^{x_j \sqrt{2v(T-u)} - \left(r + \frac{v}{2}\right)(T-u)} \quad (5.8)$$

Where  $v = \sigma^2 + \lambda(m^2 + \sigma_u^2)$  is the annualized variance of the asset return under risk neutral measure. The portfolio weights are then given by

$$q_j = \frac{q(K_j) K_j \sqrt{2v(T-u)}}{e^{-x_j^2}} w_j \quad (5.9)$$

## 6. Results in Black-Scholes framework

The results presented below were obtained starting from the following parameters:

Number of simulation	10,000
Maturity of insurance portfolio	15 years
Volatility of underlying asset	25 %
Drift of underlying asset	8.5%
Risk-free interest rate	5%
Guarantee	100
Initial value of underlying asset	100
Frequency of observation portfolio	Monthly
Costs of transaction	1%
Number of short-term options using in Carr hedging	2
Maturity of short-term options using in Carr hedging	1 year
Insurance portfolio	1,000 insured aged 45
Interval of re-hedging in corrected delta hedging	[-1,1]

### 6.1. Results

The results obtained are summarized in Table 1.

**Table 1 - Risk indicators of costs related of Hedging strategies in Black-Scholes framework**

	Expected	Median	Standard deviation	VaR 99,75%	VaR 99%	CTE 99,75%	CTE 99%	maximum
No Hedging	950	400	1 177	5 006	4 459	5 007	4 463	6 124
Static minimisation	980	660	923	4 487	3 841	4 488	3 845	5 902
Delta Hedge	1 406	1 398	260	2 158	2 040	2 158	2 041	2 581
Delta2	1 140	1 137	328	2 179	1 965	2 179	1 966	2 605
Semi-Static Hedging	984	807	1 093	4 803	4 102	4 804	4 106	5 398

We remark that:

1. the taking into account of the hedging strategies will generate future costs on average higher than liabilities of insurer;
2. all the strategies of hedging reduce volatility of costs of the insurance portfolio. This result consolidates analyze of Frantz et al (2003);
3. the hedging strategies reduce the extreme losses (Value-at-risk: VaR and Conditional Tail Expectation: CTE). But Static minimisation generates maximum future cost higher than no hedging;

4. delta hedging, with rebalancing frequently, costs more expensive to the insurer. It provides the lowest volatility of the future costs;
5. the delta hedging with rebalancing the portfolio since the error of hedging is strong (Delta2), generates costs future on average weaker than frequently rebalancing, but generates also a greater volatility. However, the extreme losses are reduced.

In conclusion, the strategies in delta provide the best results that the other strategies. The strategy delta2 reduces the costs and the indicators of extreme risks (the VAR and CTE), but the potential maximal loss is more raised.

Figure 1 compares on the plan Mean-Variance<sup>13</sup> strategies of hedging.

**Figure 1 – Representation in the plan Mean-Variance of the hedging strategies**



According to the mean-variance criterion, we note that all the strategies are preferable to no hedging. We can also note that the delta strategy with frequently rebalancing provides the lowest volatility of the future costs whereas the semi-static strategy provides the weakest expected costs.

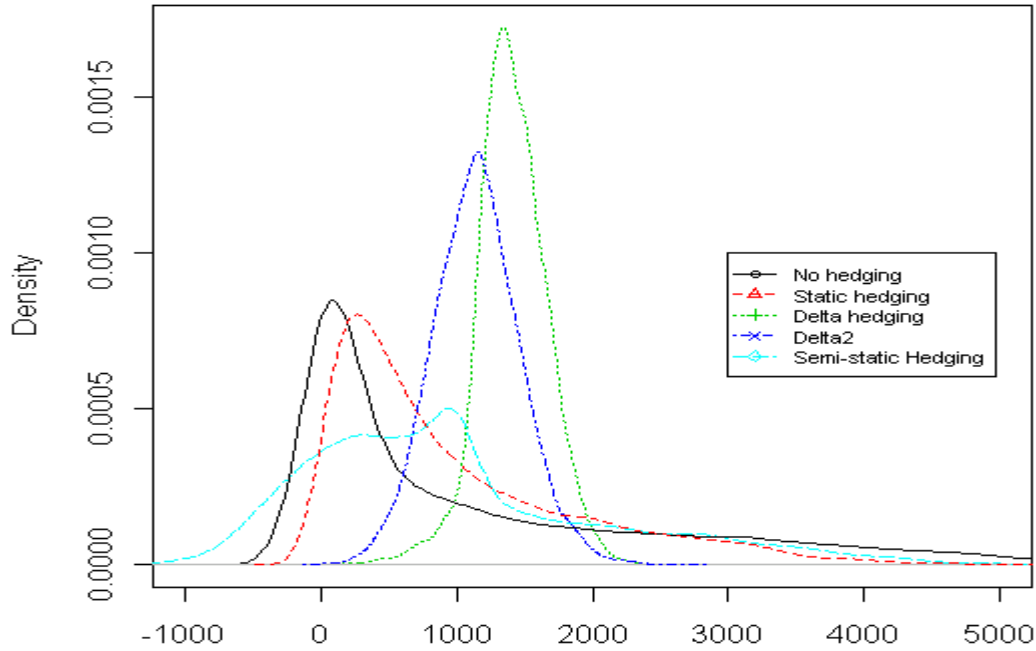
The delta strategies are always preferred to the static strategy and semi-static hedging.

If we are satisfied only with the mean-variance criterion, the choice between delta with frequently rebalancing and delta according to interval of hedging depends on the preference between a stronger volatility or a weaker expectation of costs.

If we are interested in the tails of distribution, the strategy in delta according to interval of hedging has the least thick tail of distribution, which is materialized by the weakest indicators of extreme risk.

The following Figure 2 shows density of costs of hedging strategies.

<sup>13</sup> S.D. : Standard Deviation

**Figure 2 – Density of discounted future costs of hedging strategies**

## 6.2. Robustness

Now, we are going to analyse risk indicators in case of modification of parameters and in case of shocks on mortality and underlying asset.

### 6.2.1. Impact of parameters choice

Firstly, we will compare hedging strategies in case of error in the choice of the parameters.

#### 6.2.1.1 Period of reallocation

The Table 2 shows variation of risk indicators between annual reallocation and monthly reallocation.

**Table 2 – Variation of Risk indicators of costs between annual and monthly reallocation**

	Expected	Median	Standard deviation	VaR 99,75%	VaR 1%	CTE 99,75%	CTE 99%	maximum
No Hedging	3%	8%	4%	4%	3%	4%	3%	4%
Static minimisation	2%	0%	5%	2%	3%	2%	3%	48%
Delta Hedge	-22%	-25%	80%	30%	17%	30%	17%	45%
Delta2 Hedging	-2%	-5%	66%	39%	30%	39%	30%	35%
Semi-Static Hedging	1%	-2%	4%	0%	5%	0%	5%	11%

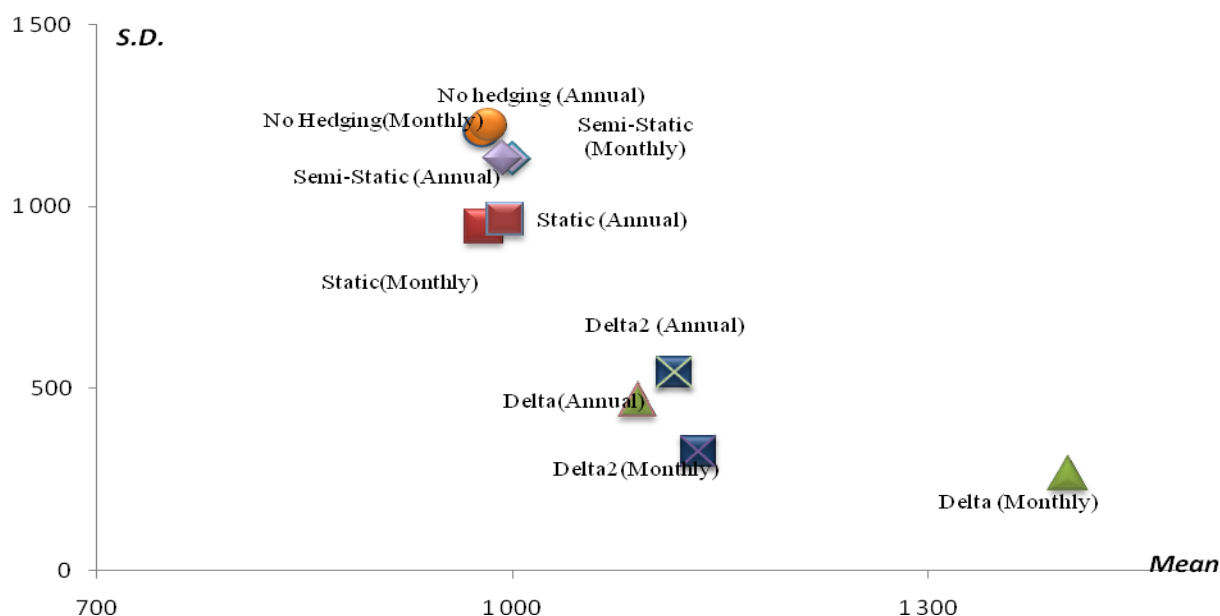
We can note that annual management is slightly expensive than monthly management of insurance portfolio. That is illustrated by a little rise of expectation of future costs (No Hedging) and volatility.



We can see that the passage to an annual management increases the indicators of risk of the whole of the strategies, except the strategies in delta, where we attend a reduction of the discounted future costs.

Figure 3 shows in the plan mean-variance evolution of hedging strategies related of two kinds of reallocation: annual reallocation and monthly reallocation.

**Figure 3 – Impact of period of reallocation in the plan Mean-Variance**



We also see that delta strategies are also better than others strategies.

#### 6.2.1.2 Costs of transaction

We can see in Table 3 impact of costs transaction increase of risks indicators.

**Table 3 – Variation of Risk indicators of costs related of 1% increase of costs of transaction**

	Expected	Median	Standard deviation	VaR 99,75%	VaR 99%	CTE 99,75%	CTE 99%	maximum
No Hedging	0%	0%	0%	0%	0%	0%	0%	0%
Static minimisation	4%	5%	1%	3%	0%	3%	0%	-35%
Delta Hedge	31%	31%	32%	29%	30%	29%	30%	26%
Delta2 Hedging	15%	15%	5%	14%	12%	14%	12%	12%
Semi-Static Hedging	1%	-1%	0%	-4%	-2%	-4%	-2%	2%

We note that the strategy in delta with frequently rebalancing is more sensitive to a rise of the costs of transactions.

The Table 4 shows risk indicators in situation of rise in the costs of transaction from 1% to 2%.

**Table 4 – News Risk indicators of COSTS with 2% cost of transaction**

	Expected	Median	Standard deviation	VaR 99,75%	VaR 99%	CTE 99,75%	CTE 99%	maximum
No Hedging	950	400	1 177	5 006	4 459	5 007	4 463	6 124
Static minimisation	1 018	684	954	4 649	3 988	4 651	3 994	9 007
Delta Hedge	1 840	1 832	352	2 814	2 639	2 814	2 640	3 088
Delta2 Hedging	1 310	1 295	342	2 453	2 191	2 454	2 192	2 909
Semi-Static Hedging	1 009	805	1 128	4 845	4 261	4 846	4 265	6 082

We can see that the delta according to interval of hedging gives lowest volatility and extreme risk indicators.

#### 6.2.1.3 Delta 2 hedging: impact of interval of re-hedging

Here, we analyze the impact of the interval of re-hedging on the risk indicators of the strategy in delta2. In Table 5 we have the results of simulations for the intervals of re-hedging  $[-1,1]$ ,  $[-2,2]$  and  $[-3,3]$ .

**Table 5 - Risk indicators of costs related of Hedging strategies in Black-Scholes framework**

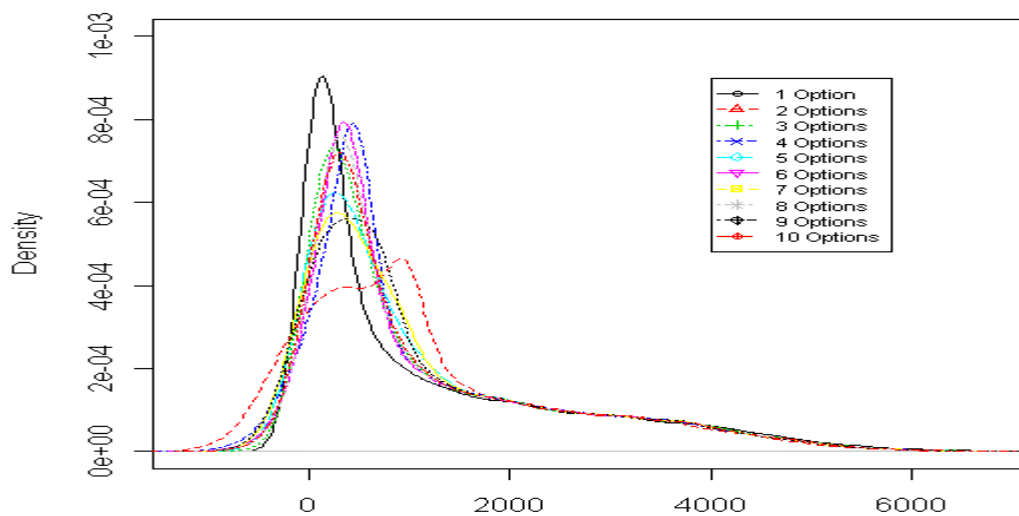
	Expected	Median	Standard deviation	VaR 99,75%	VaR 99%	CTE 99,75%	CTE 99%	maximum
No Hedging	950	400	1 177	5 006	4 459	5 007	4 463	6 124
Static minimisation	980	660	923	4 487	3 841	4 488	3 845	5 902
Delta Hedge	1 406	1 398	260	2 158	2 040	2 158	2 041	2 581
Delta2 [-1,1]	1 140	1 137	328	2 179	1 965	2 179	1 966	2 605
Delta2 [-2,2]	1 102	1 100	396	2 279	2 060	2 279	2 062	2 869
Delta2 [-3,3]	1 082	1 084	457	2 420	2 158	2 420	2 160	3 103
Semi-Static Hedging	984	807	1 093	4 803	4 102	4 804	4 106	5 398

Globally, we note that increase of the interval of re-hedging degrades quality of volatility and tail risk indicators (VaR and CTE). However, we note a light decrease of expectation of future costs.

#### 6.2.1.4 Semi static hedging

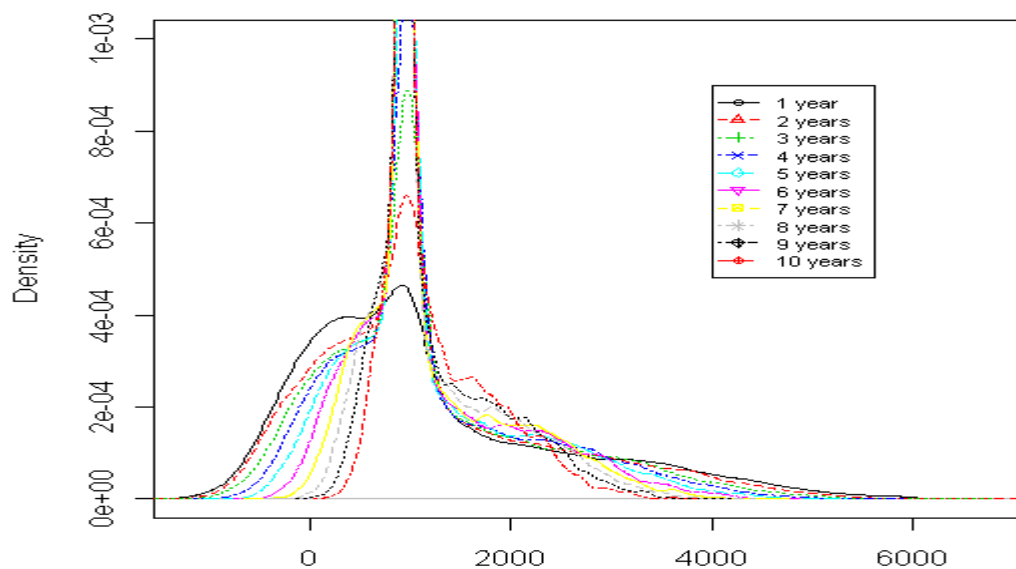
Here we analyse impact of number and maturity of short-terms option on costs in semi-static hedging. Figure 4 and Figure 5 show evolution of empirical density related of rising of number and maturity of short term options.

**Figure 4 – Impact of number of shorter-options**



We can see that impact of short term options on density distribution is not clear.

**Figure 5 – Impact of maturity of shorter-options**

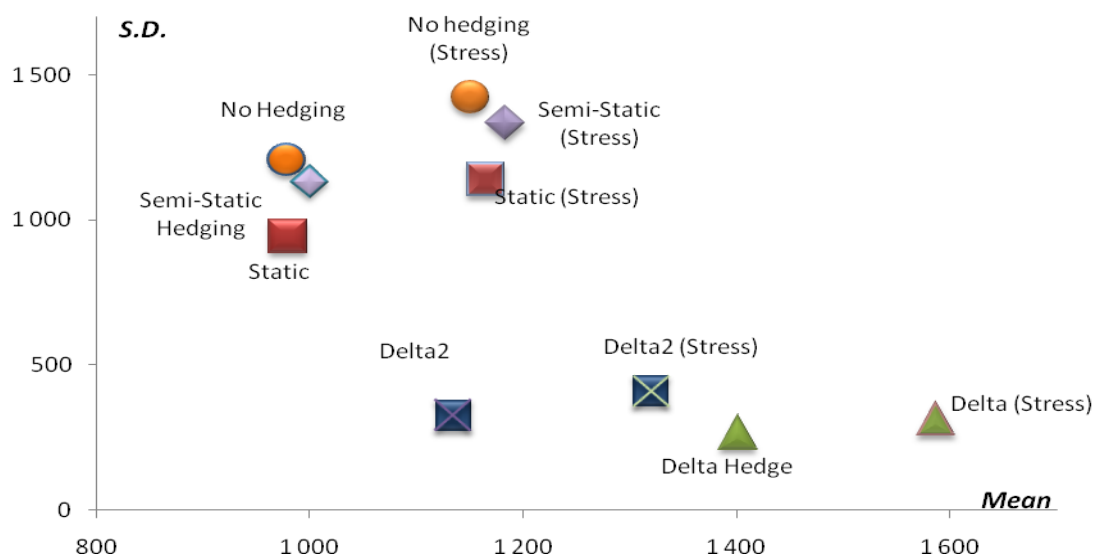


It is noted that the density of distribution condenses with the increase in the maturity of the options of hedging. That represented a reduction of the volatility of the future costs. We also note that the tail of distribution narrowed.

We conclude that the indicators of risks decrease with the maturity of the hedging options but impact of the number of option is not clear.

### 6.2.2. Impact of future mortality

In the following, we will stress the hedging strategies to a rise of future mortality. The Figure 6 presents evolution of plan Mean-variance related of 20% increasing of future mortality.

**Figure 6 – Evolution of plan Mean-Variance for 20% increasing of future mortality**

We can notice that, as awaited, the rise of future mortality increased expectation of future costs of all strategies. However, we note a fall of standard deviation of Delta and Delta2 strategies.

If we are interested in risk extreme indicators, (Table 6), Delta with frequently rebalancing and delta with rebalancing according to interval of error are always weakest VaR, CTE and Maximum loss than others strategies.

**Table 6 –Risk indicators related to 20% increasing of future mortality**

	Expected	Median	Standard deviation	VaR 99,75%	VaR 99%	CTE 99,75%	CTE 99%	maximum
No Hedging	1 150	486	1 424	6 124	5 433	6 125	5 438	7 712
Static minimisation	1 165	742	1 140	5 375	4 724	5 376	4 730	13 822
Delta Hedge	1 587	1 548	313	2 761	2 465	2 762	2 467	3 330
Delta2 Hedging	1 320	1 269	408	2 931	2 524	2 932	2 526	3 634
Semi-Static Hedging	1 183	886	1 333	5 734	5 160	5 734	5 164	6 877

But, in case of error of estimation of future mortality, delta hedging with frequently rebalancing of hedging portfolio provides lowest risk extreme indicators.

### 6.2.3. What about actual financial crisis?

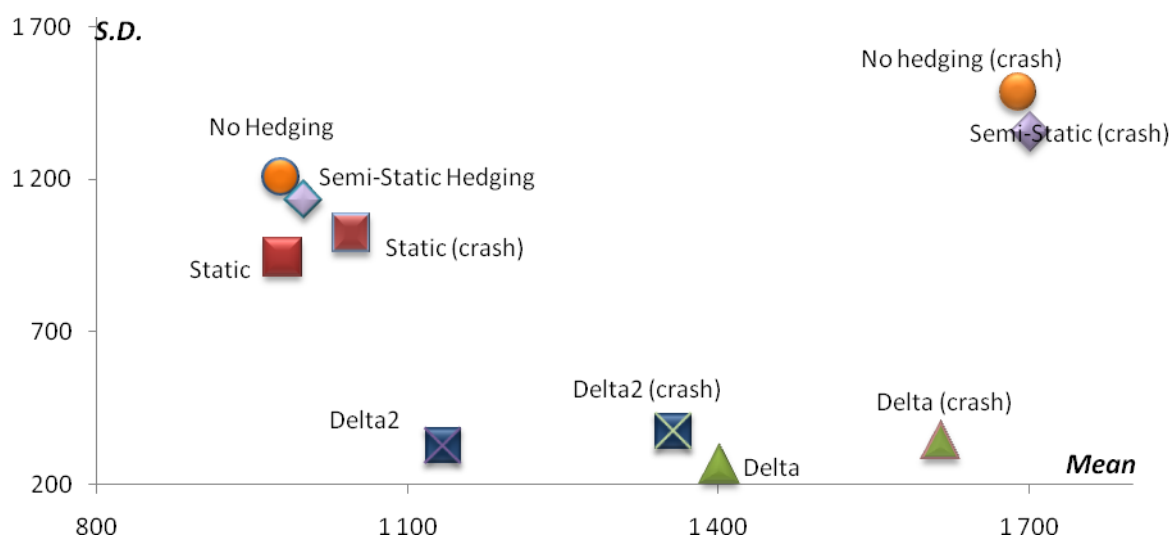
Since the beginning of the crisis of Subprime in September, 2007, we attended a collapse of the main world markets. This crisis achieved are highlight in September, 2008, with the bankruptcy of one of the first investment banks of Wall Street and the repurchase by the American state of the first world insurer.

The fear of a contagion of all the economy urged the main governments of industrial nations to take measures without the precedents to stop the crisis. This crisis had a devastating impact on the shares of the main financial institutions. As example during this period, the CaC40 lost more than 30% of its value.

In this sub-section, we are going to try to highlight the impact of this crisis on our insurance portfolio. For that purpose, we will create a shock in the decline of 30 % on the price of the underlying asset.

The first results are resumed on Figure 7.

**Figure 7 – Impact on plan Mean-Variance of 30% decline of the Underlying asset**



Firstly, we note that the shock has a very violent impact on the liabilities of the insurer in case of No Hedging. We note a strong increase of mean and volatility of future costs.

Secondly, we note that the strategy semi static is completely vulnerable in a violent shock on the share market. Indeed, the portfolio of the insurer being hedged by short-term options, the long-term payments are completely at the mercy of a not foreseen planned fluctuation in the underlying asset.

Thirdly, the impact is very weak when the hedging strategy is a static hedging. Indeed, a strong fall of the price of the underlying asset increases the liabilities of the insurer but also increases the value of the hedging portfolio, moreover the hedging portfolio is not to modify frequently; the costs of re-hedging are so reduced. In fine, the impact of the fall of the asset price is less violent than no hedging or semi-static hedging.

As for the delta strategies, we note an increase of costs of hedging and its volatility. However, the delta with recombining according to the objective of cover generates weak cost while the DRF generates lowest volatility.

According to the mean-variance criterion, It is more difficult to make a choice between the static hedging and the delta hedging because the costs generate by static hedging are weaker than delta hedging but with a stronger volatility.

However, if the indicators of risk extreme are analyzed (see Table 7), the strategies in delta provide weakest.

**Table 7 – Risk indicators related to 30% decline of underlying asset**

	Expected	Median	Standard deviation	VaR 99,75%	VaR 99%	CTE 99,75%	CTE 99%	maximum
No Hedging	1 689	1 320	1 486	5 838	5 258	5 839	5 262	6 586
Static minimisation	1 046	789	1 024	4 815	3 787	4 821	3 797	12 750
Delta Hedge	1 614	1 640	343	2 495	2 352	2 495	2 353	2 932
Delta2	1 357	1 347	374	2 486	2 297	2 486	2 298	3 044
Semi-Static	1 701	1 362	1 354	5 471	4 972	5 471	4 976	6 500

The delta hedging with rebalancing according to an interval of error gives weakest VaR and CTE, but the delta with frequently rebalancing of hedging portfolio gives the weakest maximum Loss.

## 7. Results in Merton jumps framework

The results presented below were obtained starting from the following parameters:

Number of simulation	1000
Maturity of insurance portfolio	15 years
Volatility of underlying asset	20 %
Intensity of poisson process	1
Volatility of discontinuous part (Jump volatility)	15%
Mean of amplitude of jump	0%
Drift of underlying asset	8.5%
Risk-free interest rate	5%
Guarantee	100
Initial value of underlying asset	100
Frequency of observation portfolio	Monthly
Costs of transaction	1%
Number of short-term options using in Carr hedging	2
Maturity of short-term options using in Carr hedging	1 year
Insurance portfolio	1,000 insured aged 45
Interval of re-hedging in corrected delta hedging	[-1,1]

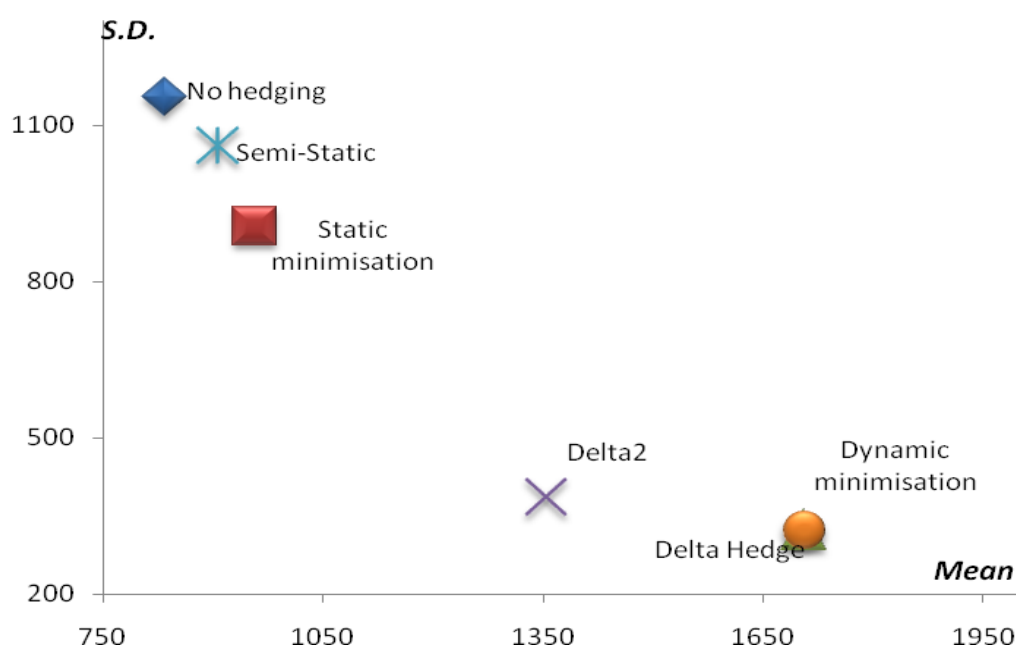
The results are summarized in Table 8.

**Table 8 - Risk indicators of costs related of Hedging strategies in Merton framework**

	Expected	Median	Standard deviation	VaR 99,75%	VaR 99%	CTE 99,75%	CTE 99%	Maximum
No Hedging	834	259	1154	5167	4587	5168	4591	5841
Static minimisation	958	629	906	4381	3895	4382	3899	5142
Dynamic minimisation	1706	1652	322	3016	2605	3017	2607	3421
Delta Hedge	1706	1652	322	3016	2605	3017	2607	3421
Delta2 Hedging	1355	1338	385	2748	2397	2748	2399	3011
Semi-Static	907	743	1060	4458	3851	4459	3857	5730

We can note that, as in Black-Scholes framework, the implementation of hedging strategies will generate future costs on average higher than liabilities of insurer. But hedging strategies reduce standard deviation and tails distribution of liabilities. We see that dynamic minimisation and delta hedging with frequently rebalancing provide same risk indicators. They give the weakest volatility of costs. Delta hedging with rebalancing according to an interval of hedging provides weakest tail risk indicators.

We can also compare hedging strategies on plan Mean-Variance (Figure 8).

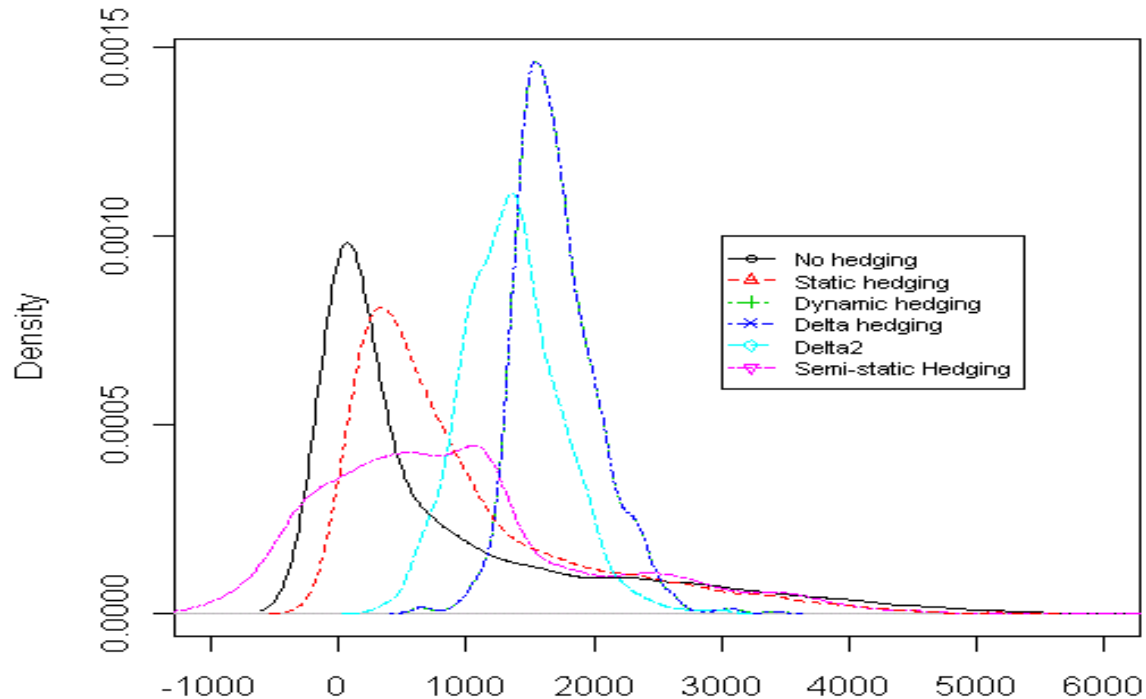
**Figure 8 – Representation in the plan Mean-Variance of the hedging strategies in Merton framework**

As in Black-Scholes model, all the hedging strategies are better than no hedging according to Mean-Variance criterion. Static hedging is better than semi-static hedging with short-term options.

Delta hedging and dynamic minimisation provide lower volatility but higher expectation of cost than static and semi-static hedging. Delta hedging, with rebalancing according to interval of hedging, gives weaker expectation and higher volatility of costs than dynamic minimisation and delta hedging with frequently rebalancing.

In Figure 9, we represent empirical density of distribution of hedging strategies.

**Figure 9 – Density of costs of hedging strategies in Merton framework**



## 8. Conclusion

In our study, we analyzed the optimality of some hedging strategies allowing the insurer to reduce the risk related to a portfolio of unit-linked life insurance contracts with minimum death guarantee. We noted that all the hedging strategies developed of generate costs on average higher than no hedging, but all these strategies reduce the volatility of the future costs and the indicators of extreme risks (VaR and CTE).

We were interested in 3 types of hedging strategies, the strategies in neutral delta, which consist to match the sensibilities of the hedging portfolio and the liabilities of the insurer, the strategies by minimization the variance of the error of hedging and the semi-static strategy of hedging using options of short term.

In the environments of Black-Scholes and Merton, we notice that the optimal strategies are the strategies in neutral delta and the dynamic strategy by minimization of the variance which provide the same results (If we just content ourselves with a portfolio constituted by the underlying asset and the free-risk asset).

The static strategy reduced volatility, the VaR and the CTE, but in the model of BS it can generate a maximum loss stronger than no hedging. This strategy is little sensitive to the increase of cost of transaction, to the periodicity of compensation for the policyholders, and to a strong fall of the price of the underlying asset. On the other hand, the costs and its volatility progress in the same way as no hedging cover in case of future abnormally high death rate.

The semi-static hedging is not very sensitive to an increase in the costs of transaction but it is at mercy, as well as no hedging and static hedging, at the risk of a strong future mortality. Although the increase of the maturity of the short-term options reduces the risk of the



portfolio, this strategy is not satisfactory. Indeed, the semi-static hedging is based on a very strong assumption, it makes the assumption of the existence of a continuum of options of maturity  $u$ . For example, our semi-static portfolio of hedging is composed of 420 options of maturity 1 year. The availability of all short-term options on the market with maturity considered is not guaranteed. Moreover, results provided by this strategy are not better than others hedging strategies.

Certainly, the strategies in delta are more expensive to the insurer, but they provide the best risk indicators. In case of the insurer pays according to a strong frequency, the delta hedging with rebalancing according to interval of error must be privileged; otherwise the delta hedging with frequently rebalancing reorganization is better. These strategies are certainly sensitive to an increase of the costs of transaction, but in case of increase of the future mortality or the strong fall in price of underlying asset, the impact on the risks indicators is less strong than in the case of the other hedging strategies.

Insofar as we are satisfied with a hedging portfolio made up only of the risky asset and risk-free asset, it seems logical that the hedging strategy in delta with frequently rebalancing and the dynamic strategy by minimization give same results (see Gabriel F. & Surlas P. (2006)). However, the framework of the model of Merton is an incomplete market; the underlying asset is not enough to hedging the risk of the portfolio. The insurer is subjected to the risk of jump. The introduction of a second instrument of cover allows stage this insufficiency. Let us also note that the options in the semi-static hedging portfolio are exerted in their maturity and the value thus made up is reinvested in acquisition of underlying asset and risk-free asset. An alternative of this strategy would be to reinvest this amount in the purchase of new hedging portfolio of short term options and to reproduce the operation. These aspects are not treated here. However, we will return there in a later study.

## References

- Black F. & Scholes M. (1973) « The Pricing of Options and Corporate Liabilities », *Journal of Political Economy* 81(3) 637-54.
- Breedon D. T. & Litzenberger R. H (1978) « Prices of State-Contingent Claims Implicit in Option Prices », *Journal of Business* 51(4), 621–51
- Cont, R. (2001) « Empirical Properties of Asset Returns: Stylized Facts and Statistical Issues », *Quantitative Finance* 1, 223–36.
- Cont R. Tankov P. & Voltchkova E.(2005) « hedging with options in models with jumps », Ecole polytechnique centre de mathématiques appliquées. R.I 601
- Carr P. & Wu L. (2004) « Static Hedging of Standard Options », Bloomberg L.P. and Courant Institute Zicklin School of Business, Baruch.
- Desmedt, X., Chenut, X. and Walhin, J.F.(2005). « Actuarial Pricing of Minimum Death Guarantees in Unit-Linked Life Insurance: a Multi-Period Capital Allocation Problem ». *Proceedings of the 3rd Actuarial and Financial Mathematics Day*, Brussels.
- Deville L. (2001) « Estimation des coûts de transaction sur un marché gouverné par les ordres : le cas des composantes du CAC 40 ». Laboratoire de recherche en gestion et économie Université Louis Pasteur Pole européen de gestion et d'économie.
- Frantz C., Chenut X. & Walhin J.-F. (2003) , « Pricing and capital allocation for unit-linked life insurance contracts with minimum death guarantee, Institut des Sciences Actuarielles », Université catholique de Louvain.
- Gabriel F. & Sourlas P. (2006) « Couverture d'options en présence de sauts », Mémoire d'Actuariat, ENSAE.
- Markowitz H. (1952) « Portfolio Selection » *The Journal of Finance*, Vol. 7, No. 1. pp. 77-91
- Kou S. & Wang H. (2004) « Option pricing under a double exponential jump-diffusion model », *Management Science* 1178-92.
- Merton, R. C. (1976) « Option Pricing When Underlying Stock Returns are Discontinuous », *Journal of Financial Economics* 3, 125–44.
- Milliman Consultants and actuaries (2005) *Financial risk management for insurance liabilities –the theoretical and practical aspects of hedging*.
- Moller, T. (1998). « Risk-Minimizing hedging strategies for unit-linked life insurance contracts », *Astin Bulletin* 28(1), pp. 17-47.
- Planchet F., Thérond P .E. [2005] « L'impact de la prise en compte des sauts boursiers dans les problématiques d'assurance », *Proceedings of the 15<sup>th</sup> AFIR Colloquium* .
- Planchet F., Thérond P.-E. & Jacquemin J. (2005) « Modèles financiers en assurance », *Economica*.
- Randrianarivony R. (2006) « Implémentation et extension de Kou pour l'évaluation d'options », ISFA, Université de Lyon 1.
- Tankov P.(2008) « Calibration de Modèles et Couverture de Produits Dérivés », Université Paris VII.
- Tankov P. & Voltchkova E. (2006) « Jump-diffusion models: a practitioner's guide »
- Therond P.-E. (2007) *Mesure et gestion des risques d'assurance: analyse critique des futurs référentiels prudentiel et d'information financière*, PhD thesis, Université de Lyon, ISFA Actuarial School.

## Appendix A. The theorem of CARR et Wu [2004]

### A1. Risk-neutral density

We assume that the random variable  $S_T$  for the maturity  $T$  admits a density under the risk-neutral measure  $\mathcal{Q}$ . We notice  $f_{S,T,t}(k) = f(S,t,k,T)$  this density. We also assume that  $P_K(k) = P(S,t,k,T)$  is  $C^2$  function. We can write:

$$\begin{aligned} P_K(k) &= P(S,t,k,T) = e^{-r(T-t)} E\left([k - S_T]^+\right) \\ &= e^{-r(T-t)} \int_0^\infty [k-s]^+ f_{S,T,t}(s) ds \\ P_K(k) &= e^{-r(T-t)} \int_0^k (k-s) f_{S,T,t}(s) ds \\ &= e^{-r(T-t)} \left( k \times \int_0^k f_{S,T,t}(s) ds - \int_0^k s \times f_{S,T,t}(s) ds \right) \end{aligned}$$

If  $F_1(s)$  is the primitive<sup>1</sup> of  $f_{S,T,t}(s)$  and  $F_2$  is the primitive of  $s \times f_{S,T,t}(s)$ , we can write

$$P_K(k) = e^{-r(T-t)} \left[ k(F_1(k) - F_1(0)) - (F_2(k) - F_2(0)) \right]$$

We derive the two members of this equation by the strike  $k$ , we have:

$$\frac{\partial P_K(k)}{\partial k} = e^{-r(T-t)} \left[ (F_1(k) - F_1(0)) + k \times f_{S,T,t}(k) - k \times f_{S,T,t}(k) \right] = e^{-r(T-t)} \left[ (F_1(k) - F_1(0)) \right]$$

We derive one second and obtain the formula of Breeden & Litzenberger:

$$\frac{\partial^2 P_K(k)}{\partial k^2} = e^{-r(T-t)} f_{S,T,t}(k) = e^{-r(T-t)} f(S,t,k,T)$$

$$\Leftrightarrow f(S,t,k,T) = e^{r(T-t)} \frac{\partial^2 P(S,t,k,T)}{\partial k^2} \quad \square$$

### A2. Proof of the theorem of CARR et Wu [2004]

We assume frictionless markets and no arbitrage. No arbitrage implies that there exists a risk-neutral probability measure  $\mathcal{Q}$  defined on a probability space  $(\Omega, \mathcal{F}, \mathcal{Q})$  such that this instantaneous expected rate of return on every asset equals the instantaneous riskfree rate  $r$ .

Analysis is restricted to the class of models in which the risk-neutral evolution of the stock price is Markov in the stock price  $S$  and the calendar time  $t$ .

We use  $P_t(K,T)$  to denote the time- $t$  price of a European put with strike  $K$  and maturity  $T$ . Our assumption that the state is fully described by the stock price and time implies that there exists a put pricing function  $P(S, t; K, T)$  such that:  $P_t(K,T) = P(S_t, t; K, T)$ ,  $t \in [0, T]$  and  $K \geq 0$

---

<sup>1</sup>  $\frac{\partial F_1(s)}{\partial s} = f_{S,T,t}(s)$  and  $\frac{\partial F_2(s)}{\partial s} = s \times f_{S,T,t}(s)$

Using the standard argument of financial theory,  $\bar{P}(S,t,K,T) = e^{-r(T-t)}P(S,t,K,T)$  is martingale, under the risk neutral measure. Then for all  $u \in [t,T]$  we have:

$\bar{P}(S,t,K,T) = E_Q(\bar{P}(S,u,K,T)|F_t)$  where  $(F_t)_{t \geq 0}$  is the filtration associated of the probability space  $(\Omega, F, Q)$ .

$$\begin{aligned}\bar{P}(S,t,K,T) &= E_Q(\bar{P}(S,u,K,T)|F_t) \\ &\Leftrightarrow P(S,t,K,T) = e^{-r(u-t)}E_Q(\bar{P}(S,u,K,T)|F_t) \\ P(S,t,K,T) &= e^{-r(u-t)} \int_0^\infty f(S,t,k,u)P(k,u,K,T)dk \\ &= e^{-r(u-t)} \int_0^\infty e^{r(u-t)} \frac{\partial^2 P}{\partial K^2}(S,t,k,u)P(k,u,K,T)dk, \text{ Using formula of Breeden \& Litzenberger.} \\ &= \int_0^\infty \frac{\partial^2 P(S,t,k,u)}{\partial k^2} P(k,u,K,T)dk\end{aligned}$$

We integrate that equation by parts twice and obtain:

$$\begin{aligned}P(S,t,K,T) &= \int_0^\infty \frac{\partial^2 P(S,t,k,u)}{\partial k^2} P(k,u,K,T)dk \\ &= \left[ P_u(K,T) \times \frac{\partial P(S,t,k,u)}{\partial k} \right]_{k=0}^{k=\infty} - \left[ P_t(k,u) \times \frac{\partial P(k,u,K,T)}{\partial k} \right]_{k=0}^{k=\infty} \\ &\quad + \int_0^\infty \frac{\partial^2 P(k,u,K,T)}{\partial k^2} P(S,t,k,u)dk\end{aligned}$$

Using the following boundary conditions

$$\left. \frac{\partial P(S,t,k,u)}{\partial k} \right|_{k \rightarrow \infty} = e^{-r(u-t)}, \left. \frac{\partial P(k,u,K,T)}{\partial k} \right|_{k \rightarrow \infty} = 0, \quad P(0,u,K,T) = e^{-r(T-u)}K$$

$$\text{we obtain the final result: } P(S,t,K,T) = \int_0^\infty \frac{\partial^2 P(k,u,K,T)}{\partial k^2} P(S,t,k,u)dk$$

### A3. Proof of the theorem of CARR et Wu [2004] using no arbitrage arguments

Assume that at time  $t$ , the market price of a put option with strike  $K$  and maturity  $T$  exceeds the price of a gamma weighted portfolio of put options for some earlier maturity  $u$ , then, conditional on the validity of the Markovian assumption, the arbitrage is to sell the put option of strike  $K$  and maturity  $T$ , and to buy the gamma weighted portfolio of all put options maturing at the earlier date  $u$ . The cash received from selling the  $T$  maturity call exceeds the

cash spent buying the portfolio of nearer dated calls. At time  $u$ , the portfolio of expiring puts pays off are  $\int_0^{\infty} \frac{\partial^2 P(k, u, K, T)}{\partial k^2} (k - S_u)^+ dk$ .

But we have:

$$\int_0^{\infty} \frac{\partial^2 P(k, u, K, T)}{\partial k^2} (k - S_u)^+ dk = \int_{S_u}^{\infty} \frac{\partial^2 P(k, u, K, T)}{\partial k^2} (k - S_u) dk$$

Integrating by parts twice this payoff, we obtain:

$$\begin{aligned} \int_{S_u}^{\infty} \frac{\partial^2 P(k, u, K, T)}{\partial k^2} (k - S_u) dk &= \left[ (k - S_u) \times \frac{\partial P(k, u, K, T)}{\partial k} \right]_{S_u}^{\infty} - \int_{S_u}^{\infty} \frac{\partial P(k, u, K, T)}{\partial k} dk \\ &= \left[ (k - S_u) \times \frac{\partial P(k, u, K, T)}{\partial k} \right]_{S_u}^{\infty} - [P(k, u, K, T)]_{S_u}^{\infty} \\ &= P(S_u, u, K, T) \end{aligned}$$

If the price of a gamma weighted portfolio of put options for some earlier maturity  $u$  exceeds the price of a put option with strike  $K$  and maturity  $T$ , the arbitrage is to sell the gamma weighted portfolio of all put options maturing at the earlier date  $u$ , and to buy the put option of strike  $K$  and maturity  $T$ .

Under the assumption of no arbitrage opportunity, we have necessarily:

$$P(S, t, K, T) = \int_0^{\infty} \frac{\partial^2 P(k, u, K, T)}{\partial k^2} P(S, t, k, u) dk$$

## Appendix B. Gauss-hermite Quadrature

The Gauss-Hermite quadrature rule is designed to approximate an integral of the form

$$\int_{-\infty}^{+\infty} f(x) e^{-\frac{x^2}{2}} dx$$

where  $f(x)$  is an arbitrary smooth function. After some rescaling, the integral can be regarded as an expectation of  $f(x)$  where  $x$  is a normally distributed random variable with zero mean and variance of two.

For a given target function  $f(x)$ , the Gauss-Hermite quadrature rule generates a set of weights  $w_i$  and nodes  $x_i$ ,  $i = 1, 2, \dots, a$ . The abscissas for quadrature order  $a$  are given by the roots  $x_i$  of the hermite polynomials  $H_a(x)$ , which occur symmetrically about 0. The weights are

$$w_i = \frac{A_{a+1}\phi_a}{A_a H'_a(x_i) H_{a+1}(x_i)}$$

$$= \frac{A_a}{A_{a-1}} \frac{\phi_a}{H'_a(x_i) H_{a-1}(x_i)}$$

with  $A_a = 2^a$ ;  $\frac{A_{a+1}}{A_a} = 2$ ; and  $\phi_a = \sqrt{\pi} 2^a a!$

$$H'_a(x) = 2aH_{a-1}(x) = 2aH_a(x) - H_{a+1}(x)$$

that are defined by

$$\int_{-\infty}^{+\infty} f(x) e^{-\frac{x^2}{2}} dx = \sum_{j=1}^a w_j f(x_j) + \frac{a! \sqrt{\pi}}{2^a} \frac{f^{(2a)}(\xi)}{(2a)!} \text{ where } \xi \in (-\infty, +\infty)$$

The approximation error vanishes if the integrand  $f(x)$  is a polynomial of degree equal or less than  $2a-1$ .

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